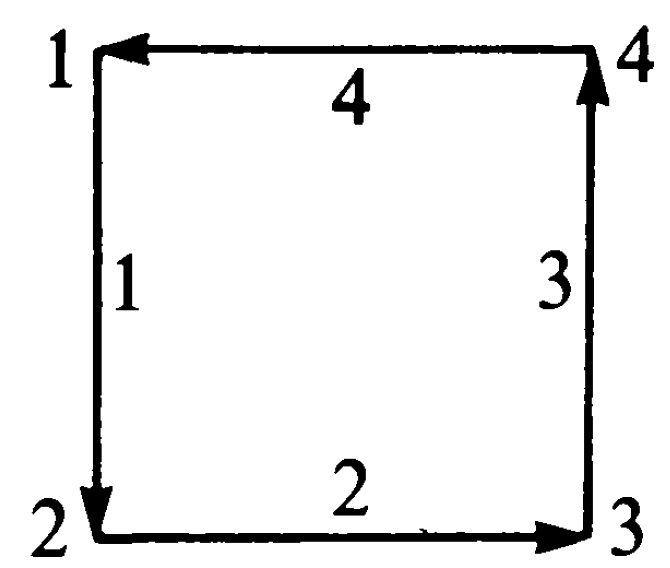
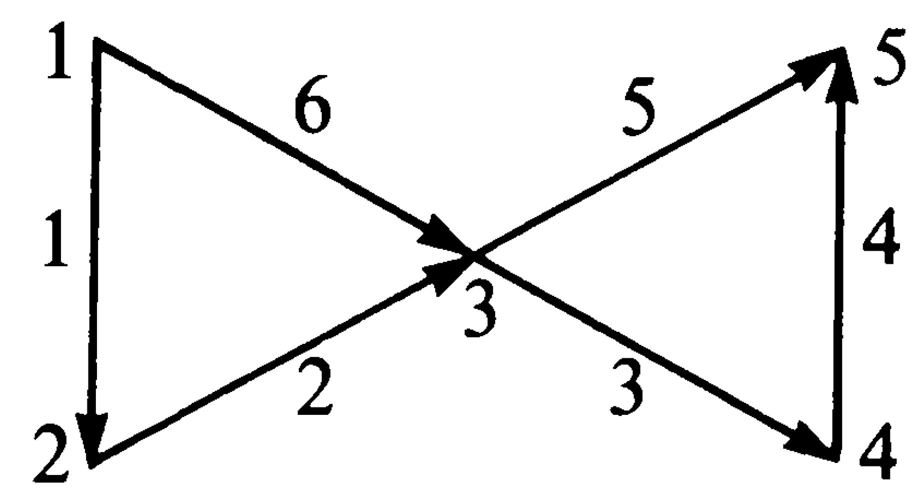
contact with the combinatorial problems of Chapter 7—which are matrix problems of an important and special kind.

EXERCISES

1.6.1 Suppose you are given a list of the edges of a graph; for edge i you are given its two node numbers $a(i)$ and $b(i)$. Describe the steps of an algorithm (or write a code) to decide whether or not the graph is connected.

1.6.2 Write down the incidence matrices A_1 and A_2 for the following graphs:





For which right sides does $A_1x=b$ have a solution? Which vectors are in the nullspace of A_1^T?

1.6.3 The previous matrix A_2 should have $n-1$ independent rows; which are they? There should also be $m-n+1$ independent vectors in the nullspace of A_2^T, one from each loop; which are they?

1.6.4 If those two graphs are the disconnected pieces of a single graph with 9 nodes and 10 edges, what is the rank of its incidence matrix? With n nodes, m edges, and p unconnected pieces, find the number of independent solutions to $Ax=0$ and $A^Ty=0$.

1.6.5 If A is the incidence matrix of a connected graph and $Ax=0$, show that $x_1=x_2=\cdots=x_n$. Each row of $Ax=0$ is an equation $x_j-x_k=0$; how do you prove that $x_j=x_k$ even when no edge goes from node j to node k?

1.6.6 In a graph with N nodes and N edges show that there must be a loop.

1.6.7 For electrical networks x represents potentials, Ax represents potential differences, y represents currents, and $A^Ty=0$ is Kirchhoff's current law (Section 2.3). Tellegen's theorem says that Ax is perpendicular to y. How does this follow from the fundamental theorem of linear algebra?

1.6.8 By transposing $AA^{-1}=I$, show that the transpose of A^{-1} is the inverse of A^T. Verify $(A^{-1})^T=(A^T)^{-1}$ for the 2 by 2 matrix

$$A=\begin{bmatrix}1 & 0\\ 4 & 1\end{bmatrix}.$$

1.6.9 Describe the vectors in the column space, nullspace, row space, and left nullspace of

$$A=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}.$$

1.6.10 In the list below, which classes of matrices contain

$$A=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{bmatrix} \quad\text{and}\quad B=\begin{bmatrix}\frac{1}{3}&\frac{1}{3}&\frac{1}{3}\\\frac{1}{3}&\frac{1}{3}&\frac{1}{3}\\\frac{1}{3}&\frac{1}{3}&\frac{1}{3}\end{bmatrix}?$$

Symmetric, orthogonal, triangular, invertible, projection, permutation, Jordan form, diagonalizable. Find the eigenvalues of A and B.

1.6.11 Find the LU factorizations of A_1 and PA_2:

$$A_1=\begin{bmatrix}1&-1&0\\0&1&-1\\1&0&-1\end{bmatrix},\quad A_2=\begin{bmatrix}-1&1&0&0\\-1&0&1&0\\0&-1&1&0\\0&-1&0&1\\0&0&-1&1\end{bmatrix},\quad P=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&0&1&0\\0&0&1&0&0\\0&0&0&0&1\end{bmatrix}.$$

L contains the multipliers l_{ij}, with $l_{ii}=1$, and the permutation P exchanges rows 3 and 4 to avoid a zero in the pivot position.

1.6.12 Invert $PA=LDU$ to find a formula for A^{-1}. It allows the inverse to be computed with n^3 multiplications—or about $n^3/2$ when A is symmetric.

1.6.13 Factor A into LU and find $A^{-1}=U^{-1}L^{-1}$ if

$$A=\begin{bmatrix}1&-1&0&0\\-1&2&-1&0\\0&-1&2&-1\\0&0&-1&2\end{bmatrix}.$$

Solve $Ax=b$ if b has components $1, 0, 0, 0$.

1.6.14 (a) Find two vectors that are orthogonal to $(1, 1, 1, 0)$ and $(0, 1, 1, 1)$, by making these the rows of A and solving $Ax=0$.

(b) Find 2 equations in 4 unknowns whose solutions are the linear combinations of $(1, 1, 1, 0)$ and $(0, 1, 1, 1)$.

1.6.15 The fundamental theorem says that either b is in the column space or it is not orthogonal to the left nullspace: Either (1) $Ax=b$ for some x, or (2) $A^Ty=0$, $y^Tb\neq 0$ for some y. Show directly that (1) and (2) cannot both be true.

1.6.16 Find matrices A for which the number of solutions to $Ax=b$ is

(1) 0 or 1 depending on b
(2) ∞ for every b
(3) 0 or ∞ depending on b
(4) 1 for every b.

How is the rank r related to m and n in each of your examples?

1.6.17 Why is the plane $x + 2y + 3z = 0$ perpendicular to the vector with components 1, 2, 3? Give the equation of a plane parallel to that one but containing the point $x = y = z = 1$.

1.6.18 Given 4 vectors in 5-dimensional space, how could a computer decide if they are linearly independent?

1.6.19 True or false: **1.** There is no matrix whose row space contains $[1 \quad 2 \quad 1 \quad 1]^T$ and whose nullspace contains $[1 \quad -2 \quad 1 \quad 1]^T$.

2. Exactly one vector is in both the row space and the nullspace.
3. If A and B have rank 3 then $A + B$ has rank at most 6.
4. The rank of the matrix with every $a_{ij} = 1$ is $r = 1$.
5. The rank of the n by n matrix with $a_{ij} = i + j$ is $r = n$.

1.6.20 From $A = S\Lambda S^{-1}$ construct the matrix that has eigenvalues 3 and 4 with eigenvectors (2, 0) and (1, 1).

1.6.21 Let $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $w = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ so that $v^T w = 0$.

(a) What is $A = vw^T$?
(b) What are the eigenvalues of A?
(c) What are its eigenvectors?
(d) What is its Jordan form?

1.6.22 If $A = S\Lambda S^{-1}$ show by transposing that A and A^T have the same eigenvalues. What matrix contains the eigenvectors of A^T? They agree with the eigenvectors of A only if $AA^T = A^T A$.

1.6.23 If the pivots of A are $d_1 = 2$ and $d_2 = 3$ (without a row exchange) what can you say about the eigenvalues? What if you also know that the trace is $a_{11} + a_{22} = 6$?

1.6.24 Show that $A^T A$ can never have a negative eigenvalue: if $A^T A x = \lambda x$ then $\lambda = \|Ax\|^2 / \|x\|^2$.

1.6.25 Prove (using the trick of multiplying by y^T) that if $AA^T y = 0$ then $A^T y = 0$. The matrices AA^T and A^T have the same nullspace, the same row space, and the same rank. Find all three if

$$A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

Note The following three exercises all involve the matrix

$$A = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}.$$

1.6.26 Factor A into a product QR of an orthogonal matrix times an upper triangular matrix.

1.6.27 Factor A into its polar form $A = QB$.

1.6.28 Find the singular value decomposition $A = Q_1 \Sigma Q_2^T$.

1.6.29 If u has components $\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}$, find

(a) the rank-one projection matrix $A = uu^T$
(b) the projection of $b = (1, 0, 0, 0)$ onto the line through u
(c) the eigenvalues of A
(d) A^2.

1.6.30 If u has components $\frac{2}{3}, \frac{2}{3}, \frac{1}{3}$, find

(a) the rank-two projection matrix $P = I - uu^T$
(b) the projection Pb of $b = (1, 0, 0)$ onto the plane perpendicular to u
(c) the eigenvalues of P
(d) P^2.

1.6.31 If u has components $\frac{2}{3}, \frac{2}{3}, \frac{1}{3}$, find

(a) the Householder matrix $H = I - 2uu^T$
(b) the eigenvalues of H
(c) H^2.

1.6.32 Describe each of the four fundamental subspaces (lines or planes?) of

$$A = \begin{bmatrix} 1 \\ 1 \end{bmatrix} [2 \quad 1 \quad 1] = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

1.6.33 For a rank-one matrix $A = vw^T$ show that A^TA is a multiple of ww^T, and that it has the eigenvalue $\|v\|^2 \|w\|^2$ with eigenvector w. (The square root $\|v\| \, \|w\|$ is the nonzero singular value of A.)

1.6.34 The inverse of $B = I - vw^T$ has the form $B^{-1} = I - cvw^T$. By multiplication find the number c. Under what condition on v and w is B not invertible?

1.6.35 (a) If A is invertible then the inverse of $B = A - vw^T$ has the form $B^{-1} = A^{-1} - cA^{-1}vw^TA^{-1}$. By multiplication (watching for a scalar inside $vw^TA^{-1}vw^TA^{-1}$) find the number c.

(b) If you subtract 1 from the first entry a_{11} of A, what matrix is subtracted from A^{-1}? In A^{-1} let q be the first column, r^T be the first row, and s be the first entry.