EXERCISES

1.4.1 Find the minimum value (and the minimizing x) for

(i) $P(x) = \frac{1}{2}(x_1^2 + x_2^2) - x_1 b_1 - x_2 b_2$
(ii) $P(x) = \frac{1}{2}(x_1^2 + 2x_1 x_2 + 2x_2^2) - x_1 + x_2$.

1.4.2 What equations determine the minimizing x for

(i) $P = \frac{1}{2}x^T A x - x^T b$
(ii) $P = \frac{1}{2}x^T A^T A x - x^T A^T b$
(iii) $E = \| Ax - b \|^2$?

1.4.3 Find the best least squares solutions to all three problems

$$\begin{bmatrix} 2 & -1 \\ 2 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}.$$

1.4.4 Find the best least squares solution to

$$Ax = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} = b.$$

Graph the measurements 0, 4, 2 at times $t = 1, 2, 3$, and the best straight line.

1.4.5 The best fit to b_1, b_2, b_3, b_4 by a horizontal line (a constant function $y = C$) is their average $C = (b_1 + b_2 + b_3 + b_4)/4$. Confirm this by least squares solution of

$$Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} [C] = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

From calculus, which C minimizes the error $E = (b_1 - C)^2 + \cdots + (b_4 - c)^2$?

1.4.6 Which three equations are to be solved by least squares, if we look for the line $y = Dt$ through the origin that best fits the data $y = 4$ at $t = 1$, $y = 5$ at $t = 2$, $y = 8$ at $t = 3$? What is the least squares solution $\bar{D}$ that gives the best line?

1.4.7 For the three measurements $b = 0, 3, 12$ at times $t = 0, 1, 2$, find

(i) the best horizontal line $y = C$
(ii) the best straight line $y = C + Dt$
(iii) the best parabola $y = C + Dt + Et^2$.

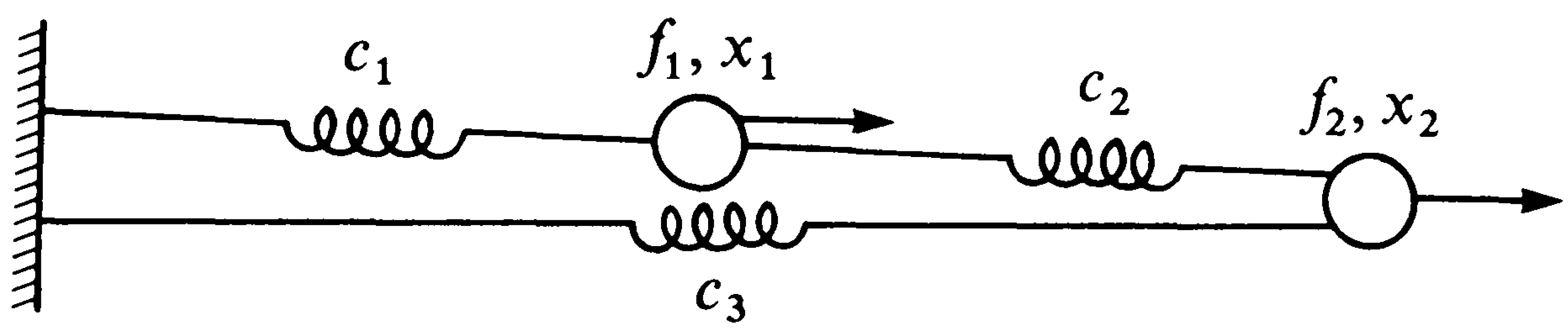
1.4.8 Multiple regression fits two-dimensional data by a plane $y = C + Dt + Ez$, instead of one-dimensional data by a line. If we are given $b_1 = 2$ at $t = z = 0$; $b_2 = 2$ at $t = 0$, $z = 1$; $b_3 = 1$ at $t = 1$, $z = 0$; and $b_4 = 5$ at $t = z = 1$, write down the 4 equations in the 3 unknowns C, D, E. What is the least squares solution from the normal equations?

1.4.9 For a matrix with more columns than rows, like

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix} \quad \text{or even} \quad A = [1 \quad 2],$$

the matrix A^TA is not positive definite. Why is it impossible for these columns to be independent? Compute A^TA in both cases and check that it is singular.

1.4.10 In a system with three springs and two forces and displacements write out the equations $e = Ax$, $y = Ce$, and $A^Ty = f$. For unit forces and spring constants, what are the displacements?



1.4.11 Suppose the lowest spring in Fig. 1.7 is removed, leaving masses m_1, m_2, m_3 hanging from the three remaining springs. The equation $e = Ax$ becomes

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Find the corresponding equations $y = Ce$ and $A^Ty = f$, and solve the last equation for y. This is the ***determinate*** case, with square matrices, when the factors in A^TCA can be inverted separately and y can be found before x.

1.4.12 For the same 3 by 3 problem find $K = A^TCA$ and A^{-1} and K^{-1}. If the forces f_1, f_2, f_3 are all positive, acting in the same direction, how do you know that the displacements x_1, x_2, x_3 are also positive?

1.4.13 If the matrix A is a column of 1's as in problem 5 above, can you invent a four spring-one displacement system which produces this matrix? With spring constants c_1, c_2, c_3, c_4 and force f, what is the displacement x?

1.4.14 If two springs with constants c_1 and c_2 hang (i) from separate supports, or (ii) with one attached below the other, what masses m_1 and m_2 in each case will produce equal displacements x_1 and x_2?

1.4.15 Suppose the three equations are changed to $e = Ax - b$, $y = Ce$, and $A^Ty = 0$. Eliminate e and y to find a single equation for x.