determinants are positive. And conversely, if a pivot $d_k \leq 0$ appears then the sequence of positive determinants is broken. Thus we have a fourth test for positive definiteness, very close to the second one:

(4) All the submatrices $A^{(k)}$ have positive determinants.

This completes the matrix theory.

EXERCISES

1.5.1 Find the eigenvalues of $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ and show that $(1, -1)$ and $(1, 1)$ are always eigenvectors. Confirm that $\lambda_1 + \lambda_2$ equals the sum of diagonal entries (the trace) and $\lambda_1 \lambda_2$ equals the determinant. Under what conditions on a and b is this matrix positive definite?

1.5.2 Write the preceding matrix in the form $A = S\Lambda S^{-1} = Q\Lambda Q^T$.

1.5.3 Find all eigenvalues and all eigenvectors (there are more than usual) of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

1.5.4 Find the eigenvalues and eigenvectors of

$$A_1 = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Check the trace and determinant.

1.5.5 Solve the first-order system

$$\frac{du}{dt} = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} u \quad \text{with} \quad u_0 = \begin{bmatrix} 0 \\ 6 \end{bmatrix}.$$

1.5.6 Solve the second-order system

$$\frac{d^2u}{dt^2} + \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} u = 0 \quad \text{with} \quad u_0 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \quad \text{and} \quad u_0' = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

These initial conditions do not activate the zero eigenvalue (see the following exercises).

1.5.7 Suppose each column of A adds to zero, as in

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix}.$$

(a) Prove that zero is an eigenvalue and A is singular, by showing that the vector of ones is an eigenvector of A^T. (A and A^T have the same eigenvalues, but not the same eigenvectors.)

(b) Find the other eigenvalues of this matrix A, and all three eigenvectors.

1.5.8 With this 3 by 3 matrix, add the three equations $du/dt = Au$ to show that $u_1 + u_2 + u_3$ is a constant. What is the general solution (as in equation (10)) for this example? *Note*: When $\omega = 0$, $\sin \omega t$ is replaced by t in the general solution—just as the 1 by 1 model problem $d^2u/dt^2 = 0$ is solved by $u = a + bt$.

1.5.9 The $x - y$ axes are rotated through an angle θ by

$$Q = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

(1) Verify that $Q^T = Q^{-1}$, so that Q is orthogonal.

(2) The rotated vector Qx is never in the same direction as x, so Q has no real eigenvalues. Find the (complex) eigenvalues and eigenvectors.

1.5.10 (a) Find the eigenvectors of $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and show that they are perpendicular—remembering that the inner product of complex vectors is $x_1^H x_2 = \bar{x}_1^T x_2$ instead of $x_1^T x_2$.

(b) Solve the system $du/dt = Au$ with $u_0 = (3, 4)$.

1.5.11 Why is the sum of entries on the diagonal of AB equal to the sum along the diagonal of BA? In other words, what terms contribute to the trace of AB?

1.5.12 Show that the determinant equals the product of the eigenvalues by imagining that the characteristic polynomial is factored into

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda), \qquad (*)$$

and making a clever choice of λ.

1.5.13 Show that the trace equals the sum of the eigenvalues, in two steps. First, find the coefficient of $(-\lambda)^{n-1}$ on the right side of (*). Next, look for all the terms in

$$\det(A - \lambda I) = \det \begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix}$$

which involve $(-\lambda)^{n-1}$. Explain why they all come from the main diagonal, and find the coefficient of $(-\lambda)^{n-1}$ on the left side of (*). Compare.

1.5.14 (a) Show how the equation $u' = Au$ becomes $v' = Jv$ if $A = SJS^{-1}$ and $v = S^{-1}u$.
(b) By back substitution (second equation first) solve

$$\begin{aligned} dv_1/dt &= 3v_1 + v_2 \\ dv_2/dt &= 3v_2 \end{aligned} \quad \text{with} \quad v(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The term involving te^{3t} enters because of the repeated eigenvalue.

1.5.15 With masses $m_1 = m_2 = 1$ and spring constants $c_1 = c_3 = 4$, $c_2 = 6$, the differential equation is

$$u_{tt} + Ku = 0 \quad \text{with} \quad K = \begin{bmatrix} 10 & -6 \\ -6 & 10 \end{bmatrix}.$$

Find its natural frequencies ω_1 and ω_2, and from the eigenvectors find its two pure oscillations $u = (a \cos \omega t + b \sin \omega t)x$.

1.5.16 If the first mass starts at equilibrium and the second is displaced to $u_2 = 6$, with initial velocities $v_1 = v_2 = 0$, find their motions u_1 and u_2.

1.5.17 If $Kx = \omega^2 x$, show that $u = (ce^{i\omega t} + de^{-i\omega t})x$ solves the differential equation $u_{tt} + Ku = 0$. This exponential form is an alternative to the trigonometric form $u = (a \cos \omega t + b \sin \omega t)x$.

1.5.18 Solve the example in the text, with frequencies $\omega_1 = 1$ and $\omega_2 = \sqrt{3}$ as in (15), if the masses start at $u_1 = u_2 = 0$ with velocities $v_1 = 1 - \sqrt{3}$ and $v_2 = 1 + \sqrt{3}$. Show that $u(t)$ is never again zero.

1.5.19 If K is negative instead of positive, and $u_{tt} = u$ instead of $u_{tt} + u = 0$, solutions will grow or decay rather than oscillating. Solve $u_{tt} = u$ with $u(0) = 2$, $du/dt(0) = 0$.

1.5.20 Suppose there is a damping term proportional to velocity in

$$M\frac{d^2u}{dt^2} + F\frac{du}{dt} + Ku = 0.$$

When will $u = e^{\lambda t}x$ be a solution?

1.5.21 If $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ then the determinant of $A - \lambda I$ is $(a - \lambda)(c - \lambda) - b^2$. From the formula for the roots of a quadratic, show that both eigenvalues are real.

1.5.22 Multiplying columns times rows, $A = Q\Lambda Q^T$ is

$$A = x_1\lambda_1 x_1^T + x_2\lambda_2 x_2^T + \cdots + x_n\lambda_n x_n^T.$$

This is the *spectral theorem*: Every symmetric matrix is a combination with weights λ of projections xx^T onto the eigenvectors. Write out this combination after rescaling to unit length the eigenvectors in the two text examples

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

1.5.23 (***Positive definite square root***) Suppose A is positive definite: $A = Q\Lambda Q^T$ with $\lambda_i > 0$. Let $A^{1/2} = Q\Lambda^{1/2}Q^T$ be the matrix with the same eigenvectors in Q and with eigenvalues $\lambda_i^{1/2}$. Explain why this $A^{1/2}$ is symmetric positive definite, and its square is A. Find $A^{1/2}$ if

$$A = \begin{bmatrix} 10 & -6 \\ -6 & 10 \end{bmatrix}.$$

1.5.24 If K and M are positive definite and $Kx = \lambda Mx$, prove that λ is positive. This is the ***generalized eigenvalue problem***, with two matrices. Find the two eigenvalues when

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

1.5.25 For the same matrices M and K, coming from masses $m_1 = 1$ and $m_2 = 2$ and spring constants $c_1 = c_2 = c_3 = 1$, find the two pure oscillations $u = (a \cos \omega t + b \sin \omega t)x$ of the system $Mu_{tt} + Ku = 0$. Since $M^{-1}K$ is no longer symmetric, its eigenvectors x_1 and x_2 are no longer perpendicular; verify that now $x_1^T M x_2 = 0$.

1.5.26 Suppose a single mass m is between two springs with constants c_1 and c_2; their other ends are fixed. Write down (1) the second-order equation (Newton's law) for the displacement u of the mass, and (2) the frequency ω in the solution $u = a \cos \omega t + b \sin \omega t$.