This creates three factors instead of two, but now they preserve the symmetry:

$$\begin{bmatrix} 1 & & & \\ \frac{1}{2} & 1 & & \\ & \frac{2}{3} & 1 & \\ & & \frac{3}{4} & 1 \end{bmatrix} \begin{bmatrix} 2 & & & \\ & \frac{3}{2} & & \\ & & \frac{4}{3} & \\ & & & \frac{5}{4} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & & \\ & 1 & \frac{2}{3} & \\ & & 1 & \frac{3}{4} \\ & & & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & & \\ 1 & 2 & 1 & \\ & 1 & 2 & 1 \\ & & 1 & 2 \end{bmatrix}.$$

That is the ***triple factorization*** $A = LDL^T$.

Before ending this section we must emphasize the practical importance of $A = LU$:

> Once L and U are known, every new right side b involves only two triangular systems $Lc = b$ and $Ux = c$ (n^2 steps). The elimination on A is not repeated, and A^{-1} is not needed.

Suppose the components of b are changed to 0, 0, 4, 8. Forward elimination in $Lc = b$ leads to $c = (0, 0, 4, 5)$. Then back substitution gives $x = (0, 0, 0, 4)$. This solution is correct, since 4 times the last column of A does equal b. The essential point is that every vector b can be processed in the same extremely fast way.

In the next section we see how the relation of U to L recovers the symmetry of A, and how positive pivots are linked to a minimization.

EXERCISES

1.2.1 Solve

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix},$$

using elimination to reach $Ux = c$ and then back substitution to compute x.

1.2.2 In the preceding problem, describe the graph of the second equation $x_1 + 3x_2 + 3x_3 = 0$. Find its line of intersection with the first surface $x_1 + x_2 + x_3 = 2$, by giving two points on the line.

1.2.3 Solve $Ax = b$ with the same A as in the text, but a different b:

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 12 \\ 11 \end{bmatrix}.$$

Without repeating the steps from A to U, apply them only to solve $Lc = b$. Then solve $Ux = c$.

1.2.4 (a) Find the value of q for which elimination fails, in the system

$$\begin{bmatrix} 3 & 6 \\ 6 & q \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

(b) For this value of q, what happens to the first geometrical interpretation (two intersecting lines) in Fig. 1.1?

(c) What happens to the second interpretation, in which b is a combination of the two columns?

(d) What value should replace $b_2 = 4$ to make the system solvable for this q?

1.2.5 For the small angle in Fig. 1.1, find $\cos^2\theta$ from the inner product formula

$$\cos^2\theta = \frac{(v^T w)^2}{(v^T v)(w^T w)}, \quad \text{with} \quad v^T w = [2 \quad 4] \begin{bmatrix} 4 \\ 11 \end{bmatrix}.$$

1.2.6 By applying elimination to

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 7 & 7 \\ 2 & 7 & 9 \end{bmatrix},$$

factor it into $A = LU$.

1.2.7 From the multiplication LS show that

$$L = \begin{bmatrix} 1 & & \\ l_{21} & 1 & \\ l_{31} & 0 & 1 \end{bmatrix} \quad \text{is the inverse of} \quad S = \begin{bmatrix} 1 & & \\ -l_{21} & 1 & \\ -l_{31} & 0 & 1 \end{bmatrix}.$$

S subtracts multiples of row 1 and L adds them back.

1.2.8 Unlike the previous exercise, show that

$$L = \begin{bmatrix} 1 & & \\ l_{21} & 1 & \\ l_{31} & l_{32} & 1 \end{bmatrix} \quad \text{is *not* the inverse of} \quad S = \begin{bmatrix} 1 & & \\ -l_{21} & 1 & \\ -l_{31} & -l_{32} & 1 \end{bmatrix}.$$

If S is changed to

$$E = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & -l_{32} & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ -l_{21} & 1 & \\ -l_{31} & 0 & 1 \end{bmatrix},$$

show that E is the correct inverse of L. E contains the elimination steps as they are actually done—subtractions of multiples of row 1 followed by subtraction of a multiple of row 2.

1.2.9 Find examples of 2 by 2 matrices such that

(a) $LU \neq UL$
(b) $A^2 = -I$, with real entries in A
(c) $B^2 = 0$, with no zeros in B
(d) $CD = -DC$, not allowing $CD = 0$.

1.2.10 (a) Factor A into LU and solve $Ax = b$ for the 3 right sides:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

(b) Verify that your solutions x_1, x_2, x_3 are the three columns of A^{-1}. (A times this inverse matrix should give the identity matrix.)

1.2.11 True or false: Every 2 by 2 matrix A can be factored into a lower triangular L times an upper triangular U, with nonzero diagonals. Find L and U (if possible) when

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

1.2.12 What combination of the vectors

$$v_1 = \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 3 \\ 4 \\ 9 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 2 \\ 0 \\ 7 \end{bmatrix} \quad \text{gives} \quad b = \begin{bmatrix} 2 \\ -8 \\ 7 \end{bmatrix}?$$

1.2.13 What is the intersection point of the three planes $2x_1 + 3x_2 + 2x_3 = 2$, $4x_2 = -8$, $6x_1 + 9x_2 + 7x_3 = 7$?

1.2.14 What are the possibilities other than Fig. 1.2 for a 3 by 3 matrix to be singular? Draw a different figure and find a corresponding matrix A.

1.2.15 Solve by elimination and back-substitution (and row exchange):

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 4 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -4 \end{bmatrix}.$$

1.2.16 Write down a 3 by 3 matrix with row 1 − 2 row 2 + row 3 = 0 and find a similar dependence of the columns—a combination that gives zero.