EXERCISES

1.3.1 Write $A = \begin{bmatrix} 3 & -3 \\ -3 & 5 \end{bmatrix}$ in the forms $A = LDL^T$ and $A = l_1 d_1 l_1^T + l_2 d_2 l_2^T$. Are the pivots positive, so that A is symmetric positive definite? Write $3x_1^2 - 6x_1x_2 + 5x_2^2$ as a sum of squares.

1.3.2 Factor $A = \begin{bmatrix} 3 & 6 \\ 6 & 8 \end{bmatrix}$ into $A = LDL^T$. Is this matrix positive definite? Write x^TAx as a combination of two squares.

1.3.3 Find the triangular factors L and U of

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

In this case U is the same as L^T. What is the pivot matrix D? Solve $Lc = b$ and $Ux = c$ if $b = (1, 0, 0, 0)$.

1.3.4 How do you know from elimination that the rows of L always start with the same zeros as the rows of A? *Note*: Zeros inside the central band of A may be lost by L; this is the "fill-in" that is painful for sparse matrices.

1.3.5 Write $f = x_1^2 + 10x_1x_2 + x_2^2$ as a difference of squares, and $f = x_1^2 + 10x_1x_2 + 30x_2^2$ as a sum of squares. What symmetric matrices correspond to these quadratic forms by $f = x^TAx$?

1.3.6 In the 2 by 2 case, suppose the positive coefficients a and c dominate b in the sense that $a + c > 2b$. Is this enough to guarantee that $ac > b^2$ and the matrix is positive definite? Give a proof or a counterexample.

1.3.7 Decide for or against the positive definiteness of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad A' = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

Write A as $l_1 d_1 l_1^T$ and write A' as LDL^T.

1.3.8 If each diagonal entry a_{ii} is larger than the sum of the absolute values $|a_{ij}|$ along the rest of its row, then the symmetric matrix A is positive definite. How large would c have to be in

$$A = \begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix}$$

for this statement to apply? How large does c actually have to be to assure that A is positive definite? Note that

$$x^T A x = (x_1 + x_2 + x_3)^2 + (c-1)(x_1^2 + x_2^2 + x_3^2);$$

when is this positive?

1.3.9 (i) The determinant of a triangular matrix is the product of the entries on the diagonal. Thus det $L = 1$ and

$$\det A = \det LDL^T = (\det L)(\det D)(\det L^T) = \det D.$$

The determinant is the *product of the pivots*. Show that det $A > 0$ if A is positive definite.
(ii) Give an example with det $A > 0$ in which A is *not* positive definite.
(iii) What is the determinant of A in Exercise 1.3.3?

1.3.10 Inverting $A = LDL^T$ gives $A^{-1} = MD^{-1}M^T$, where M is the inverse of L^T. Is M lower triangular or upper triangular? How could you factor A itself, so that the first factor is upper and not lower triangular?

1.3.11 A function $F(x, y)$ has a local minimum at any point where its first derivatives $\partial F/\partial x$ and $\partial F/\partial y$ are zero and the matrix of second derivatives

$$A = \begin{bmatrix} \dfrac{\partial^2 F}{\partial x^2} & \dfrac{\partial^2 F}{\partial x \partial y} \\ \dfrac{\partial^2 F}{\partial x \partial y} & \dfrac{\partial^2 F}{\partial y^2} \end{bmatrix}$$

is positive definite. Is this true for $F_1 = x^2 - x^2y^2 + y^2 + y^3$ and $F_2 = \cos x \cos y$ at $x = y = 0$? Does F_1 have a global minimum or can it approach $-\infty$?

1.3.12 Find the inverse of the 2 by 2 symmetric matrix $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$. Verify by direct multiplication that the inverse of a 2 by 2 block symmetric matrix is

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + A^{-1}BSB^TA^{-1} & -A^{-1}BS \\ -SB^TA^{-1} & S \end{bmatrix},$$

where $S = (C - B^TA^{-1}B)^{-1}$. A and C are square but B can be rectangular.

1.3.13 For the block quadratic form

$$f = [x^T \quad y^T] \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x^TAx + x^TBy + y^TB^Tx + y^TCy,$$

find the term that completes the square:

$$f = (x + A^{-1}By)^T A (x + A^{-1}By) + y^T(\quad ? \quad)y.$$

The block matrix is positive definite when A and $C - B^TA^{-1}B$ are positive definite.

1.3.14 The rule for block multiplication of AB seems to be: Vertical cuts in A must be matched by horizontal cuts in B, while other cuts (horizontal in A or vertical in B) can be arbitrary. Examples for 3 by 3 matrices are

$$\left[\begin{array}{c|c|c} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{array}\right]\left[\begin{array}{ccc} \times & \times & \times \\ \hline \times & \times & \times \\ \hline \times & \times & \times \end{array}\right] \qquad \left[\begin{array}{ccc} \times & \times & \times \\ \hline \times & \times & \times \\ \hline \times & \times & \times \end{array}\right]\left[\begin{array}{c|c|c} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{array}\right]$$

column times row row times column

$$\left[\begin{array}{cc|c} \times & \times & \times \\ \times & \times & \times \\ \hline \times & \times & \times \end{array}\right]\left[\begin{array}{cc|c} \times & \times & \times \\ \times & \times & \times \\ \hline \times & \times & \times \end{array}\right]$$

matching blocks

Give two more examples and put in numbers to confirm that the multiplication succeeds.

1.3.15 Find the LDL^T factorization, and Cholesky's $\bar{L}\bar{L}^T$ factorization with $\bar{L}=LD^{1/2}$, for the matrix

$$A=\begin{bmatrix} 4 & 12 \\ 12 & 45 \end{bmatrix}.$$

What is the connection to $x^TAx=(2x_1+6x_2)^2+(3x_2)^2$?

1.3.16 Suppose

$$A=\begin{bmatrix} 1 & & \\ 2 & 1 & \\ 4 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ & 3 & \\ & & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 4 \\ & 1 & 2 \\ & & 1 \end{bmatrix} \quad \text{and} \quad b=\begin{bmatrix} 0 \\ 6 \\ 1 \end{bmatrix}.$$

Solve $Ax=b$ by solving two triangular systems. How do you know that A is symmetric positive definite?

1.3.17 If $a_{11}=d$ is the first pivot of A, under what condition is a_{22} the second pivot? Are the pivots of A^{-1} equal to the reciprocals $1/d_i$ of the pivots of A?

1.3.18 Write down in words the sequence of column operations (!) by which the following code computes the usual multipliers l_{ij}. It overwrites a_{ij} with these multipliers for $i>j$, using no extra storage. The notation $:=$ signals a definition in terms of existing quantities.

Symmetric factorization LDL^T for positive definite A

For $j=1, \ldots, n$

For $p=1, \ldots, j-1$

$r_p := d_p a_{jp}$

$d_j := a_{jj} - \sum_{p=1}^{j-1} a_{jp} r_p$

If $d_j = 0$

then quit

else

For $i = j + 1, \ldots, n$

$$a_{ij} := \left(a_{ij} - \sum_{p=1}^{j-1} a_{ip} r_p\right) \Big/ d_j$$

The algorithm requires about $n^3/6$ multiplications.

1.3.19 Explain (and if possible code) the following solution of $Ax = b$ for *positive definite tridiagonal* A. The diagonal of A is originally in $d_1, \ldots, d_n$ and the subdiagonal and superdiagonal in $l_1, \ldots, l_{n-1}$; the solution x overwrites b.

For $k = 2, \ldots, n$

$t := l_{k-1}$

$l_{k-1} := t/d_{k-1}$

$d_k := d_k - t l_{k-1}$

For $k = 2, \ldots, n$

$b_k := b_k - l_{k-1} b_{k-1}$

For $k = 1, \ldots, n$

$b_k := b_k / d_k$

For $k = n - 1, \ldots, 1$

$b_k := b_k - l_k b_{k+1}$

Show how this uses $5n$ multiplications or divisions, and give an example of failure when A is not positive definite.

1.3.20. If a new row v^T is added to A, what is the change in $A^T A$?