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Elementary Geometric and Algebraic Topology

1/19/2011

Textbook: Essential Topology by M.D. Crossley

Lecture 1: Continuity

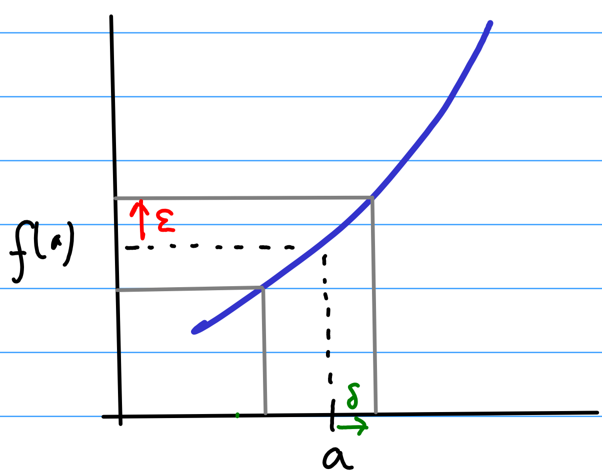
The essential concept underlying topology.

1st year version: $S \subset \mathbb{R}$, $f : S \to \mathbb{R}$

CONTINUOUS at a point a of its domain S if, $\forall \varepsilon > 0$, $\exists\, \delta > 0$ s.t.

$$|f(x) - f(a)| < \varepsilon \quad \text{whenever} \quad |x - a| < \delta$$

Simply "continuous" if continuous at every pt in S.



Lots of "debris" in this definition: ε, δ, $|\cdot|$, ...

Can streamline by introducing OPEN SETS

OPEN INTERVAL: $(a,b) = \{x \in \mathbb{R},\ a < x < b\}$

More generally, $S \subset \mathbb{R}$ is OPEN if, $\forall x \in S$, there is some open interval $(x-\delta_x, x+\delta_x)$ $(\delta_x > 0)$ contained within S.

open neighbourhood

Crossley calls this "breathing space"

examples: $(0,2)$, $(1,\infty)$, $(0,2) \cup (3,5)$.

- The union of open sets is open.
- Any finite intersection of open sets is open

Counterexample: $S_1 = (-1,1)$, $S_2 = (-1, \frac{1}{2})$, $S_2 = (-1, \frac{1}{3})$, ...

$$I = S_1 \cap S_2 \cap S_3 \cap \cdots = (-1, 0]$$

NOT OPEN!

0 is in every set S_i

$\mathbb{R}$ and $\emptyset$ are open sets.

A subset $S \subset \mathbb{R}$ is CLOSED if $\mathbb{R} - S$ is open.

NOTE: Many sets are neither open or closed.

- $(-\infty, 2]$ is closed, $(-\infty, 2)$ is open.

If $f: D \to C$ is a function, S subset of C,

PREIMAGE of S under f, $f^{-1}(S)$, is

$$f^{-1}(S) = \{x \in D : f(x) \in S\}$$

(Not necessary for inverse function to exist!)

THEOREM: $f: \mathbb{R} \to \mathbb{R}$;

f is continuous $\iff$ $f^{-1}(S)$ is open whenever $S \subset \mathbb{R}$ is open.

Proof: see p. 13 of Crossley.

Lecture 2: Topological Spaces 1/21/11

A TOPOLOGICAL SPACE is a set X with a collection $\mathcal{T}$ of subsets of X, called "open sets", s.t.

1. X is "open"
2. ϕ is "open"
3. Unions of "open" sets are "open"
4. Intersections of "open" sets are "open"

"open" not quotes, allowing for abstract def'n of open. (Analogy w. $\mathbb{R}$)

Collection of open sets called A TOPOLOGY on X.

$\mathbb{R}$ w. open sets as in Lect. 1 is of course a topological space.

Another example: let's topologize $\{0,1\}$ in two ways.

1) ϕ, $\{0\}$, $\{1\}$, $\{0,1\}$ open: DISCRETE TOPOLOGY: all subsets open

2) ϕ, $\{0,1\}$ open: INDISCRETE TOPOLOGY: only ϕ and S open.

A topology defines open sets. Define closed sets as before (complement open)

Note: in the indiscrete topology ϕ and $\{0,1\}$ are closed.

discrete topology all subsets also closed.

General def'n of continuity: $f: S \to T$ with S, T topological spaces, then

f continuous if $f^{-1}(Q)$ open in S for any open set Q in T.

Note: "map" usually means a continuous function. (At least for Crossley)

example: $B = \{0,1\}$ w. discr. topo.

$T = \{0,1\}$ w. indiscr. topo.

a) $g: B \to T$; $g(0) = 0$, $g(1) = 1$

Two open sets in T: ϕ and $\{0,1\}$.

$g^{-1}(\phi) = \phi$ open in B

$g^{-1}(\{0,1\}) = \{0,1\}$ open in B

$\therefore$ g continuous

b) $h: T \to B$: $h(0) = 0$, $h(1) = 1$ $\quad h^{-1}(\{0\}) = \{0\}$ NOT open in T.

$\therefore$ h discontinuous

More generally: • if S has discrete topology and T any topological space, then any $f: S \to T$ is continuous.

• if T has indiscrete topology and S any topological space, then any $f: S \to T$ is continuous.

(because S and ϕ are open in S)

Remark: composition respects continuity.

Topologizing $\mathbb{R}^2$:

this is a metric. GEOMETRIC topology.

$$B_\delta(x,y) = \{ (x',y') \in \mathbb{R}^2 : \sqrt{(x-x')^2 + (y-y')^2} < \delta \}$$

OPEN BALL of radius δ around (x,y).

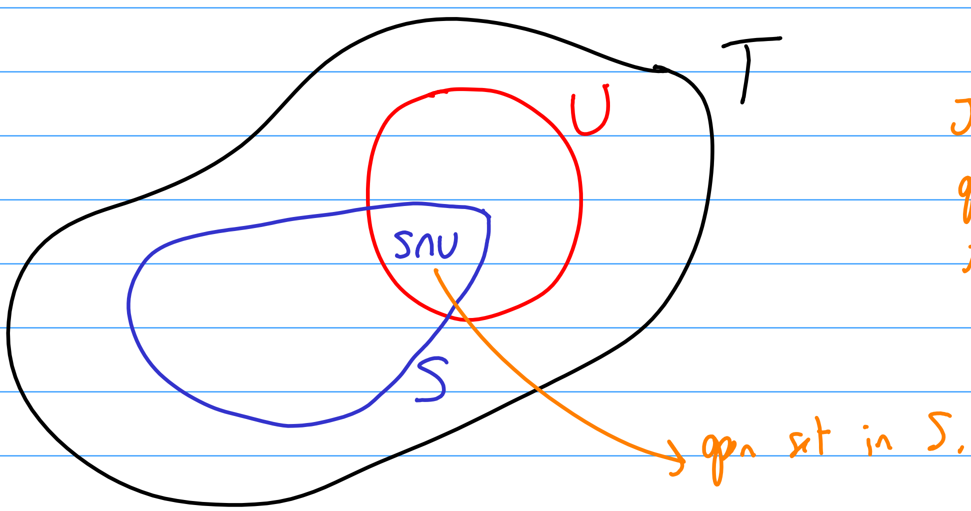
Obvious generalization to $\mathbb{R}^n$

$Q \subset \mathbb{R}^2$ is open if, $\forall (x,y) \in Q$, $\exists \delta > 0$ s.t. $B_\delta(x,y) \subset Q$.

Under this topology addition and multiplication ($\mathbb{R}^2 \to \mathbb{R}$) are continuous.
$\Rightarrow$ polynomials are continuous.

SUBSPACE TOPOLOGY: T topological space, $S \subset T$ any subset.

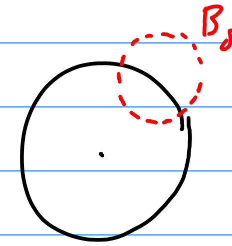
Subspace topology is a way of topologizing S.
A subset of S is open in the subspace topology if it is the intersection of S with an open set in T.



Just slice off the open sets of T to fit in S.

Examples: a) circle $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$

Subspace topology leads to "open intervals" on S^1!



b) $S^0 = \{1, -1\}$ inherits the discrete topology from $\mathbb{R}$

c) $\mathbb{R}^n - \{0\}$ topologized in this way (for $1/x$ style functions)

d) $GL(n, \mathbb{R})$ General Linear Group (matrices with $\det \neq 0$)
subspace of $\mathbb{R}^{n^2}$

e) $O(n)$ Orthogonal Group (matrices P satisfying $P^T P = I$)

f) $SO(n)$ ($O(n)$ with $\det P = +1$) "rotations" in $\mathbb{R}^n$ (no reflections)

g) $\mathbb{RP}^2$ (real projective plane)

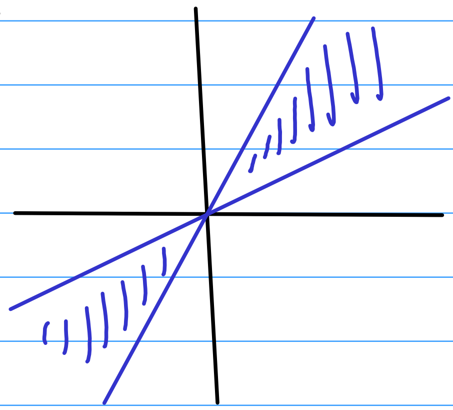
Lecture 3: Continuity in Subspace Topology 1/28/11

(1/24 – 1/26 — travel)

Projective spaces: $\mathbb{RP}^n$ = lines through origin in $\mathbb{R}^{n+1}$

one higher

Elements of $\mathbb{RP}^n$ are lines, which are not elements of $\mathbb{R}^{n+1}$.
However, define topology by looking at unions of lines as subsets in $\mathbb{R}^{n+1}$:



← potential "open set" in $\mathbb{PR}^1$.

However, this gives the indiscrete topology on $\mathbb{RP}^n$!

The problem is the origin: union must contain 0, but cannot contain an open ball around 0, unless it is the whole space.

Remedy: consider union of lines as a subspace of $\mathbb{R}^{n+1} - \{0\}$.

$\mathbb{RP}^0 = 1$ point, $\mathbb{RP}^1 \simeq S^1$ (circle), $\mathbb{RP}^2 \simeq SO(3)$

Continuity: mercifully, a simple result holds. For S, T topological spaces, $f: S \to T$ continuous map, suppose $Q \subset S$ subset whose image under f is contained in $R \subset T$.
Now give Q and R subspace topology.

Then $f|_Q$ is continuous. (f restricted to Q)

$$\begin{array}{ccc} S & \xrightarrow{\;f\;} & T \\ \cup & & \cup \\ Q & \xrightarrow[f|_Q]{} & R \end{array}$$

Proof: Let $P \subset R$ open set in subsp. topol.

Then $f|_Q^{-1}(P) = Q \cap f^{-1}(P)$. P open, so $\exists$ open set $U \in T$ in subsp. topol.

such that $P = R \cap U$. Hence,

$$f^{-1}(P) = f^{-1}(R \cap U) = f^{-1}(R) \cap f^{-1}(U)$$

$$\Rightarrow f|_Q^{-1}(P) = Q \cap f^{-1}(R) \cap f^{-1}(U) = Q \cap f^{-1}(U)$$

f cont., U open, so $f^{-1}(U)$ also open.

$\Rightarrow Q \cap f^{-1}(U)$ open in subsp. topo. on Q.

$f|_Q^{-1}(P)$ is open in subs. topo. on Q, so $f|_Q$ continuous. $\blacksquare$

example of use: $\det : GL(n,\mathbb{R}) \to \mathbb{R}$ continuous, since $GL(n,\mathbb{R}) \subset \mathbb{R}^{n^2}$ with subspace topo, and $\det : \mathbb{R}^{n^2} \to \mathbb{R}$ is continuous.

Basis for a topological space T: a collection $\mathcal{B}$ of open subsets of T such that every open subset of T is a union of sets in $\mathcal{B}$.

example: • For the real line, finite open intervals (a,b) form a basis.

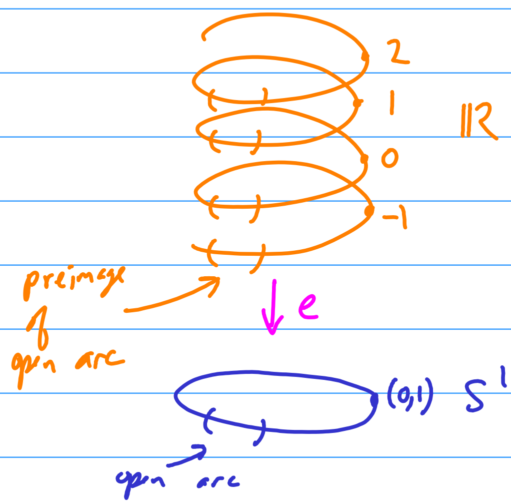
• $\{1,2,3\}$ with discr. topo: $\{1\}, \{2\}, \{3\}$ form a basis.

Easier to check continuity: just verify $f^{-1}(B)$ open for every $B \subset \mathcal{B}$.

Proof follows because unions lift to unions under preimage.

example: exponential map $e: \mathbb{R} \to S^1$, $e(x) = (\cos(2\pi x), \sin(2\pi x))$.

Subspace topology on S^1 leads to



With subspace topo. a basis for S^1 consists of open arcs.

f^{-1}(open arc) = infinite union of open arcs on $\mathbb{R}$, separated by integer

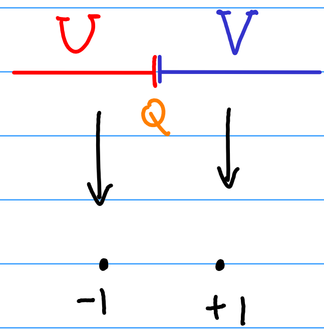
$\Rightarrow$ open in $\mathbb{R}$

Interlude: constructions vs invariants.

Connectivity: (Chapter 4)

Theorem: There is no continuous surjective map $\mathbb{R} \to S^0$.

Proof:



$\{-1\}$, $\{1\}$ open sets in S^0.

$U = f^{-1}(\{-1\})$, $V = f^{-1}(\{+1\})$

are open in $\mathbb{R}$.

surjective $\Rightarrow$ U, V nonempty. $U \cup V = \mathbb{R}$

Since $f^{-1}(\underbrace{\{-1\} \cap \{+1\}}_{\phi}) = f^{-1}\{-1\} \cap f^{-1}\{+1\}$

$\Rightarrow U \cap V = \phi$ disjoint

But by continuity U and V must meet "at a point" Q.
Q cannot be in both U and V, and if it is in U then
V cannot be closed & vice versa

$\Rightarrow$ contradiction. $\blacksquare$

(Proof in book is more thorough.)

Lecture 4: Connectivity 1/31/11

Last time: no continuous surjection $\mathbb{R} \to S^0$.

Central fact about proof used to define connectivity generally:

Def'n: T disconnected if can find two open sets U, V s.t.

1. $U \cap V = \phi$
2. $U \cup V = T$
3. $U \neq \phi$ and $V \neq \phi$

Otherwise, T is connected.

Lemma: if T is connected, no continuous surjection $T \to S^0$. (same proof as before)

Fixed point theorem for $[0,1]$: $f: [0,1] \to [0,1]$ cont. map. Then f has a fixed pt.

that is, $\exists\, x \in [0,1]$ s.t. $f(x) = x$.

Proof: Assume no such point. Define: $g(x) = \dfrac{x - f(x)}{|x - f(x)|}$, $x \in [0,1]$.

g is continuous (composition of addition & division, $f(x) \neq x$)

$g(x) = +1$ or -1, so $g: [0,1] \to S^0$.

$f(0) \in [0,1]$, $f(0) \neq 0$, so $f(0) > 0 \Rightarrow g(0) = -1$

$f(1) < 1 \Rightarrow g(1) = +1. \quad \Rightarrow g$ surjective.

Cannot be true for connected space! So $f(x) = x$ for some x. ∎

Another consequence: intermediate value theorem (similar proof)

Note: $\mathbb{R}^n$ is connected, $\mathbb{R}^n - \{0\}$ connected for $n > 1$.

Complementary lemma: if T disconnected, then there $\exists$ cont. surjection $T \to S^0$.

Proof. $\exists\, U, V \in \mathcal{T}$ s.t. $U \cap V = \phi$, $U \cup V = T$.
Define f by $f(U) = 1$, $f(V) = -1$.

Corollary: S connected, T disconnected. No cont. surjection $S \to T$.

Proof: T disconnected $\Rightarrow \exists$ cont. surj. $T \to S^0$. Assume $\exists$ cont. surj. $S \to T$. Can combine these to get cont. surj. $S \to S^0$, which cannot happen.

Classroom discussion: other way?

example: $GL(3,\mathbb{R})$ disconnected, because $\mathbb{R} - \{0\}$ disconnected and cont. surj. $GL(3,\mathbb{R}) \to \mathbb{R} - \{0\}$ defined by $M \mapsto \det(M)$.

(surjective: $\lambda \in \mathbb{R}$ is det. of $\begin{pmatrix} \lambda & & \\ & 1 & \\ & & 1 \end{pmatrix}$)

Lemma: S connected, T discrete, then any cont. map $f: S \to T$ is constant.

Proof: $u \in$ image of f, $u = f(x)$ for some $x \in S$. $\{u\}$ open (T discrete).

$T - \{u\}$ open as well. $\Rightarrow$ $\left.\begin{matrix} f^{-1}(\{u\}) \\ f^{-1}(T-\{u\}) \end{matrix}\right\}$ open in S

$f^{-1}(\{u\}) \cap f^{-1}(T-\{u\}) = \phi$

$f^{-1}(\{u\}) \cup f^{-1}(T-\{u\}) = S$

But then either $f^{-1}(\{u\})$ or $f^{-1}(T-\{u\})$ must be empty! (connected)

$f^{-1}(\{u\})$ cannot be empty, so $f^{-1}(T-\{u\})$ must be empty.

$\Rightarrow f^{-1}(\{u\}) = S$, or $f(x) = u$ for all x.
$\Rightarrow$ constant. $\blacksquare$

COMPACTNESS:

Proposition: $f: [0,1] \to \mathbb{R}$ continuous, the f is bounded

i.e., $\exists\, j, k \in \mathbb{R}$ s.t. $\text{Im} f \subset (j,k)$. $\quad (j < f(x) < k \quad \forall x \in [0,1])$

Lecture 5: Compactness

2/4/11

(2/2 → snow day)

An open cover of a top. sp. T is a collection of open subsets of T s.t. every point in T lies in at least one of these open subsets.

T is compact if every open cover of T admits a finite refinement.

↳ keep only a finite number

Prop: $[0,1]$ is compact.

will lead to contradiction

Proof: Assume an open cover of $[0,1]$ with no finite refinement.

Consider $[0,\frac{1}{2}]$ and $[\frac{1}{2},1]$. $\cap$ w. open cover gives open cover for each interval.

One of them must not have a finite refinement, say $I_1 = [0,\frac{1}{2}]$.

Divide I_1 into two intervals: $[0,\frac{1}{4}]$ and $[\frac{1}{4},\frac{1}{2}]$.

Again one of these must not have a finite refinement, say I_2.

Divide I_2 in half, ... and so on.

$$[0,1] \supset I_1 \supset I_2 \supset \cdots \qquad I_n \text{ has length } \frac{1}{2^n}$$

None admit a finite refinement.

Intersection is a single point $c \in [0,1]$.

c is also in one of the sets of the open cover.

open so "breathing space" around c: $(c-\delta_c, c+\delta_c) \cap [0,1]$, $\delta_c > 0$.

But this contains any interval around c of length $< \delta_c$, i.e., I_n for $n > \log_2\left(\frac{1}{\delta_c}\right)$.

n large enough $\Rightarrow$ all I_n contained in $(c-\delta_c, c+\delta_c)$.

Oops! Can drop all the other $I_n \Rightarrow$ finite refinement.

Contradiction $\Rightarrow$ $[0,1]$ is compact ▨

Prop: T compact, $f: T \to \mathbb{R}$ continuous, then f bounded.

Proof sketch: For an open cover of $\mathbb{R}$, f^{-1} of this cover forms an open cover of T. But since T is compact, this can be refined to a finite cover. Then map back to $\mathbb{R}$ and take the maximal endpoints of intervals.

S^1 is also compact (using exponential map to relate to $[0,1]$).

Not compact: $\mathbb{R}$, $(0,1)$

Easy to show this now by constructing unbounded cont. functions

$f: \mathbb{R} \to \mathbb{R}$, $\quad f(x) = x$.

$f: (0,1) \to \mathbb{R}$, $\quad f(x) = \frac{1}{x}$

Prop: $f: S \to T$ continuous map, S compact. Image of f is compact.

Corollary: no continuous surjection $f: [0,1] \to (0,1)$.

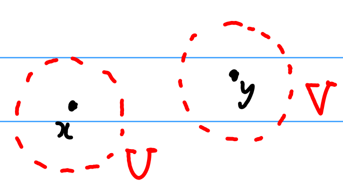
Heine-Borel theorem: subspace T of $\mathbb{R}^n$ compact iff T closed and bounded.

Proof: see book.

In particular, this proves that S^n is compact.

Hausdorff property: we say T is Hausdorff if, for any two distinct points x, y in T, there are open subsets U, V of T s.t. $x \in U$, $y \in V$, and $U \cap V = \phi$.

"housed off" from each other



Most spaces we will meet are Hausdorff.

Counterexample: $\{1,2\}$ with the indiscrete topology is not Hausdorff.

Easy to see that $[0,1]$ is Hausdorff: take $x<y$, $\delta=(y-x)/2$ then

$$U = (x-\delta, x+\delta) \cap [0,1] \qquad U \cap V = \phi$$
$$V = (y-\delta, y+\delta) \cap [0,1]$$

$\mathbb{R}^n$ is Hausdorff for any n.

Prop: T Hausdorff, $f: T \to T$ continuous, then

$$\mathrm{Fix}(f) = \{x \in T : f(x) = x\} \qquad \text{fixed-point set}$$

is a closed subset of T.

Corollary: $f: \mathbb{R} \to \mathbb{R}$ continuous, $f(x) \neq x$, there is an open interval $(x-\delta, x+\delta)$ $(\delta>0)$ containing no fixed points.

Prop.: $f: S \to T$ continuous and injective and T is Hausdorff, then S is Hausdorff.

Proof: $x, y \in S$ distinct $\Rightarrow f(x) \neq f(y)$. $\exists\, U, V \in T$ s.t. $x \in U$, $y \in V$, and $U \cap V\ \phi$. Preimage open, which houses-off x and y. ▨

Lecture 6: Homeomorphisms 2/7/11

Two topological spaces S and T are homeomorphic if there is a continuous bijection $f: S \to T$, s.t. f^{-1} is also continuous.

f (and f^{-1}) are homeomorphisms. Write $S \cong T$.

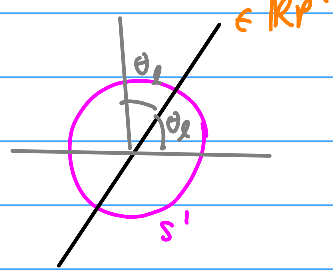
example: $(-1,1)$ and $\mathbb{R}$ homeomorphic through $f: (-1,1) \to \mathbb{R}$

$$f(x) = \tan\left(\frac{\pi x}{2}\right), \quad f^{-1}(x) = \frac{2}{\pi}\arctan x.$$

(true for any open interval)

$\cong$ is transitive: if $S \cong T$ and $T \cong U$, then $S \cong U$.

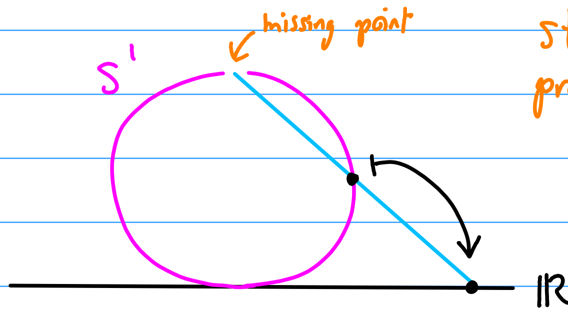
example: $\mathbb{RP}^1$ homeomorphic to S^1.



Associate point on S^1 with angle $2\theta_\ell$, to cover the whole circle.

example: $S^1 - \{(0,1)\}$ homeomorphic to $\mathbb{R}$.

(S^n with missing pt. homeomorphic to $\mathbb{R}^n$)



stereographic projection

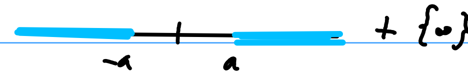
$$f: S^1 - \{(0,1)\} \to \mathbb{R}$$

$$f(x,y) = \frac{2x}{1-y}$$

Can extend to bijection $f: S^1 \to \mathbb{R} \cup \{\infty\}$ by adding "point at ∞" to $\mathbb{R}$.

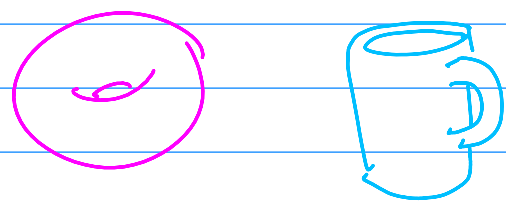
Need a topology on $\mathbb{R} \cup \{\infty\}$: open sets containing ∞ are of the form.

$$(-\infty, -a) \cup (a, \infty) \cup \{\infty\}$$



$\mathbb{R}^2 \cup \{\infty\}$ is called the Riemann sphere.

Spaces that can be continuously "deformed" into each other are homeomorphic.



Possible that f continuous bijection, but not f^{-1}:

example: $S = \{1, 2\}$ w. discr. topo., $T = \{1, 2\}$ w. indiscr. topo.

$f: T \to S$ bijection, $g^{-1}\{1\}$ contains one point $\to$ not open in T.

Prop: Let S and T be two homeomorphic spaces. Then:

1. If S connected, so is T
2. " compact, "
3. " Hausdorff, "

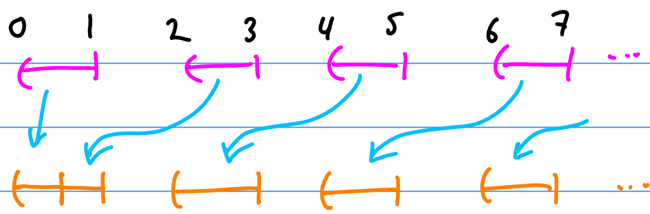
Proof: 1. No cont. surjection from connected to disconnected space.
2. $f: S \to T$ homeo. $\operatorname{Im} f$ is compact, so T is compact. (surjective)
3. $f^{-1}: T \to S$ continuous and injective, so T Hausdorff.

example: S^1 not homeo to $[0,1)$, since after removing a point S^1 still connected, but not $[0,1)$.

Even when two spaces are homeomorphic, may be possible to define bijection which is not homeomorphism.

example: $V = (0,1] \cup (2,3] \cup (4,5] \cup \dots$

$$f: V \to V, \quad f(x) = \begin{cases} x/2, & x \in (0,1] \\ (x-1)/2, & x \in (2,3] \\ x-2, & \text{otherwise} \end{cases}$$



f continuous bijection

But $f^{-1}(0,1] = (0,1] \cup (2,3] \longrightarrow$ connected to disconnected
Not possible if f^{-1} continuous

The problem is that f does not necessarily map open sets to open sets (open map)

$(0,1]$ is open in V, but not $f((0,1])$.

But often it is true that all bijections are homeomorphisms:

Theorem: X compact, Y Hausdorff, $f: X \to Y$ continuous bijection. Then f homeomorphism.

Lemma: X compact, $U \subset X$ closed, then U is compact.

Proof: inherit finite cover from X.

Lemma: Y Hausdorff, $V \subset Y$ compact, then V closed.

Proof: Show $Y - V$ is open. Hausdorff comes in to be able to select an open set containing $y \in Y - V$ which does not contain $v \in V$. Do this for all v, intersect with V, get open cover of V. V compact $\to$ refine to finite cover. Then can take complement and finite intersection and still have open set.

Lemma: $f: S \to T$ continuous iff $f^{-1}(U)$ closed whenever U closed.

Proof of theorem: $f: X \to Y$ cont. bijection. Show $g = f^{-1}$ continuous. Let $U \subset X$ closed. X compact, so U compact. $g^{-1}(U) = f(U) \to$ compact, also $f(U) \subset Y$ Hausdorff, so $f(U)$ closed.

Lecture 7: Disjoint Unions and Product Spaces 2/9/11

(Finish proof of Thm 5.24, p.66, at beginning.)

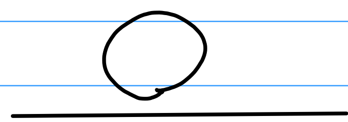
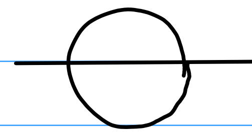
$S^0 = \{-1, +1\}$ is disjoint union of $\{-1\}$ and $\{+1\}$.

$$S^0 = \{-1\} \amalg \{+1\}$$

Open sets in $S \amalg T$ are just a "doubling up" of open sets in S, T.

Q open in $S \amalg T$ if $Q \cap S$ open in S
$Q \cap T$ open in T

When $S \cap T \neq \phi$, points in intersection get counted twice.

$S^1 \amalg [-2,2]$ homeo. to  NOT 

Theorem: 1) R topol. space, $S \amalg T \to R$ continuous corresponds to a pair of continuous maps $(S \to R,\ T \to R)$.

2) Q connected topol. space, $Q \to S \amalg T$ continuous corresponds to either $Q \to S$ continuous or $Q \to T$ continuous.

Proof: 1) $g: S \amalg T \to R$ continuous, define $g_S : S \to R$ and $g_T : T \to R$:

$$g_S(x) = g(x),\ x \in S, \quad g_T(y) = g(y),\ y \in T.$$

$h_S : S \to R$, $h_T : T \to R$ continuous. Define $h : S \amalg T \to R$ by

$$h(x) = \begin{cases} h_S(x), & x \in S \\ h_T(x), & x \in T \end{cases}$$

2) $f : Q \to S \amalg T$ cont., then $f^{-1}(S)$ and $f^{-1}(T)$ both open.

Union is $f^{-1}(S \amalg T) = Q$, intersection is ϕ.

Q connected $\Rightarrow$ $f^{-1}(S)$ or $f^{-1}(T)$ empty.

example: $\mathbb{R} - \{0\} \cong \mathbb{R} \amalg \mathbb{R}$

$$O(3) \cong \underbrace{SO(3)}_{\text{map to } \det = +1} \amalg \underbrace{SO(3)}_{\text{map to } \det = -1}$$

Theorem: S, T compact $\Leftrightarrow$ $S \amalg T$ compact

S, T Hausdorff $\Leftrightarrow$ $S \amalg T$ Hausdorff

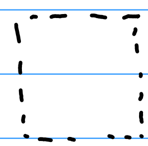
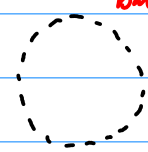
Lemma: S disconnected, then nonempty Q, R s.t. $S \cong Q \amalg R$
Conversely, given nonempty Q, R $\longrightarrow$ $Q \amalg R$ disconnected.

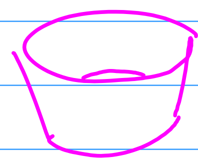
Product spaces:

S, T: Cartesian product $S \times T$ contains pairs (s,t), $s \in S$, $t \in T$.

example: $\mathbb{R}^2$.

Product topology on $S \times T$ has basis consisting of all products $P \times Q$, $P \subset S$ open, $Q \subset T$ open.

Same topology on $\mathbb{R}^2$ as before: square  interval × interval same as ball 

example: $S^1 \times [0,1] \cong$  cylinder

Theorem: A continuous map $f: Q \to S \times T$ corresponds to a pair of continuous maps $f_1: Q \to S$, $f_2: Q \to T$.

Proof: $p_1 : S \times T \to S$, $p_1(s,t) = s$
$p_2 \; S \times T \to T$, $p_2(s,t) = t$
} easy to show these are continuous
$p^{-1}(U) = U \times T$ open in product topo.

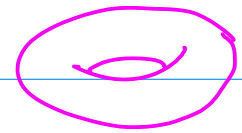
$f_1 = p_1 \circ f$, $f_2 = p_2 \circ f$ continuous since composition of cont. maps.

Conversely, given f_1, f_2, define $f(q) = (f_1(q), f_2(q))$

$$f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V) \quad \text{open}$$

$f^{-1}(U)$ = elements that map into U, all V.

example: $T^2 \cong S^1 \times S^1$

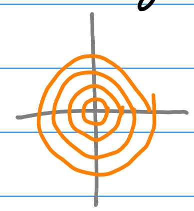


$\mathbb{R}^2 - \{0\} \cong S^1 \times (0,\infty)$ infinite tube

$f: \mathbb{R}^2 - \{0\} \to S^1 \times (0,\infty)$ by $f(x,y) = \left(\frac{(x,y)}{\sqrt{x^2+y^2}}, \sqrt{x^2+y^2}\right)$

$f^{-1}: S^1 \times (0,\infty) \to \mathbb{R}^2 - \{0\}$

$g((x,y),t) = (tx, ty)$



Theorem: S, T topological spaces

1. $S \times T$ Hausdorff iff S and T Hausdorff
2. " connected " connected
3. " compact " compact $\Leftarrow$ Tychonov's theorem

Lecture 8: Quotient Spaces

2/11/11

$$D^2 = \{(x,y) \in \mathbb{R}^2 : \sqrt{x^2+y^2} \leq 1\}$$ unit disk

∂D^2 = boundary = S^1

$D^2/\partial D^2$ = quotient space obtained by identifying all points on ∂D^2 with one point.

one point, representing ∂D^2

$$D^2/\partial D^2 = D^2 - \partial D^2 \cup \{*\}$$

Define $\pi : D^2 \longrightarrow D^2/\partial D^2$ by $\pi(D^2 - \partial D^2) = D^2 - \partial D^2$
$\pi(\partial D^2) = \{*\}$

No topology yet, but define one such that π is continuous.

If $S \in D^2/\partial D^2$ does not contain $\{*\}$, then think as subset of $D^2 - \partial D^2$ and define open sets as open in $D^2 - \partial D^2$.

If $S \in D^2/\partial D^2$ does contain $\{*\}$, complement $D^2/\partial D^2 - S$ does not. S is then open if $D^2/\partial D^2 - S \subset D^2 - \partial D^2$ is closed.

Definition: X topological space, $A \subset X$, quotient space X/A is the set $(X - A) \amalg \{*\}$. $U \subset (X-A) \amalg \{*\}$ is open iff $\pi^{-1}(U)$ is open in X, where $\pi : X \to (X-A) \amalg \{*\}$ defined by $\pi(x) = x$, $x \notin A$, $\pi(x) = *$, $x \in A$.

example: check $D^2/\partial D^2 \cong S^2$.

stereographic projection

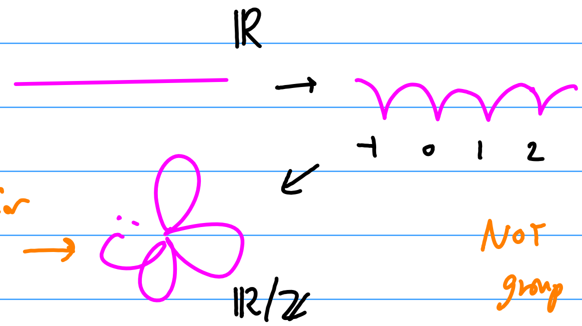
$D^2 - \partial D^2 \cong \mathbb{R}^2 \cong S^2 - \{(0,0,1)\}$ gives $g: D^2 - \partial D^2 \to S^2 - \{(0,0,1)\}$

Extend by defining $g(*) = (0,0,1)$.

Similarly, take $X = [0,1]^2$, $\partial X = \{(x,y) \in X : x(1-x)y(1-y) = 0\}$

Again $X/\partial X \cong S^2$.

$\mathbb{R}/\mathbb{Z}$ is a "bouquet"

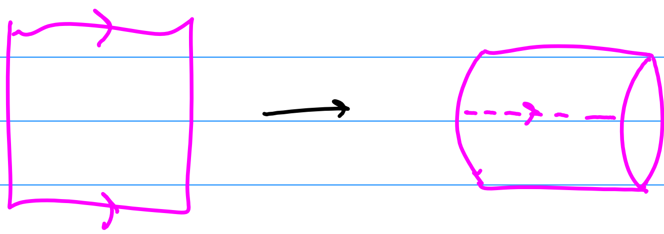


One loop for each integer. →

NOT the same as group $\mathbb{R}/\mathbb{Z} \cong S^1$

For more general "gluings", define an equivalence relation $\sim$:

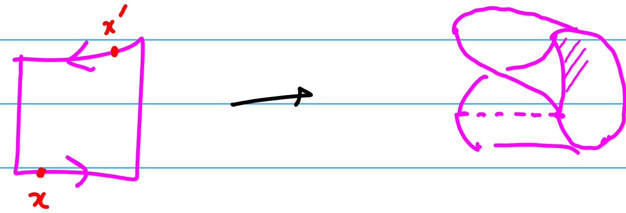
example: $X = [0,1]^2$. Define $(x,y) \sim (x',y')$ iff $x = x'$, $y - y' \in \mathbb{Z}$



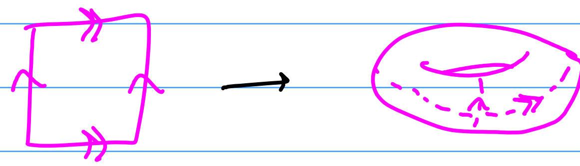
cylinder

Write $X/\sim \; \cong$ cylinder.

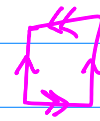
If instead we define: $(x,y)\sim(x',y') \Leftrightarrow x=1-x',\ y-y'=\pm1$, obtain Möbius band:



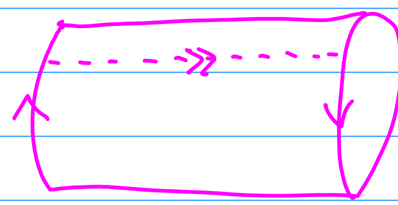
If we glue both edges, $(x,y)\sim(x',y') \Leftrightarrow x-x'\in\mathbb{Z},\ y-y'\in\mathbb{Z}$, get torus T^2:



If we reverse orientation of one pair of edges,

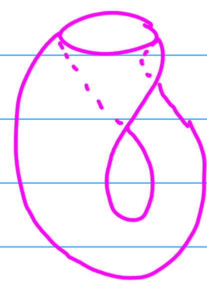


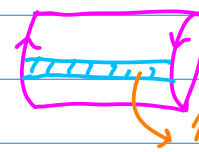
$(x,y)\sim(x',y') \Leftrightarrow x=1-x',\ y-y'=\pm1$
OR $x-x'\in\mathbb{Z},\ y=y'$



Klein bottle

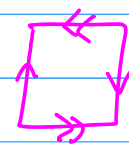
Cannot be embedded in $\mathbb{R}^3$ w/o intersection.



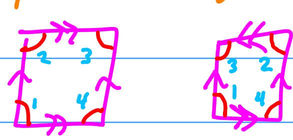


Möbius strip

Q: What if we reverse both sets of arrows?



Try to draw a little loop around origin:



torus Klein

Lecture 9: Quotient Spaces (cont'd) 2/14/11

More examples:

example: $\mathbb{R}^n - \{0\}$ with $x \sim y \Leftrightarrow x = \lambda y$ for some $\lambda \in \mathbb{R} - \{0\}$.

$$(\mathbb{R}^n - \{0\})/\sim \;\cong\; \mathbb{RP}^{n-1}$$

example: $S^1 \cong S^1/\sim$, $x \sim y \Leftrightarrow x = \pm y$.

$$z \mapsto z^2$$

Prop: $A \subset X$, then X/A is $X/\sim$ with

$$x \sim y \Leftrightarrow x, y \in A$$

Theorem: X with $\sim$, $Q = X/\sim$. Let S be any other topo. space and $f: Q \to S$ continuous. Then we can compose f with $\pi: X \to Q$ to get continuous function $X \to S$.

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, dashed, "f \circ \pi"] & \\ Q \arrow[r, "f"'] & S \end{tikzcd}$$

Converse is true: Given π, g, $\exists$ continuous f.

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "g"] & \\ Q \arrow[r, dashed, "f"'] & S \end{tikzcd}$$

$g(x) = g(y)$ when $x \sim y$

Proof: First part: follows from composition of continuous functions.

Converse: $g: X \to S$ continuous s.t. $g(x) = g(y)$ when $x \sim y$.

Define: $f: Q \to S$ by $f(E) = g(x)$, E equivalence class, x member of E.

(f is well defined $\to$ respects equivalence) Then $f \circ \pi = g$.

Now let R be an open subset of S:

$$\underbrace{g^{-1}(R)}_{\text{open, since } g \text{ cont.}} = (f \circ \pi)^{-1}(R) = \pi^{-1}(f^{-1}(R))$$

Topology on Q defined such that preimage π^{-1} open iff set is open. Hence $f^{-1}(R)$ is open, and f continuous. ▨

Thus there is a one-to-one correspondence between:

$$f: (X/\sim) \to S$$

and $g: X \to S$, $g(x) = g(y)$ whenever $x \sim y$.

example: How do we describe cont. functions $\mathbb{R}P^3 \to T^2$?

$\mathbb{R}P^3$ is $S^3/\{\text{opposite pts identified}\}$, T^2 is $S^1 \times S^1$

Take $f, g: S^3 \to S^1$, both of which satisfy $f(x) = f(-x)$, $g(x) = g(-x)$.

Prop: $Q = X/\sim$ and X is compact, then Q is compact.

Proof: Get surjection $X \to X/\sim$ which is continuous (π). Then follows from 4.27.

Since $\mathbb{RP}^n$ is $S^n/\sim$, $x \sim y \Leftrightarrow x = -y$, then $\mathbb{RP}^n$ compact.

$X/\sim$ may be Hausdorff even if X isn't, or may not be even if X is.

Prop: If X is connected, then $X/\sim$ is also connected.

Proof: Continuous surjection $\pi: X \to X/\sim$, so connected by Prop. 4.11.

Since S^n is connected for $n > 0$, $\to$ $\mathbb{RP}^n$ connected for $n > 0$.

Converse not true: can get connected $X/\sim$ w. disconnected X.

Trivial example: X/X is always only one point, so connected!

<u>Gluing Lemma:</u> T topological space, closed subsets $U_1, \dots, U_n$ s.t. every point in T is in at least one U_i.

If S any top. sp. and

$f_i : U_i \to S$ cont. w.r.t. subspace tops on U_i, $i = 1, \dots, n$

and such that $f_i(x) = f_j(x)$, $x \in U_i \cap U_j$, then f_i can be glued together to give cont. map $f : T \to S$

$$f(x) = f_i(x) \quad , \quad x \in U_i .$$

<u>Proof:</u> Show $T \cong Q = (U_1 \amalg U_2 \cdots \amalg U_n)/\sim$, where

$x \sim y$ if $x \in U_i$, $y \in U_j$, $h_i(x) = h_j(y)$

$h_i : U_i \to X$, $h_j : U_j \to X$ inclusion maps.

$$\begin{tikzcd}
U_1 \amalg U_2 \cdots \amalg U_n \arrow[d, "\pi"'] \arrow[dr, "{(f_1, \dots, f_n)\ \text{Theorem 5.31}}"] & \\
Q \arrow[r, dashed, "f"', "\text{Theorem 5.62}"] & S
\end{tikzcd}$$

Lecture 10: Homotopy

2/18/11

(4/16/11: Midterm 1 in class)

Two maps $f, g : S \to T$ are **homotopic** if there is a continuous function

$$F : S \times I \to T \qquad I = [0,1]$$

such that $F(s,0) = f(s)$ for all $s \in S$, $F(s,1) = g(s)$ for all $s \in S$.

F is a **homotopy** between f and g; write $f \simeq g$.

f and g can be "deformed" into each other continuously.

example: $f : [0,2] \to \mathbb{R}$, $f(x) = 1 + x^2(x-2)^2$ is homotopic to $g : [0,2] \to \mathbb{R}$, $g(x) = 1$ via

$$F(x,t) = 1 + (1-t)x^2(x-2)^2$$

More generally: if $f, g : \mathbb{R} \to \mathbb{R}$ any two functions,

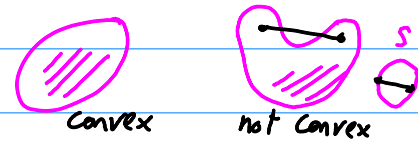
$$F(x,t) = (1-t)f(x) + t\,g(x)$$

All continuous functions on $\mathbb{R}$ are homotopic

$T \subset \mathbb{R}^n$ is convex if $\forall\, x, y \in T$,

$$tx + (1-t)y \in T, \quad t \in I$$

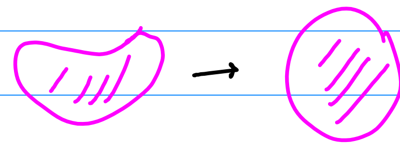
straight line between x, y is contained in T.



Prop: If $T \subset \mathbb{R}^n$ is convex, then any two $f, g : S \to T$ are homotopic.

Proof: $F(x,t) = t\,f(x) + (1-t)\,g(x)$.

Note: converse not true:  not convex, but all f, g homotopic. It's just that homotopy not a linear interpolation.

In fact just show:  continuous.

Lemma: $f : S \to T$ continuous. Then $f \simeq f$.

Proof: trivial: $F(x,t) = f(x)$.

Lemma: $f, g : S \to T$ continuous. $f \simeq g \Rightarrow g \simeq f$.

Proof: Given $F : S \times I \to T$, define $G : S \times I \to T$ by $G(x,t) = F(x, 1-t)$.

Lemma: $f, g, h : S \to T$ continuous. $f \simeq g$, $g \simeq h \Rightarrow f \simeq h$.

Proof: Given $F : S \times I \to T$, $F(x,0) = f(x)$, $F(x,1) = g(x)$
$G : S \times I \to T$, $G(x,0) = g(x)$, $G(x,1) = h(x)$

Define:
$$H(x,t) = \begin{cases} F(x,2t) & , \ 0 \leq t \leq \frac{1}{2} \\ G(x,2t-1) & , \ \frac{1}{2} \leq t \leq 1. \end{cases}$$

Continuous by the Gluing Lemma. $H(x,0) = f(x)$, $H(x,1) = g(x)$. $\blacksquare$

Conclude: we can form equivalence classes of homotopic functions.

Write $[S,T]$ for the set of homotopy classes of maps $S \to T$.

$[S,T]$ only has one element if T is convex.

Prop: $f \simeq g : S \to T$ and $h \simeq j : T \to U$, then
$$(h \circ f) \simeq (j \circ g) : S \to U.$$

Proof: If F homotopy from f to g, H homotopy from h to j, define $G : S \times I \to U$ by
$$G(s,t) = H(F(s,t),t)$$

$G(s,0) = h(f(s)) = (h \circ f)(s)$, $G(s,1) = j(g(s)) = (j \circ g)(s)$.

G is continuous since it is a composition of cont. functions. $\blacksquare$

Lecture 11: Homotopy Equivalence 2/21/11

or "homotopic"

S, T are **homotopy equivalent** if there are continuous maps $f: S \to T$ and $g: T \to S$ s.t. $g \circ f \simeq \underbrace{\text{identity}}_{1_S}$ on S and $f \circ g \simeq$ identity on T.

f and g are called homotopy **equivalences**. Write $S \simeq T$.

Lemma: $S \simeq T$ and Q any topo. space, then

$$[S,Q] = [T,Q] \quad \text{and} \quad [Q,S] = [Q,T].$$

Proof: Given $h: S \to Q$, get $(h \circ g): T \to Q$
$j: T \to Q$, $(j \circ f): S \to Q$

$f: S \to T$
$g: T \to S$

So, any $h: S \to Q$ gives $(h \circ g): T \to Q$, and any $j: T \to Q$ gives $(j \circ f): S \to Q$. Just need to show that it's the "same function" up to homotopy.

turn into $S \to Q$

$$\underbrace{(h \circ g)}_{T \to Q} \circ f = h \circ (g \circ f) \simeq h \circ 1_S = h \quad \leftarrow \text{we're back to } h$$

$$(j \circ f) \circ g = j \circ (f \circ g) \simeq j \circ 1_T = j \quad \leftarrow \text{back to } j$$

Similarly for $[Q,S] = [Q,T]$. ∎

Lemma: $S \cong T \Rightarrow S \simeq T$.

Proof: $f: S \to T$ and $g: T \to S$ homeomorphisms.
Then $f \circ g = 1_T \simeq 1_T$ and $g \circ f = 1_S$.

example: ($S \simeq T \not\Rightarrow S \cong T$ in general)

not invertible

$S = \{0\} \simeq \mathbb{R}$. $\quad f: \mathbb{R} \to S$ = constant function
$g: S \to \mathbb{R}$ = $f(0) = 0$.

$f \circ g: S \to S$ is identity, $g \circ f: \mathbb{R} \to \mathbb{R}$ constant function $f(x) = 0$.

But all functions $\mathbb{R} \to \mathbb{R}$ are homotopic, so $g \circ f \simeq 1_{\mathbb{R}}$.

Hence $\{0\} \simeq \mathbb{R}$, but clearly not homeomorphic.

A space which is homotopic to 1 point is called <u>contractible</u>.

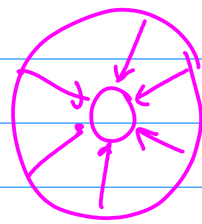
(a,b), (a,∞), $(-\infty, b)$ are all contractible.

Prop. If S contractible and T any topo. sp., then any continuous $f, g: T \to S$ are homotopic. In particular, any cont. func. to a contractible space $\simeq$ constant map.

Proof: S contractible $\Rightarrow h: S \to \{0\}$, $j: \{0\} \to S$ s.t. $h \circ j \simeq 1$, $j \circ h \simeq 1$.

$$f = 1\circ f \simeq \underbrace{j\circ \overbrace{h\circ f}^{\substack{T\to\{0\}\\ \text{constant}}}}_{\substack{\text{constant } T\to S\\ t\mapsto j(0)}}, \quad g = 1\circ g \simeq \underbrace{j\circ \overbrace{h\circ g}^{\substack{T\to\{0\}\\ \text{constant}}}}_{t\mapsto j(0)}$$

$$\Rightarrow f \simeq g \qquad \blacksquare$$

More examples: annulus $A \simeq S^1$ , $\mathbb{R}^2 - \{(0,0)\} \simeq S^1$

Proving that two spaces are not homotopic can be hard.

Prop: S^0 is not contractible.

Proof: Suppose it is, with $f: S^0 \to \{0\}$ and $g: \{0\} \to S^0$.

$f\circ g : \{0\} \to \{0\}$ identity, $g\circ f : S^0 \to S^0 \simeq 1$.

Hence, $\exists\ F: S^0 \times I \to S^0$, $F(x,0) = x$, $F(x,1) = g(f(x)) = g(0)$.

Now define $h: I \to S^0$ by $h(t) = F(-g(0), t)$.

h continuous w. $h(0) = F(-g(0), 0) = -g(0)$, $h(1) = F(-g(0),1) = g(0)$.

Since S^0 has only two points, h surjective.

Not possible by Lemma 4.3 (no cont. surjection from connected $\overset{I}{T} \to S^0$).

$\blacksquare$

More generally:

Prop: X connected and Y disconnected are not $\simeq$.

$\mathbb{R} \simeq \{0\}$, for instance

Compactness and Hausdorfness cannot be used to determine homotopy inequivalence.

example: $S = \{1,2\}$ w. indiscrete topo. (not Hausdorff)

$T = \{0\}$

Then $S \simeq T$ (see Crossley)

Lecture 12: The circle: Path lifting 2/23/11

Prove: S^1 not contractible. (But do better than that.)

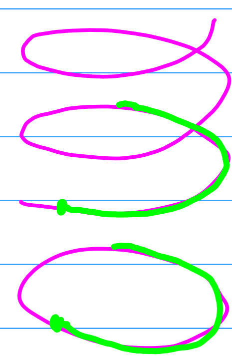
Recall exponential map $e: \mathbb{R} \to S^1$, $e(x) = (\cos(2\pi x), \sin(2\pi x))$

Prop: $g: [0,1] \to S^1$ cont., x any point $\in \mathbb{R}$ s.t. $e(x) = g(0)$.

Then $\exists$ unique cont. function $\tilde{g}: [0,1] \to \mathbb{R}$ s.t. $e(\tilde{g}(t)) = g(t)$ for all $t \in [0,1]$, with $\tilde{g}(0) = 1$.

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "e"] \\ {[0,1]} \arrow[ur, "\tilde{g}"] \arrow[r, "g"'] & S^1 \end{tikzcd}$$

PATH LIFTING



Proof: Take any proper subset (i.e., not ϕ or S^1 itself) U of S^1!

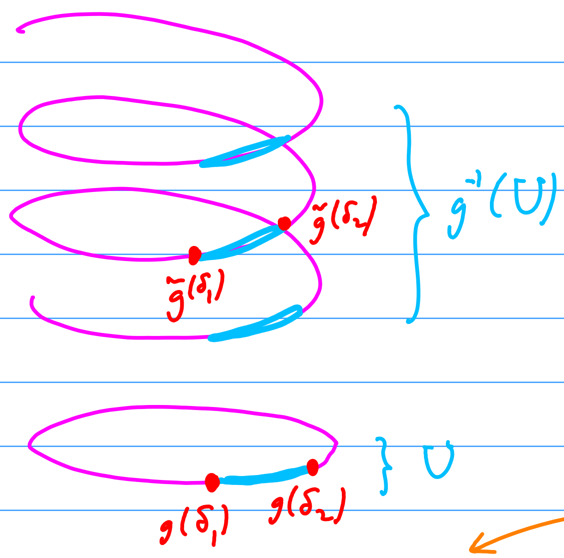
Preimage under e is disjoint union of infinitely many subsets, each homeomorphic to U.

Now take $[\delta_1, \delta_2] \subset [0,1]$ whose image under g is $\subset U$.

Suppose $\tilde{g}(\delta_1)$ defined s.t. $e(\tilde{g}(\delta_1)) = g(\delta_1)$.

Then $\tilde{g}(\delta_1)$ lies in one of these spaces homeomorphic to U.

Compose homeo with g to define $\tilde{g}$ on that space,



$$e([\tilde{g}(\delta_1), \tilde{g}(\delta_2)]) = [g(\delta_1), g(\delta_2)]$$

Then define $\tilde{g}$ over $[0,1]$ by combining intervals, say two of them

are these specific forms used?

Use $U = S^1 - \{(1,0)\}$, $V = S^1 - \{(-1,0)\}$. $S^1 = U \cup V$.

$g^{-1}(U)$ and $g^{-1}(V)$ are open in $[0,1]$, w. $g^{-1}(U) \cup g^{-1}(V) = [0,1]$.

Write $g^{-1}(U)$ and $g^{-1}(V)$ as unions of (a,b), $[0,b)$, $(a,1]$, to get open cover of $[0,1]$. Refine to finite $I_1, \dots, I_n$.

(assume as small as possible) Each set maps to either U or V.

Won't one of these preimages be just one point? Must then be 0 or 1.

Let $0 \in I_1$, so that $I_1 = [0, b_1)$, $0 \le b_1 \le 1$.

Let $b_1 \in I_2$, so that $I_2 = (a_2, b_2)$, $a_2 < b_1 < b_2 < 1$

or $I_2 = (a_2, 1]$

If the former, keep going.

Hence, can order $I_1, \dots, I_n$ from "left-to-right".

Let $\delta_0 = 0$, $\delta_n = 1$, $\delta_i = (a_{i+1} + b_i)/2$, $1 < i < n$

Then $\delta_i \in I_i \cap I_{i+1}$, $1 \leq i < n$, so $[\delta_i, \delta_{i+1}] \subset I_{i+1}$.

$\tilde{g}(0) = x$, so $\tilde{g}(\delta_0) = x$. $[\delta_0, \delta_1] \subset I_1$, and $g(I_1)$ is either $\subset U$ or in V.

$\Rightarrow$ unique open interval of $\mathbb{R}$, containing x, homeo to U or V

Compose with g to obtain continuous $\tilde{g}: I_1 \to \mathbb{R}$, $\tilde{g}(\delta_0) = x$, $e \circ g = g \mid I_1$.

Now we have $\tilde{g}(\delta_1)$, so start over on $[\delta_1, \delta_2]$.
(Gluing Lemma guarantees continuity)
In the end get cont. map $\tilde{g}: [0,1] \to \mathbb{R}$, s.t. $\tilde{g}(0) = x$, $e \circ \tilde{g} = g$.

Uniqueness: $\bar{g}: [0,1] \to \mathbb{R}$ another lift w. $\bar{g}(0) = x = \tilde{g}(0)$.

$e \circ \bar{g} = e \circ \tilde{g} \Rightarrow \bar{g}(y) - \tilde{g}(y) \in \mathbb{Z}$, for all y.

Get continuous map $\bar{g} - \tilde{g}: [0,1] \to \mathbb{Z}$, so must be constant by Lemma 4.18.

But $\bar{g}$ and $\tilde{g}$ agree at one point, so const. is 0 and $\bar{g} = \tilde{g}$. ∎

Lecture 13: The circle: Homotopy lifting 2/25/11

$\pi : [0,1] \to S^1$

Take cont. $f : S^1 \to S^1$ and form $g = f \circ \pi : [0,1] \to S^1$.

$\Rightarrow \quad g(0) = g(1)$, since $\pi(0) = \pi(1)$.

Lift $\tilde{g}$ satisfies $e\tilde{g}(0) = e\tilde{g}(1)$. $e(t) = e(s)$ iff $t - s \in \mathbb{Z}$.

$\Rightarrow \quad \tilde{g}(1) - \tilde{g}(0) \in \mathbb{Z}$. degree or winding number of f.

$\deg(f)$

The degree does not depend on the choice of lift.
(the lift is determined only by $\tilde{g}(0)$, but this cancels out.)

example: $f : S^1 \to S^1$, $f(\cos\theta, \sin\theta) = (\cos m\theta, \sin m\theta)$, $m \in \mathbb{Z}$

$g : [0,1] \to S^1$ is $t \mapsto (\cos(2\pi m t), \sin(2\pi m t))$

Take lift $\tilde{g} : [0,1] \to \mathbb{R}$, $\tilde{g}(t) = mt$. Hence, $\deg(f) = m$.

This means the identity map has degree 1. $(m=1)$
A constant map has degree 0 $(m=0)$

We will see that homotopic maps have equal degree.

Prop: (Homotopy lifting) $F: I \times I \to S^1$ cont., $x \in \mathbb{R}$ any pt.
s.t. $e(x) = F(0,0)$. $\exists$ unique cont. $\tilde{F}: I \times I \to \mathbb{R}$
s.t. $e\tilde{F}(s,t) = F(s,t)$ $\forall s,t \in I$ with $\tilde{F}(0,0) = x$.

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "e"] \\ I \times I \arrow[ur, "\tilde{F}"] \arrow[r, "F"'] & S^1 \end{tikzcd}$$

Commutes

Prove this using two results. Subset of $\mathbb{R}^n$ has diameter $< d$ if dist. between any two points $< d$.

Prop: (Domain splitting) $f: X \to Y$, X compact subset of $\mathbb{R}^n$, open cover $\mathcal{U}$ of Y. $\exists \delta > 0$ s.t. for $V \subset X$ with diameter $< \delta$, $f(V) \subset$ one of the sets in $\mathcal{U}$.

Proof: f cont., so preimages of open sets in $\mathcal{U}$ are open in X.

$$f^{-1}(\mathcal{U}) = \mathcal{W} = \text{open cover of } X.$$

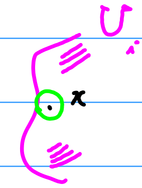
Any $V \subset X$ which is contained in one of the $\mathcal{W}$ will have $f(V) \subset$ one of the sets in $\mathcal{U}$. Get δ from lemma: $\blacksquare$

Lemma: (Lebesgue) Compact $X \subset \mathbb{R}^n$, open cover $\mathcal{U}$ of X, $\exists \delta > 0$
s.t. any subset U of X of diameter less than δ
is contained in one of the sets in $\mathcal{U}$.

Proof: Refine $\mathcal{U}$ to $U_1, \dots, U_n$. Define $f_i : X \to \mathbb{R}$ by

$$f_i(x) = \begin{cases} \text{largest radius } r \text{ s.t. } B_r(x) \subset U_i, & x \in U_i \\ 0, & x \notin U_i \end{cases}$$

f_i is continuous: largest radius $\to 0$ as $x \to$ edge of U_i.



$f : X \to \mathbb{R}$, $\quad f(x) = \max\limits_{1 \leq i \leq n} f_i(x)$ also continuous.

If there is $\delta > 0$ s.t. $f(x) \geq \delta$ for all x, then every open ball of radius $< \delta$ is contained in some open set U_i. Every set of diameter $< \delta$ $\subset$ open ball of radius δ, so lemma follows.

Does δ exist? $f(x) > 0$, so $f(x) \neq 0$. X compact, so image of f is compact subset of $\mathbb{R}$ $\Rightarrow$ closed (Heine-Borel).

$\hookrightarrow$ complement open and contains 0.

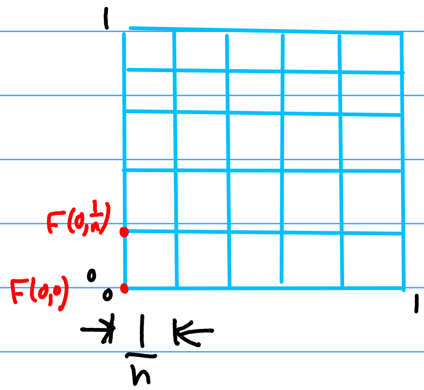
$\hookrightarrow$ interval $(-\delta, \delta)$ around 0. $(\delta > 0)$

$\hookrightarrow$ $f(x) \geq \delta \quad \forall x$.

Basically, to form an open cover open sets have to overlap to some thickness.

Proof of Homotopy lifting: (sketch)

Cover S^1 with $U = S^1 - \{(1,0)\}$, $V = S^1 - \{(-1,0)\}$, as before. There is a $\delta > 0$ s.t. any subset of $I \times I$ of diam $< \delta$ is mapped into either U or V by F.



Split $I \times I$ into an $n \times n$ grid, with

$$\frac{1}{n} < \frac{\delta}{\sqrt{2}} \quad (\text{diameter} < \delta)$$

Each square mapped by F into either U or V.

$\tilde{F}(0,0) = x$, so construct $\tilde{F}$ over lower-left square, using homeo between component of $e^{-1}(U)$ or $e^{-1}(V)$ and U or V.

Get $\tilde{F}(0, \frac{1}{n})$, which allows def'n of $\tilde{F}$ on next square.
The boundary between the two squares has well-defined value of $\tilde{F}$, by homotopy lifting! etc. for the other squares.
Uniqueness same as before. ▨

Lecture 14: The circle and Brouwer's FPT 2/28/11

Last time: homotopy lifting property.

Corollary: f, g homotopic $\Rightarrow \deg(f) = \deg(g)$.

Proof: Let $H: S^1 \times I \to S^1$ homotopy between f and g.
Lift to $\tilde{H}: I \times I \to \mathbb{R}$.

$\tilde{H}\big|_{I \times \{0\}}$ = lift for f, so $\deg(f) = \tilde{H}(1,0) - \tilde{H}(0,0)$

$\tilde{H}\big|_{I \times \{1\}}$ = lift for g, so $\deg(g) = \tilde{H}(1,1) - \tilde{H}(0,1)$

Define $D: I \to \mathbb{Z}$ by $D(t) = \tilde{H}(1,t) - H(0,t)$

$\deg(f) = D(0)$, $\deg(g) = D(1)$. But D continuous so constant!
(since I connected)

Hence $\deg(f) = \deg(g)$. ▨

Corollary: The circle is not contractible.

Proof: Suppose $f: S^1 \to \{0\}$, $g: \{0\} \to S^1$ homotopy equivalences.
$g \circ f \simeq 1_{S^1}$

$(g \circ f)(x,y) = g(0)$, $\forall\, (x,y) \in S^1$. constant
↓
degree 0

But identity map has degree 1. ▨

$\pi : [0,1] \to S^1$ cont. surjective $(= e|_I$, say$)$

Theorem: $f, g : S^1 \to S^1$ have $\deg(f) = \deg(g) \Leftarrow f, g$ homotopic.

Proof: Assume $(f \circ \pi)(0) = (g \circ \pi)(0)$. (See below if not the case)

$$\text{Lift } f, g \text{ to } \tilde{f}, \tilde{g} : [0,1] \to \mathbb{R}, \quad \tilde{f}(0) = \tilde{g}(0)$$

$$\tilde{f}(1) = \deg(f) + \tilde{f}(0) = \deg(g) + \tilde{g}(0) = \tilde{g}(1)$$

Define: $H : I \times I \to \mathbb{R}$ by

homotopy "upstairs"

$$\tilde{H}(s,t) = t\,\tilde{f}(s) + (1-t)\,\tilde{g}(s),$$

then $\left.\begin{aligned} \tilde{H}(0,t) &= \tilde{f}(0) = \tilde{g}(0) \\ \tilde{H}(1,t) &= \tilde{f}(1) = \tilde{g}(1) \end{aligned}\right\}$ do not depend on t.

$$\Rightarrow \quad \tilde{H}(1,t) - \tilde{H}(0,t) = \deg(f)$$ integer, so project to same point under e

$(e \circ \tilde{H})(0,t) = (e \circ \tilde{H})(1,t)$ map $H : S^1 \times I \to S^1$

homotopy between f and g. $\blacksquare$

If $(f \circ \pi)(0) \neq (g \circ \pi)(0)$, use:

Lemma: $g : S^1 \to S^1$, $(x,y) \in S^1$, then $\exists\, h : S^1 \to S^1 \simeq$ to g and s.t. $h(\pi(0)) = (x,y)$.

Proof: by rotation

Corollary: $[S^1, S^1] = \mathbb{Z}$

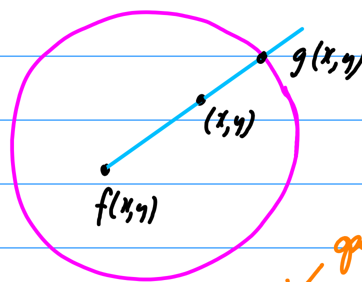
Brouwer's Fixed-Point Theorem:

$f: D^2 \to D^2$ continuous, D^2 = closed disk.
$= \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$

Then f has a fixed point. ($f(x,y) = (x,y)$ for at least one (x,y))

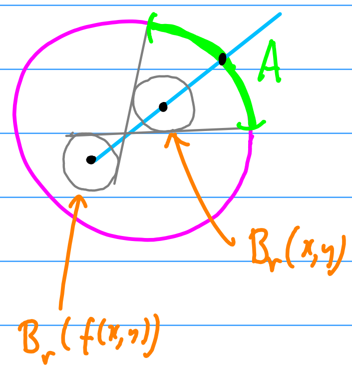
Proof: Suppose it doesn't: $f(x,y) \neq (x,y)$, $\forall (x,y) \in D^2$.

Draw line through (x,y) and $f(x,y)$:
Get $g(x,y): D^2 \to S^1$



g is continuous: A open arc around $g(x,y)$.
$\exists$ radius r s.t. whenever
$(x',y') \in B_r(x,y)$ and $f(x',y') \in B_r(f(x,y))$,
then $g(x',y')$ is in A.

open balls



f continuous, so $\exists \delta > 0$ s.t. $f(x',y') \in B_r(f(x,y))$
whenever $(x',y') \in B_\delta(x,y)$.
Hence $g^{-1}(A)$ contains $B_{\min(\delta,r)}(x,y)$.
↳ hence $g^{-1}(A)$ will be union of open balls
↳ $g^{-1}(A)$ open.

If (x,y) on boundary of D^2, then $g(x,y) = (x,y)$ no matter what $f(x,y)$ is.

Now define: $F: S^1 \times I \to S^1$ by $F((x,y),t) = g(tx,ty)$.

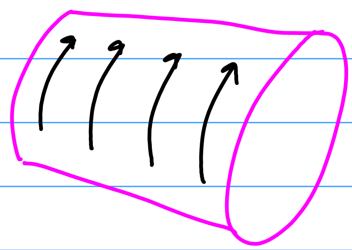
deg = 0

F continuous, so homotopy between $h: S^1 \to S^1$, $h(x,y) = F((x,y),0) = g(0,0)$
$j: S^1 \to S^1$, $j(x,y) = F((x,y),1) = g(x,y)$
$= (x,y)$ on S^1

CONTRADICTION! $\Longleftarrow$ deg = 1 $\blacksquare$

Lecture 15: Vector fields 3/2/11

Vector field on a surface $S \subset \mathbb{R}^n$: $v: S \to \mathbb{R}^n$ continuous, $v(s)$ tangent to S at s.



on S = cylinder

On the disk, we can then prove a theorem very similar to Brouwer's FPT:

Theorem: When stirring a cup of coffee, there is always some particle on the surface which is stationary.

or: A vector field $v: D^2 \to \mathbb{R}^2$ tangential to D^2 and ∂D^2 has at least one point $x \in D^2$ s.t. $v(x) = 0$.

Proof: $v: D^2 \to \mathbb{R}^2$ = velocity at point $(x,y) \in D^2$.
Assume $v \neq 0$, $\forall (x,y)$.

Define:
$$g: S^1 \to S^1 \text{ by } g(x,y) = -\frac{v(x,y)}{|v(x,y)|}, \quad (x,y) \in S^1 = \partial D^2.$$

There is a homotopy between g and id defined by

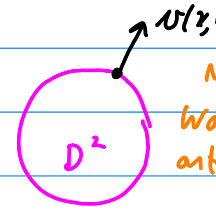
$$G((x,y),t) = \frac{(x,y)t - (1-t)v(x,y)}{|(x,y)t - (1-t)v(x,y)|}.$$

Is the denominator ever zero? Would require $(x,y)\,t = (1-t)\,v(x,y)$.

$t = 0 \Rightarrow v(x,y) = 0$, which cannot happen by assumption

$t = 1 \Rightarrow (x,y) = 0$, no since $(x,y) \in S^1$!

So $t \neq 0$ or 1, and so $v(x,y) = \underbrace{\dfrac{t}{1-t}}_{>0}(x,y)$



No! Would move out of cup.

So G is continuous.

Can also define homotopy $F : S^1 \times I \to S^1$ by

$$F((x,y),t) = -\frac{v(tx,ty)}{|v(tx,ty)|} \qquad F((x,y),1) = g(x,y)$$

$$F((x,y),0) = -\frac{v(0,0)}{|v(0,0)|} \quad \text{const}$$

Hence, $g \simeq$ constant map $-\dfrac{v(0,0)}{|v(0,0)|}$.

Conclude:

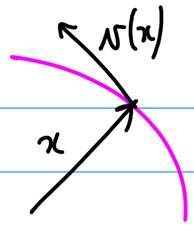
$$id \simeq g \simeq \text{const} \qquad \text{Oops! Impossible.}$$

"Hairy ball theorem": S^n has a continuous nonzero tangent vector field iff n is odd.

Proof: [Hatcher, sec. 2.2, p. 135]

Let $v : S^n \to \mathbb{R}^{n+1}$ be a cont. tangent vector field, s.t. $v \neq 0$.

(Here S^n is viewed as a subspace of $\mathbb{R}^{n+1}$.)



Regarding $v(x)$ as a vector at the origin instead of at x, tangency means that x and $v(x)$ are orthogonal in $\mathbb{R}^{n+1}$.

Let $w(x) = \dfrac{v(x)}{|v(x)|}$ (ok since $v(x) \neq 0$, by assumption.) Also a continuous tangent vector field

Define: $F(x,t) = \cos(\pi t)\, x + \sin(\pi t)\, w(x)$

$F(x,t) \in S^n$, since $|F(x,t)|^2 = \cos^2(\pi t) \underbrace{|x|^2}_{1} + \sin^2(\pi t) \overbrace{|w|^2}^{1} + 2 \sin(\pi t) \cos(\pi t) \overbrace{x \cdot w}^{0}$

$$= 1$$

Thus $F(x,t)$ is a homotopy between $F(x,0) = x$ identity map id
and $F(x,1) = -x$ antipodal map $-$id

We have $\deg(\mathrm{id}) = 1$, and $\deg(-\mathrm{id}) = (-1)^{n+1}$. (see later)

Hence, $\deg(\mathrm{id}) = \deg(-\mathrm{id})$ only if n is odd.

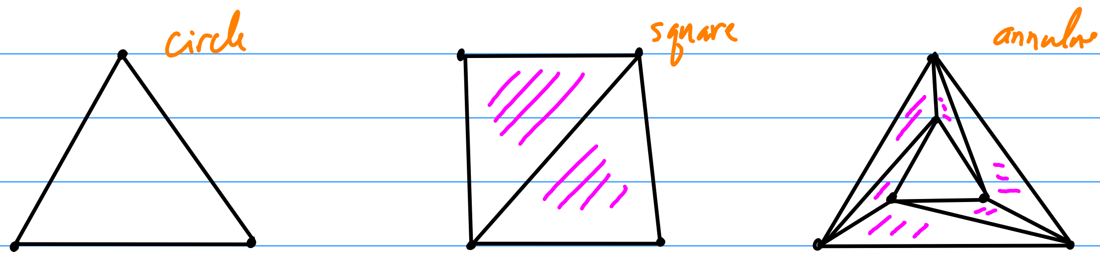
This proves the "only if" part. For the converse, let n be odd, say $n = 2k-1$, and define

$$v(x_1, x_2, \ldots, x_{2k-1}, x_{2k}) = (-x_2, x_1, -x_4, x_1, \ldots, -x_{2k}, x_{2k-1})$$

Then v is orthogonal to x, so v is a tangent vector field on S^n, and

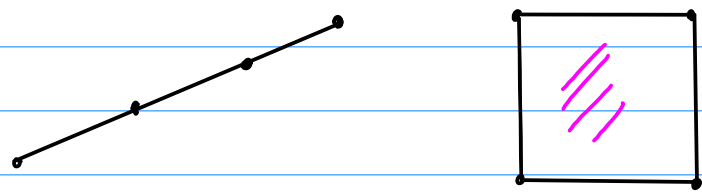
$|v(x)| = 1 \quad \forall x \in S^n$, so $v(x) \neq 0$. ∎

Lecture 16: Simplicial complexes 3/4/11



simplicial complexes made of simplices.

k-simplex is described by a list of $k+1$ vertices (points) in $\mathbb{R}^n$.
Smallest convex subspace in $\mathbb{R}^n$ containing those vertices.



Avoid bad choices such as these. These both have $k+1=4$ points, but are not 3-dimensional.

We thus insist that vertices $v_0, \dots, v_k$ be in general position:

$v_1 - v_0,\ v_2 - v_1,\ \dots,\ \dots,\ v_k - v_{k-1}$ are k linearly indep. vectors.

Ensures that a k-simplex is truly k-dimensional.

k-simplex is the smallest convex subspace of $\mathbb{R}^n$ containing the $k+1$ vertices.

$[v_0, \dots, v_k]$, spanned by $t_0 v_0 + t_1 v_1 + \dots + t_k v_k$, $t_i \in I$.

$$t_0 + \dots + t_k = 1$$

t_i are barycentric coordinates.

Subsimplex is simplex formed by (nonempty) subset of vertices:

faces

$[v_0, v_1, v_2]$ has 7 subsimplices: $[v_0]$, $[v_1]$, $[v_2]$, $[v_0, v_1]$, $[v_0, v_2]$, $[v_1, v_2]$, $[v_0, v_1, v_2]$.

Subsimplex is a face if it omits only one vertex.

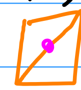
k-simplex has $k+1$ faces and $2^{k+1} - 1$ subsimplices.

The union of faces is the boundary of the k-simplex. all point with at least one $t_i = 0$

Complement of boundary is the interior. all points with $t_i \neq 0$, $\forall i$

0-simplices have empty boundary; interior is the simplex itself.

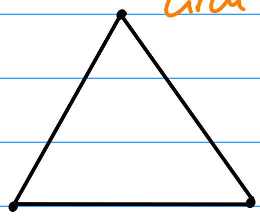
A simplicial complex K is a subspace of $\mathbb{R}^n$ together with a finite list of simplices such that

1. Union of simplices $= K$, each point in K lies in the interior of only one simplex (could be a subsimplex! )

2. Every face of every simplex is also in the list.

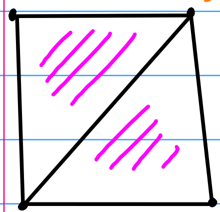
Finiteness means simplicial complexes are compact.

Dimension of a simplicial complex given by largest k-simplex in complex.

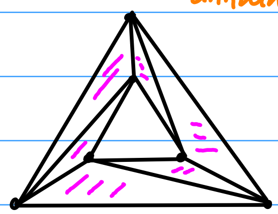


1-dim, 3 0-simp, 3 1-simp

Can't do this:  not convex



2-dim, 4 0-simp, 5 1-simp, 2 2-simp



2-dim, 6 0-simp, 12 1-simp, 6 2-simp

Prop: S and T simplices of a simplicial complex K. Then $S \cap T$ is either empty or a subsimplex of both S and T.

Proof: $S \cap T \neq \phi$, let $v_1, \dots, v_n$ set of all vertices of K that are contained in $S \cap T$. Show $S \cap T = [v_1, \dots, v_n]$.
(since $S, T = [v_1, \dots, v_n, \dots]$)

Let $x \in S \cap T$. Then x contained in the interior of exactly one simplex of K, say $[w_1, \dots, w_k]$. $x \in S$ so we can write x as linear comb. of vertices of S barycentric coords. Take only non-zero coeffs: get subsimplex of S whose interior contains x.
A subsimplex of S is a simplex of K, so this must be $[w_1, \dots, w_k]$.
Hence $w_1, \dots, w_k$ are vertices in S.

Apply same argument to T to show $w_1, \dots, w_k \in S \cap T$.

Thus $\{w_1, \dots, w_k\} \subset \{v_1, \dots, v_n\}$, so

$$x \in [w_1, \dots, w_k] \subset [v_1, \dots, v_n].$$

Hence, $S \cap T \subset [v_1, \dots, v_n]$.

On the other hand, since $v_1, \dots, v_n \in S$, S convex,

$$\Rightarrow [v_1, \dots, v_n] \subset S.$$

Applies also to T, and thus $[v_1, \dots, v_n] \subset S \cap T$.

Conclude $S \cap T = [v_1, \dots, v_n]$. ∎

Lecture 17: Triangulation of surfaces 3/7/11

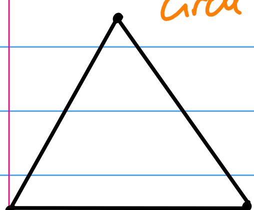
Euler number: T an n-dimensional simplicial complex

i_k = number of k-simplices in T $\quad (0 \leq k \leq n)$

$$\chi(T) = i_0 - i_1 + i_2 - i_3 + \cdots + (-1)^n i_n$$

Euler number or Euler characteristic

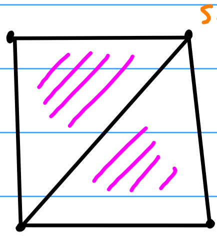
circle



1-dim,

$i_0 = 3, \; i_1 = 3 \qquad \chi = 3 - 3 = \underline{\underline{0}}$

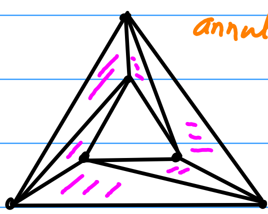
square



2-dim,

$i_0 = 4, \; i_1 = 5, \; i_2 = 2 \qquad \chi = 4 - 5 + 2 = \underline{\underline{1}}$

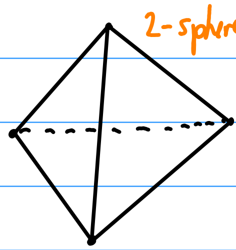
annulus



2-dim,

$i_0 = 6, \; i_1 = 12, \; i_2 = 6 \qquad \chi = 6 - 12 + 6 = \underline{\underline{0}}$

2-sphere S^2



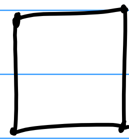
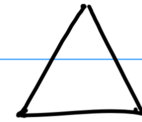
2-dim

$i_0 = 4, \; i_1 = 6, \; i_2 = 4 \qquad \chi = 4 - 6 + 4 = \underline{\underline{2}}$

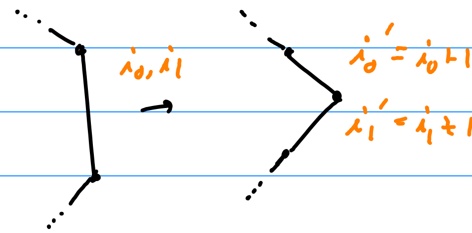
A **triangulation** of a topological space T is a simplicial complex K and a homeomorphism $K \leftrightarrow T$.

(T is **triangulable**.)

The Euler characteristic is a topological invariant.

example: S^1  $\chi = 4-4=0$ or  $\chi = 3-3=0.$

Clearly adding an edge does not change χ:



If two triangulable spaces are homeomorphic, then they have to same Euler number.

can relax to homotopic

(Converse is not true: $\chi(S^1) = \chi(T^2) = 0$, for instance.)

Surfaces: A surface is a Hausdorff space such that around every point of the space there is an open neighborhood homeomorphic with an open disk in $\mathbb{R}^2$.

Note: rules out boundaries! examples:

"surface with boundary"

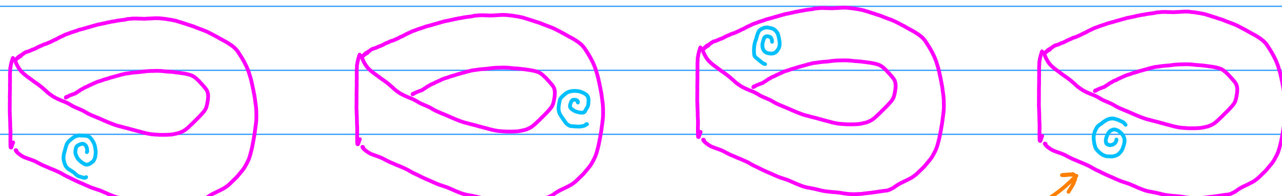
$S^1 \times \mathbb{R}$

$S^1 \times (0,1)$ (but not $S^1 \times [0,1]$)

Klein bottle

Orientability: does a "spiral" retain the same orientation?

Möbius strip is not orientable



on other side, but same point

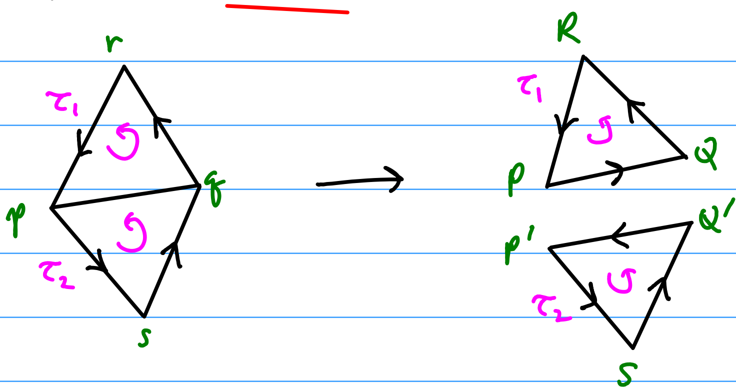
Let's prove a classification theorem: two compact orientable surfaces are homeomorphic iff they have the same Euler characteristic.

(Follow Fulton, Chapt. 17)

Fulton	here
faces	→ simplices
edges	→ edges or faces

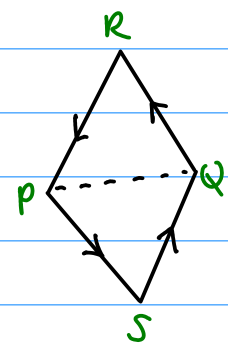
(We assume the surfaces are triangulable → no restriction at all.)

Take two adjacent simplices τ_1, τ_2 in the triangulation. Give their boundary a counterclockwise orientation:



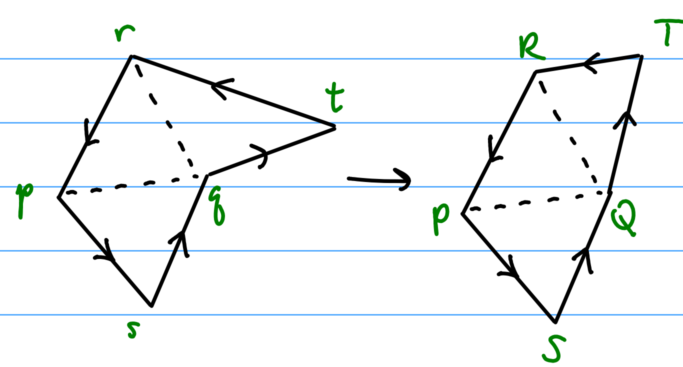
oriented in opposite direction along their common edge

Use a homeomorphism to map these two triangles in the plane to get a convex quadrilateral Π_2.

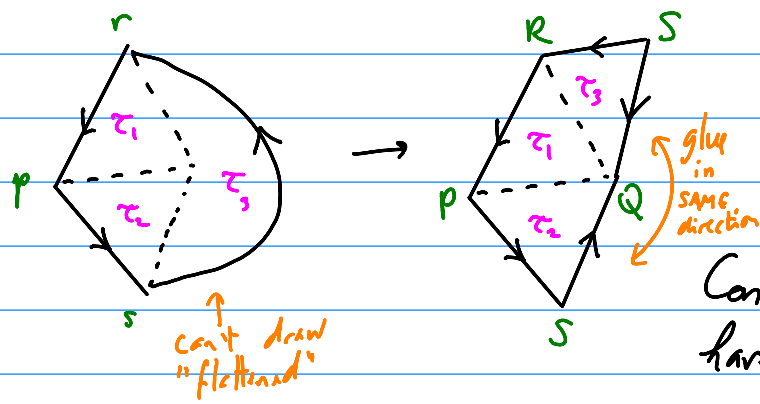


Now choose a simplex τ_3 adjacent to τ_1 or τ_2. It either shares an edge with τ_1, τ_2, or both.

If only one, (say τ_1), map to convex polygon in the plane



If τ_3 adjacent to τ_1 and τ_2, map τ_3 to triangle in the plane adjacent to one of them, say τ_1:



Get convex polygon with "edges identified"

Continue until all f simplices have been used.

Get convex polygon Π with $f+2$ sides. Each side of Π corresponds to a common edge of two simplices, and such a common edge will give rise to two sides of Π.

Hence, sides of Π are paired off, and original surface realized as quotient space of Π.

Next time: continue classification.

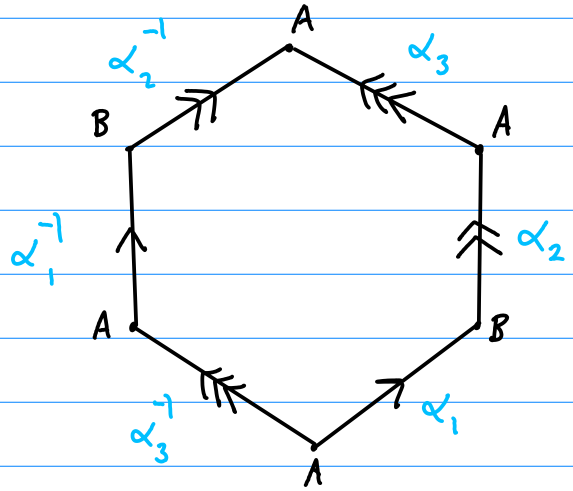
Lecture 18: Classification of surfaces (cont'd) 3/9/11

From last time: start with orientable, triangulable compact surface. (with f 2-simplices)

Get convex polygon Π with $f+2$ sides. Each side of Π corresponds to a common edge of two simplices, and such a common edge will give rise to two sides of Π.

Hence, sides of Π are paired off, and original surface realized as quotient space of Π.

Encounter corresponding sides in opposite direction when travelling around boundary of Π. Can encode the identification information by $m = \frac{1}{2}(f+2)$ letters.

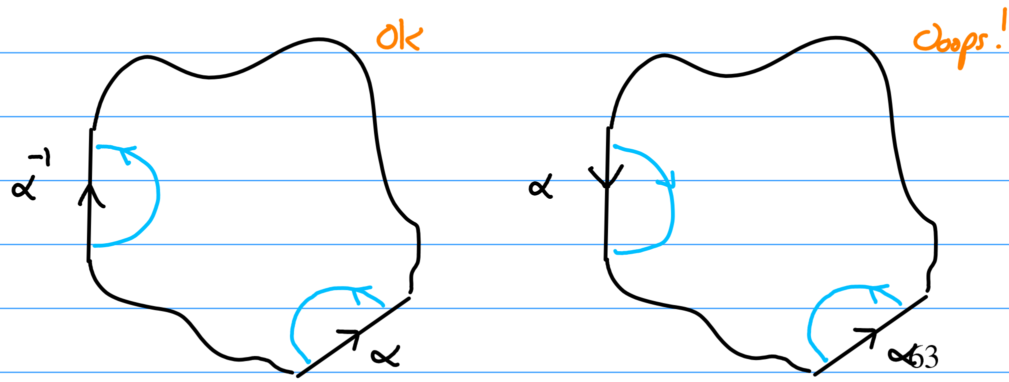


Write letters around perimeter, writing α_i^{-1} when we encounter an edge with opposite orientation.

← example: $\alpha_1 \alpha_2 \alpha_3 \alpha_2^{-1} \alpha_1^{-1} \alpha_3^{-1}$

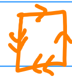
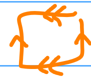
Can also write as any cyclic permutation of the sequence of symbols.

Orientability:

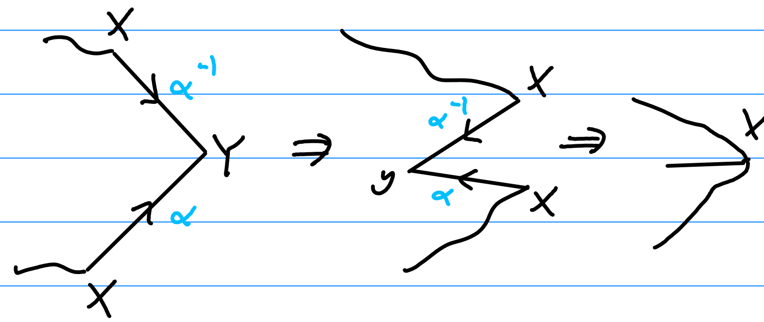


Now forget the triangulation: we just need the code.

Polygon with $2m$ sides, $m \geq 3$. We want to simplify the code.

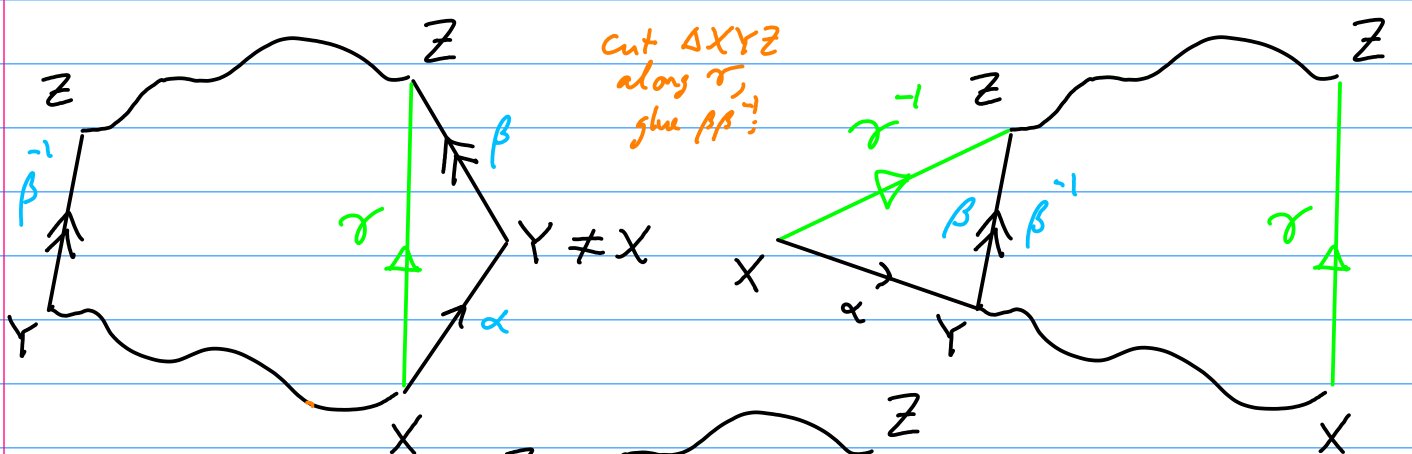
($m = 2$: sphere or torus:  or .)

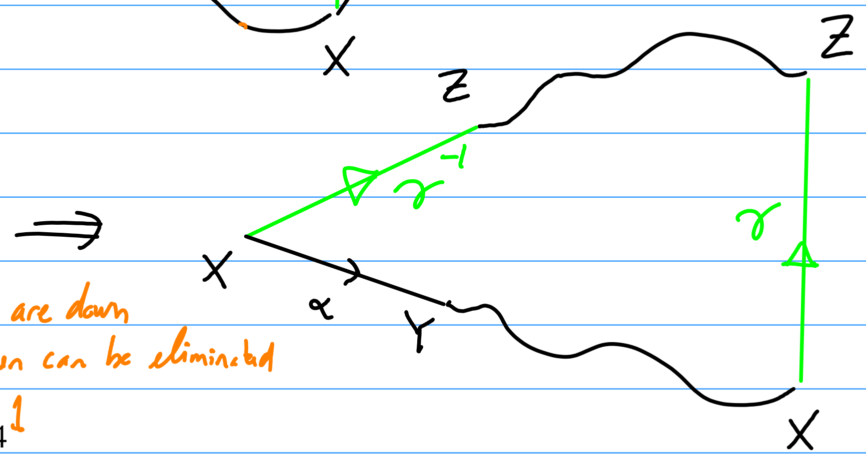
Step 1: Omit sequences $\alpha\alpha^{-1}$:



Step 2: Make all vertices map to the same point X.

Assume they do not. Then there is a vertex α joining a point that maps to X to a point that does not, say Y.
Let the next side be β.





Note that we've lost a Y and gained an X! Continue this way until only points that map to X.

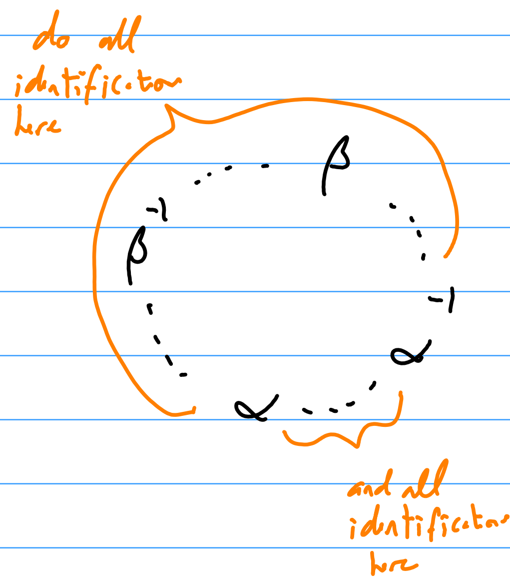
Note: if we are down to one Y, then can be eliminated using Step 1

Step 3: Now all vertices map to X (step 2), no edge adjacent to its inverse (step 1).

Claim: for any edge α, there must be β between α and α^{-1} such that β^{-1} is between α^{-1} and α.

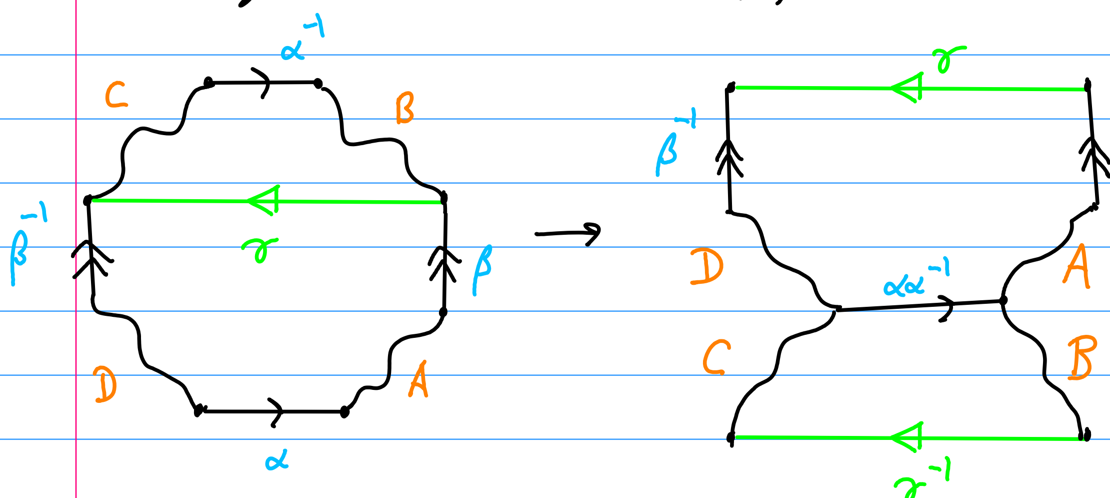
That is (possibly after cyclic perm.): $\ldots\alpha\ldots\beta\ldots\alpha^{-1}\ldots\beta^{-1}\ldots$

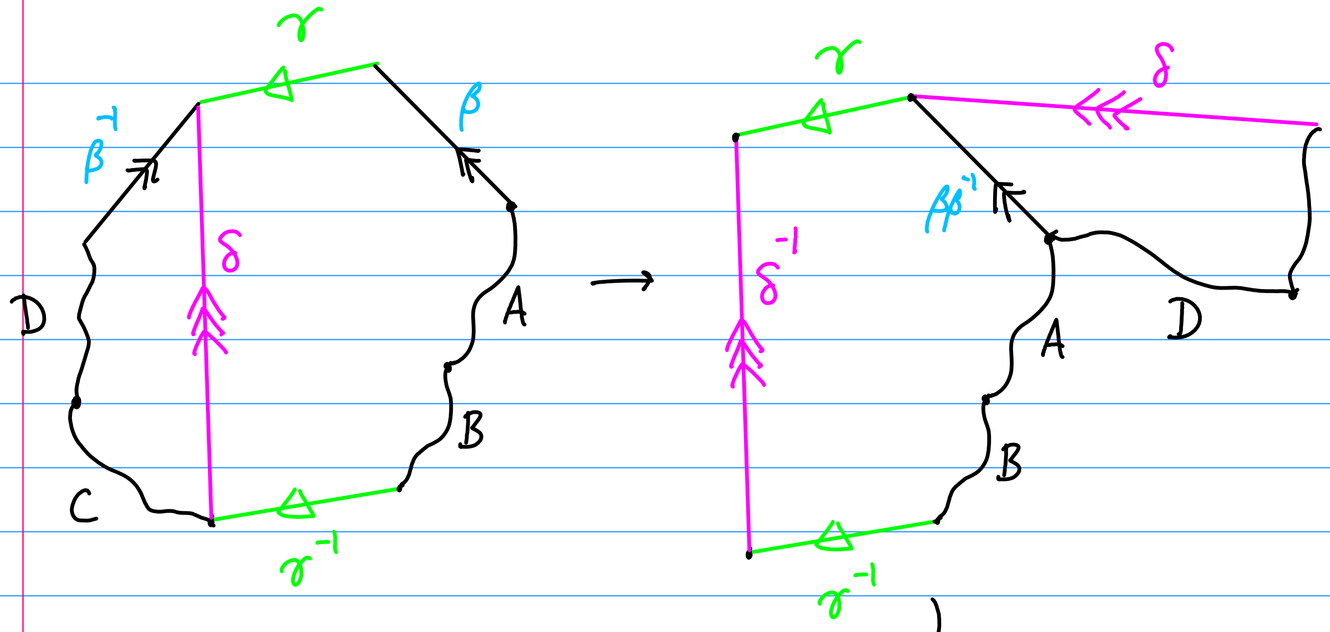
If not: $\ldots\alpha\ldots\alpha^{-1}\ldots\beta\ldots\beta^{-1}\ldots$ (note: no need to consider $\ldots\alpha\ldots\beta^{-1}\ldots\alpha^{-1}\ldots\beta\ldots$, since replace $\beta^{-1}\to\beta$.)

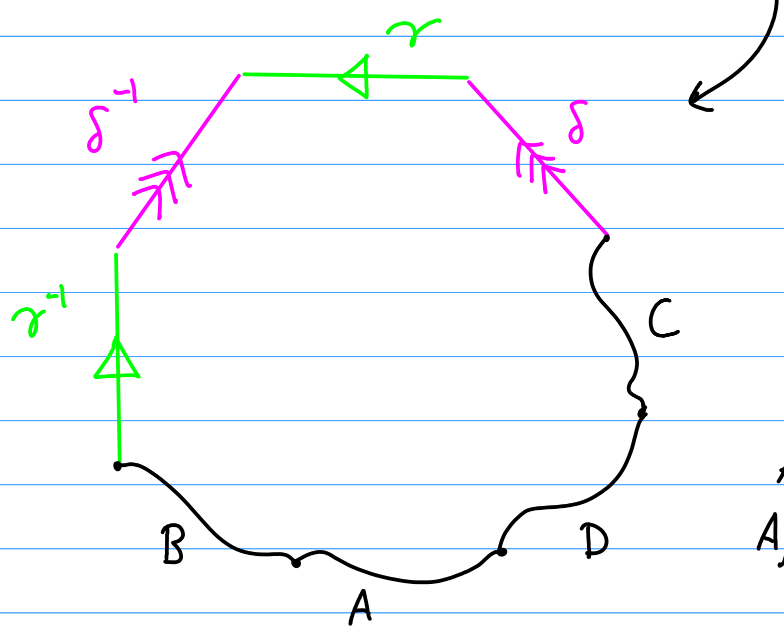


Oops! The two endpoints of α never get identified, contradicting Step 2.

Now take such a set of edges $\ldots\alpha\ldots\beta\ldots\alpha^{-1}\ldots\beta^{-1}\ldots$, possibly with other edges in between (labeled A, B, C, D.)







Same type of polygon (same number of vertices) $..\alpha..\beta...\alpha^{-1}..\beta^{-1}...$ replaced by adjacent $\delta\gamma\delta^{-1}\gamma^{-1}$.

Any such sequence occurring in A, B, C, D is undisturbed.

Thus, after steps 1-3, get a code: NORMAL FORM

$$\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\alpha_2\beta_2\alpha_2^{-1}\beta_2^{-1}\cdots\alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1}$$

g = genus

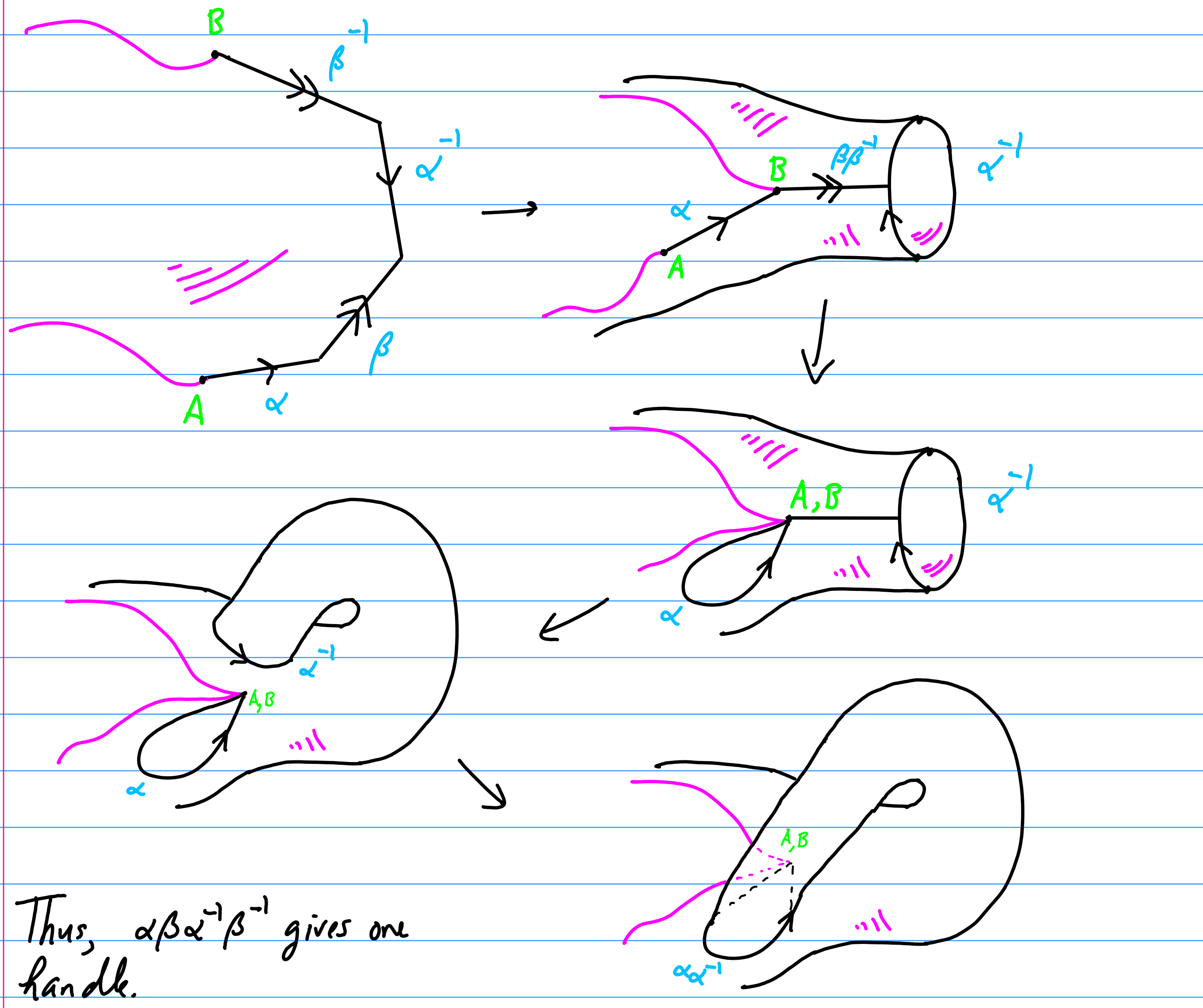
This is a sphere with g handles (next lecture).

Lecture 19: Classification of surfaces (end) 3/11/11

Last time: normal form for compact orientable surface.

$\alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \alpha_2 \beta_2 \alpha_2^{-1} \beta_2^{-1} \cdots \alpha_g \beta_g \alpha_g^{-1} \beta_g^{-1}$ $\quad g =$ genus

Now show: This is a sphere with g handles.

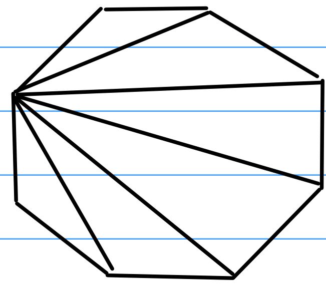


Thus, $\alpha\beta\alpha^{-1}\beta^{-1}$ gives one handle.

Assume polygon in normal form: what is Euler characteristic?

Re-triangulate:

$g=2$



$4g$ sides

$i_0 = 1$ (step 1)

$i_1 = (4g-3) + \frac{4g}{2} = 6g-3$

identification of sides

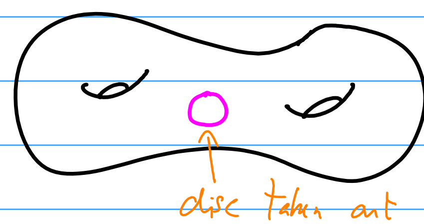
$i_2 = 4g-2$

$$\chi = 1 - (6g-3) + 4g - 2$$
$$= 2 - 2g$$

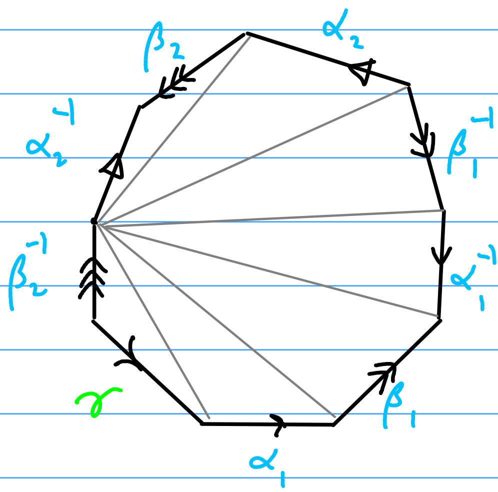
$\boxed{\chi = 2-2g}$ for a surface of genus g

Hence, 1-1 relationship between g and χ, so also true that two compact orientable surfaces $\cong$ iff same χ.

What about boundaries?



Accommodate by adding an edge.



\# of boundaries

$i_0 = 1$

$i_1 = (4g+b) - 3 + \frac{4g}{2} + b = 6g+2b-3$

$i_2 = 4g+b-2$

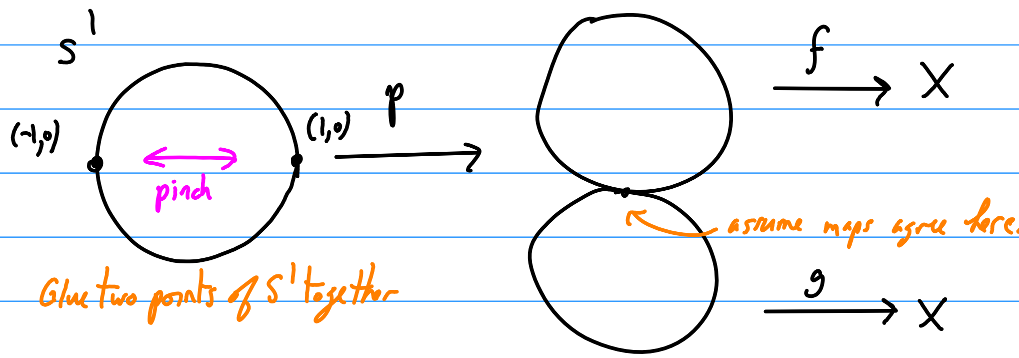
$\boxed{\chi = 2-2g-b}$

Of course, χ by itself can't distinguish surfaces with boundaries!
(But g, b can.)

Non-orientable classification: $\alpha_1\alpha_1\alpha_2\alpha_2\cdots\alpha_k\alpha_k$

Homotopy groups: (back to Crossley, chapt. 8)

Recall from chapt. 6: homotopy classes of maps $S^1\to S^1$, $[S^1,S^1]$

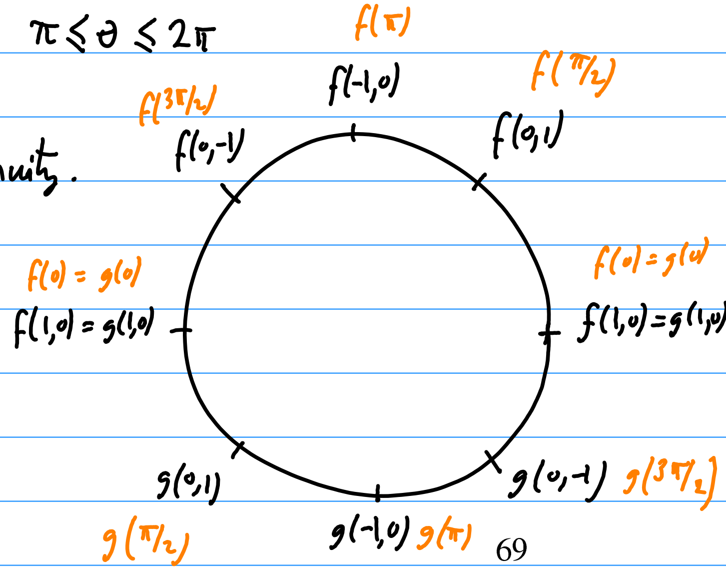


Now take two continuous maps $f,g : S^1\to X$ (X some topol. space)

Define: $f\# g : S^1 \to X$ by first mapping S^1 to the pair of circles, then mapping the top circle by f, bottom by g.

$$(f\# g)(\theta) = \begin{cases} f(2\theta), & 0\leq\theta\leq\pi \\ g(2(\theta-\pi)), & \pi\leq\theta\leq 2\pi \end{cases}$$

Need $f(0)=g(0)$ for continuity.



$(f\# g)(0) = f(0) = g(0)$
$= f(1,0) = g(1,0)$

If $f, g : S^1 \to S^1$, then $\deg f \# g = \deg f + \deg g$.

Proof: Lifts $\tilde{f}$ and $\tilde{g}$. $g(1,0) = f(1,0)$, $e(0) = (1,0)$, so arrange such that $\tilde{g}(0) = \tilde{f}(1)$ by adding integer to $\tilde{g}$.

$$\widetilde{f \# g}(t) = \begin{cases} \tilde{f}(2t) & t \leq 1/2 \\ \tilde{g}(2t-1) & t \geq \frac{1}{2} \end{cases}$$

$$\begin{aligned} \Rightarrow \widetilde{f\#g}(1) - \widetilde{f\#g}(0) &= \tilde{g}(1) - \tilde{f}(0) \\ &= \tilde{g}(1) - \tilde{g}(0) + \tilde{f}(1) - \tilde{f}(0) \\ &= \deg g + \deg f. \end{aligned}$$ ▨

Hence, $\deg(f \# g) = \deg(g \# f)$, which means $f \# g \simeq g \# f$. $(S^1 \to S^1)$

Think of $\#$ as addition operation in set of maps $S^1 \to S^1$.

Hence, $[S^1, S^1]$ with $\#$ is an Abelian group. (not always the case)

Next time: do this more generally.

Lecture 20: Homotopy Groups 3/21/11

Recall: defined "addition" on maps $S^1 \to X$.

$X = S^1$ + take homotopy classes: get Abelian group, $\mathbb{Z}$.

Now generalize this to any X (non-Abelian group in general)

First: recall $f(1,0) = g(1,0)$ (agree at point where circles "touch")

A pointed space is a topo. space X with a specific choice of point $x_0 \in X$, the base point.

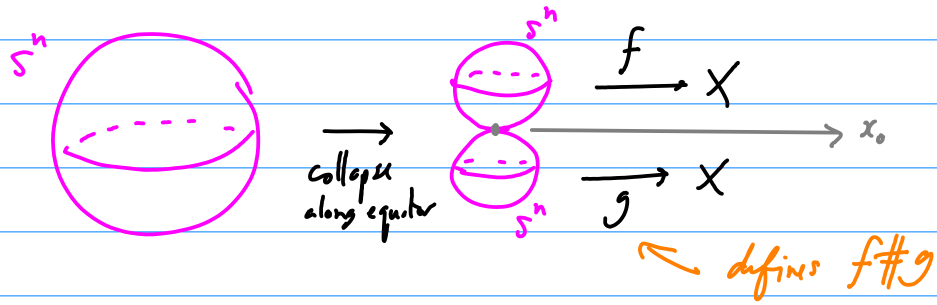
Write (X, x_0)

Then only consider pointed maps $f: (X, x_0) \to (Y, y_0)$ s.t. $f(x_0) = y_0$,
and
pointed homotopies $F: (X, x_0) \times I \to (Y, y_0)$ s.t. $F(x_0, t) = y_0$ $\forall t \in I$.

(For circle, we chose $x_0 = (1,0)$. f, g, $f \# g$ were all pointed maps.)

Hence, can define "addition" $\#$ for any pointed maps $S^1 \to X$.

In fact, can do this for pointed maps $S^n \to X$. $(n > 0)$



Make this more precise, but instead more convenient to use

$$S^n \cong I^n / \partial I^n$$ (example 5.51 in Crossley)

$$I^n = \{(x_1, \dots, x_n) \mid x_i \in I\}$$

$$\partial I^n = \{(x_1, \dots, x_n) \mid x_i \in I,\ x_i = 0 \text{ or } 1 \text{ for at least one } x_i\}$$

∂I^n gets collapsed to a point, so it is natural choice for x_0.

Hence, pointed maps $S^n \to X$ set ∂I^n to base point $x_0 \in X$.

A <u>topological pair</u> is (X, A) where X topo. sp. and A subspace of X.

$f : (X, A) \to (Y, B)$ is such that $f(A) \subset B$.

So a pointed map $f : (X, x_0) \to (Y, y_0)$ is $f : (X, \{x_0\}) \to (Y, \{y_0\})$.

(But keep writing x_0 for $\{x_0\}$)

Hence, with our def'n of the sphere above, pointed maps $S^n \to X$ become

$$f : (I^n, \partial I^n) \to (X, x_0), \quad \text{or} \quad (S_n, \partial S_n) \to (X, x_0)$$

note lower index! $S_n = I^n$

So now given $f, g : (S_n, \partial S_n) \to (X, x_0)$

$$(f \# g)(s_1, \dots, s_n) = \begin{cases} f(s_1, \dots, s_{n-1}, 2s_n), & s_n \le 1/2 \\ g(s_1, \dots, s_{n-1}, 2s_n - 1), & s_n \ge 1/2 \end{cases}$$

for $(s_1, \dots, s_n) \in S_n$.

If $s_n = 1/2$, $(s_1, \dots, s_{n-1}, 2s_n)$ and $(s_1, \dots, s_{n-1}, 2s_n - 1)$ both $\in \partial S_n$, so both map to x_0. $\Rightarrow$ $f \# g$ continuous by gluing lemma.

$\#$ is an "addition" operation for pointed maps $S^n \to X$.

Proposition: $f \simeq f'$ and $g \simeq g'$ by pointed homotopies

$$\Rightarrow \quad f \# g \simeq f' \# g' \text{ by a pointed homotopy.}$$

Proof. Homotopy given by $H_t = F_t \# G_t$. ∎

Write equivalence classes under pointed homotopy as $[f]$, set of classes by $\pi_n(X)$, and "addition" in $\pi_n(X)$ by:

$$[f] + [g] = [f \# g]$$

$\pi_n(X)$ is a group under $+$. Proof next time!

↳ n^{th} homotopy group of (X, x_0)

Lecture 21: Homotopy Groups (cont'd) 3/23/11

Let's demonstrate that $[f]+[g]=[f\#g]$ satisfies the group properties.

Prop: $c: S^n \to X$ constant map $(c(s_1,\dots,s_n)=x_0)$, $f: S^n \to X$.
Then $f\#c \simeq c\#f \simeq f$.

Proof: $f, c: (S_n, \partial S_n) \to (X, x_0)$. Show $c\#f \simeq f$.

Define $H: S_n \times I \to X$ by

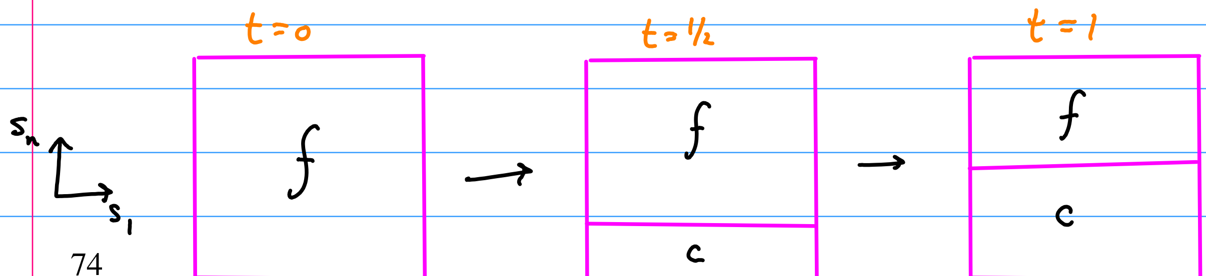
$$H((s_1,\dots,s_n),t) = \begin{cases} x_0 & s_n \leq t/2 \\ f\left(s_1,\dots,s_{n-1},\dfrac{2s_n-t}{2-t}\right) & s_n \geq t/2 \end{cases}$$

When $s_n = t/2$, $f\left(s_1,\dots,s_{n-1},\frac{2s_n-t}{2-t}\right) = f(s_1,\dots,s_{n-1},0) = x_0$, so continuous.

$$H((s_1,\dots,s_n),0) = f(s_1,\dots,s_{n-1},s_n)$$

$$H((s_1,\dots,s_n),1) = \begin{cases} c(s_1,\dots,s_n) \; (x_0) & s_n \leq 1/2 \\ f(s_1,\dots,s_{n-1},2s_n-1) & s_n \geq 1/2 \end{cases}$$

$$= c\#f(s_1,\dots,s_n)$$



Similarly, $f\#c \simeq f$ by

$$H((s_1,\dots,s_n),t) = \begin{cases} f\left(s_1,\dots,s_{n-1},\dfrac{2s_n}{2-t}\right) & s_n \leq 1-\frac{t}{2} \\ x_0 & s_n \geq 1-\frac{t}{2} \end{cases}$$

$\blacksquare$

So we have our identity element in $\pi_n(X,x_0)$.
Next we look for the inverse:

<u>Prop:</u> Given $f: S^n \to X$, $\exists\, \bar{f}: S^n \to X$ s.t.

$$f\#\bar{f} \simeq \bar{f}\#f \simeq c.$$

<u>Proof:</u> $f: (S_n, \partial S_n) \to (X, x_0)$.

Define: $\bar{f}(s_1,\dots,s_n) = f(s_1,\dots,s_{n-1},1-s_n)$

$f\#\bar{f} \simeq c$ by

$$H((s_1,\dots,s_n),t) = \begin{cases} (f\#\bar{f})\left(s_1,\dots,s_{n-1},\dfrac{1-t}{2}\right) & \dfrac{1-t}{2} \leq s_n \leq \dfrac{1+t}{2} \\ (f\#\bar{f})(s_1,\dots,s_{n-1},s_n) & \text{otherwise} \end{cases}$$

At $s_n = \dfrac{1-t}{2}$, obviously continuous

At $s_n = \dfrac{1+t}{2}$, need more work to show continuity:

$$(f\#\bar{f})\left(s_1,\dots,s_{n-1},\frac{1+t}{2}\right) = \bar{f}\left(s_1,\dots,s_{n-1},\overbrace{2\left(\tfrac{1+t}{2}\right)-1}^{t}\right)$$

$\in [\frac{1}{2},1]$, so use $2s_n-1$

$$= f(s_1,\dots,s_{n-1},1-t) \quad \text{by def'n of } \bar{f}$$

$$= (f\#\bar{f})\left(s_1,\dots,s_{n-1},\frac{1-t}{2}\right) \quad \in [0,\tfrac{1}{2}]$$

When $t=0$, H gives $f \# \bar{f}$, and maps to x_0 when $t=1$ (const. map).
Hence,

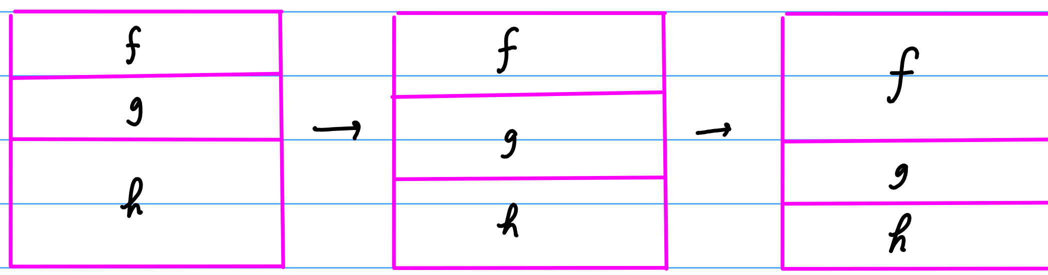
$$f \# \bar{f} \simeq c.$$

Note that $\bar{\bar{f}} = f$, so $\bar{f} \# \bar{\bar{f}} \simeq c$, or $\bar{f} \# f \simeq c$. $\blacksquare$

Now for associativity:

Prop: $f, g, h : S^n \to X$, then $(f \# g) \# h \simeq f \# (g \# h)$.

Proof: (omit explicit form, see Crossley p.133)



And we're done! $\pi_n(X, x_0)$, $n > 0$, is a group with $[f]+[g]=[f \# g]$.

Example: $X \subset \mathbb{R}^n$ convex. Then any pointed maps $f: S^n \to (X, x_0)$ is homotopic to the constant map by

$$H(x,t) = t f(x) + (1-t) x_0 \quad , \quad x \in S^n$$

Respects base: $H((1,0,\dots,0),t) = t \underbrace{f(1,0,\dots,0)}_{x_0} + (1-t)x_0 = x_0$

Hence, $\pi_n(X) = \{0\}$.

$f \simeq c$ is called null-homotopic.

Any map from S^n to a convex subset of $\mathbb{R}^n$ is null-homotopic.

Example: We know already that $\pi_1(S^1) = \mathbb{Z}$.

What about $\pi_2(S^1)$? Take $f : (S_2, \partial S_2) \to (S^1, \overset{x_0}{\widetilde{(1,0)}})$.

Lift to map $\tilde{f} : S_2 \to \mathbb{R}$ by homotopy lifting (Prop. 6.28)

$f(s_1,s_2) = (1,0)$ whenever $(s_1,s_2) \in \partial S_2$, so $\tilde{f}(s_1,s_2) \in \mathbb{Z}$ in that case.

Boundary connected, so $\tilde{f}$ constant on boundary (n=1 different here! ∂S_1 not connected)

$\mathbb{R}$ convex, so define homotopy $\tilde{F} : S_2 \times I \to \mathbb{R}$ by

$$\tilde{F}(s_1,s_2,t) = t\,\tilde{f}(s_1,s_2) + (1-t)\,\tilde{f}(0,0)$$

Compose with e to get $F = e \circ \tilde{F} : S_2 \times I \to S^1$

$t=0$: $F(s_1,s_2,0) = e(\tilde{f}(0,0)) = \text{const.}$

$t=1$: $F(s_1,s_2,1) = e(\tilde{f}(s_1,s_2)) = f(s_1,s_2)$.

For $(s_1,s_2) \in \partial S_2$, $F(s_1,s_2,t) = e(t\,\underbrace{\tilde{f}(s_1,s_2)}_{\text{same as } \tilde{f}(0,0)} + (1-t)\,\tilde{f}(0,0))$

$$= e(\tilde{f}(0,0)) = f(\underbrace{0,0}_{\in \partial S_2}) = x_0$$

Hence $f \simeq$ const. map.

Hence, all maps $S^2 \to S^1$ are null-homotopic, and $\pi_2(S^1) = \{0\}$.

The same argument holds for $\pi_i(S^1)$, $i \geq 2$.

Hence $\pi_i(S^1)$ is nontrivial only for $i = 1$.

$\Rightarrow$ Eilenberg-MacLane space.

Lecture 22: Induced homomorphisms 3/25/11

More about $\pi_n(X)$:

We saw: $\pi_n(S^1) = \{0\}, \quad n > 1$

Can show (not so easy): $\pi_i(S^n) = 0, \quad i < n$ (n-1)-connected

$\pi_n(S^n) = \mathbb{Z}$

$\pi_i(S^n) = ?, \quad i > n$ not generally known

Helpful Lemma: $\pi_3(S^2) = \mathbb{Z}$ (Hopf)

If X and Y are two pointed spaces, then $\pi_n(X \times Y) = \pi_n(X) \times \pi_n(Y)$.

Proof: $f: S^n \to X \times Y$ corresponds to pair $f_X: S^n \to X$, $f_Y: S^n \to Y$.

Suppose $g: S^n \to X \times Y$ is another map, (with g_X, g_Y)

with $F: S^n \times I \to X \times Y$ homotopy between f and g.

$\hookrightarrow F_X: S^n \times I \to X$, $F_Y: S^n \times I \to Y$ homotopies between f_X & g_X, f_Y & g_Y.

Can also do the converse, i.e., construct F given F_X, F_Y.

Pointed homotopies ok, and "addition" preserved. $\blacksquare$

examples: $C = S^1 \times I$ cylinder has $\pi_n(C) = \pi_n(S^1) \times \pi_n(I) = \pi_n(S^1)$

$T^2 = S^1 \times S^1$ has $\pi_n(T^2) = \pi_n(S^1) \times \pi_n(S^1)$

Definition: Given pointed continuous $f: X \to Y$, there is an induced function $f_*: \pi_n(X) \to \pi_n(Y)$

homomorphism, in fact

$$f_*[j] = [f \circ j] \quad \text{for any pointed cont. } j: S^n \to X.$$

Need to verify that f is well-defined: if $[j]=[k]$, then $f_*[j] = f_*[k]$. Ok since $f \circ k \simeq f \circ j$.

Theorem: f_* is a group homomorphism. (induced homomorphism)

1. $(g \circ f)_* = g_* \circ f_*$

2. $i: X \to X$ identity $\Rightarrow$ i_* identity.

3. $h_* = f_*$ if $[h] = [f]$ (well-defined: ok since $h \circ j \simeq f \circ j$)

4. $c: X \to Y$ takes X to base point of Y $\Rightarrow$ $c_* = 0$.

Proof: $j_1, j_2: S^n \to X$ pointed maps. $f_*([j_1]+[j_2]) = f_*([j_1 \# j_2])$

$(S_n, \partial S_n) \to X$

$$= [f \circ (j_1 \# j_2)]$$

$$f \circ (j_1 \# j_2)(s_1, \ldots, s_n) = \begin{cases} f(j_1(s_1, \ldots, s_{n-1}, 2s_n - 1)) & s_n \geq 1/2 \\ f(j_2(s_1, \ldots, s_{n-1}, 2s_n)) & s_n \leq 1/2 \end{cases}$$

$$= (f \circ j_1) \# (f \circ j_2)$$

$$\Rightarrow [f\circ(j_1 \# j_2)] = [(f\circ j_1)\#(f\circ j_2)]$$
$$= [f\circ j_1] + [f\circ j_2]$$
$$= f_*(j_1) + f_*(j_2)$$

1. $(g\circ f)_*[j] = [g\circ f\circ j]$

$(g_*\circ f_*)[j] = g_*(f_*[j]) = g_*([f\circ j]) = [g\circ f\circ j]$

2. $i_*[j] = [i\circ j] = [j]$

4. $c_*[j] = [\underbrace{c\circ j}_{\text{base point}}] = 0$

Now we have the power to easily prove some things

Prop: No continuous map $f: D^2 \to S^1$ s.t. $f(x,y) = (x,y)\ \forall (x,y)\in S^1$!

Proof: Assume f exists, define inclusion $i: S^1 \to D^2$.

$$S^1 \xrightarrow{i} D^2 \xrightarrow{f} S^1$$

$$\pi_1(S^1) \xrightarrow{i_*} \pi_1(D^2) \xrightarrow{f_*} \pi_1(S^1)$$

$$\mathbb{Z} \xrightarrow{i_*} 0 \xrightarrow{f_*} \mathbb{Z}$$

note: inclusion map (injective) induces zero homomorphism

$f\circ i$ identity on S^1, so $(f\circ i)_*$ id. in $\mathbb{Z}$. Oops! Contradiction. ▨

Prop: If S and T are pointed homotopy equivalent, then $\pi_n(S)$ and $\pi_n(T)$ are isomorphic.

Proof: $f: S \to T$ and $g: T \to S$ s.t. $f \circ g \simeq id_T$, $g \circ f \simeq id_S$.

$\Rightarrow$ $f_* \circ g_* = g_* \circ f_* = id$ invertible, so isomorphism. $\blacksquare$

Hence, both homeomorphic and pointed homotopy equivalent spaces have the same homotopy groups.

example: $\mathbb{R}^2 - \{0\} \simeq S^1$, so $\pi_i(\mathbb{R}^2 - \{0\}) = 0 \quad i \neq 1$

$\pi_1(\mathbb{R}^2 - \{0\}) = \mathbb{Z}$

Theorem: $\pi_n(X)$ is Abelian for $n > 1$!

Lecture 23: Mapping Class Group of the Torus — 3/30/11

Consider $f, g : T^2 \to T^2$ homeomorphisms.

f and g are isotopic if they are homotopic, but the homotopy between them is also an isotopy: $F(x,t)$ is a homeomorphism for $0 \leq t \leq 1$.

If f, g are orientation-preserving, then homotopy $\Leftrightarrow$ isotopy. (for orientable surfaces)

(Baer)

The set of homeomorphisms from $T^2 \to T^2$ is denoted $\mathrm{Homeo}(T^2)$.

Orientation-preserving: $\mathrm{Homeo}^+(T^2)$.

$\mathrm{Homeo}(T^2)$ and $\mathrm{Homeo}^+(T^2)$ are groups under composition of functions.

Define:

$$\mathrm{MCG}(T^2) = \mathrm{Homeo}^+(T^2) / \text{isotopy}$$

Mapping Class Group of T^2

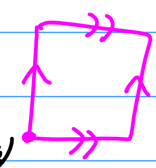
(inherits the group structure of $\mathrm{Homeo}^+(T^2)$)

assume they have a fixed point $\mathrm{Homeo}^+(T^2, x_0)$

What does $\mathrm{MCG}(T^2)$ look like?

Consider an induced homomorphism on $\pi_1(T^2, x_0)$

↳ $(0,0)$



$$f : T^2 \to T^2, \qquad f_* : \underbrace{\pi_1(T^2)}_{\mathbb{Z}^2} \to \pi_1(T^2)$$

<u>linear</u>

$f_*([f]+[g]) = f_*[f] + f_*[g]$

Hence, f_* given by matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad a,b,c,d \in \mathbb{Z}$$

$\underbrace{\in GL(2,\mathbb{Z})}$

But also $f \circ f^{-1} = id \Rightarrow f_* \circ f_*^{-1} = I$ so f_* invertible.

$ad - bc \neq 0$

Let $m = ad - bc \neq 0$. We have also $f_*^{-1} : \mathbb{Z}^2 \to \mathbb{Z}^2$, so

$$f_*^{-1} : \quad \frac{1}{m}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

, so need all entries $\in \mathbb{Z}$.

↳ m divides every entry

Let $a = m\alpha$, $b = m\beta$, $c = m\gamma$, $d = m\delta$, $\alpha, \beta, \gamma, \delta$ integers.

Then $m = ad - bc = m^2(\alpha\delta - \beta\gamma) \Rightarrow 1 = m(\alpha\delta - \beta\gamma)$.

Since all integers, need $m = \pm 1$. $m = +1 \Rightarrow$ orientable.

Hence, $\boxed{MCG(T^2) = SL(2,\mathbb{Z})}$ Why is this?

Now, how do we classify the elements of this group?

Look at eigenvalues. $\det\left[\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{M} - xI\right] = x^2 - (a+d)x + \underbrace{ad-bc}_{1}$

$(= f_*)$

Let $\tau = a+d$ (trace)

note: $p(M) = M^2 - \tau M + I = 0$
Cayley-Hamilton thm

Characteristic polynomial: $p(x) = x^2 - \tau x + 1$

Eigenvalues: $x = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4}\right)$ So $|\tau| = 2$ important.

Let's examine different cases.

1) $|\tau| < 2$. $\tau = -1, 0, 1$.

If $\tau = 0$, then $p(M) = M^2 + I = 0 \Rightarrow M^2 = -I \Rightarrow \boxed{M^4 = I}$

If $\tau = \pm 1$, $p(M) = M^2 \mp M + I \Rightarrow M^2 = \pm M - I$

$$\begin{aligned} M^3 &= M(\pm M - I) \\ &= \pm M^2 - M \\ &= \pm(\pm M - I) - M \\ &= \mp I \end{aligned}$$

$\boxed{M^6 = I}$

Either way, we can write

$$\boxed{M^{12} = I, \quad |\tau| < 2}$$

This is called finite-order. After applying f enough times, it is isotopic to the identity map.

2) $|\tau| = 2$: Then eigenvalues are both ± 1 $(= \tau/2)$

$$M^2 \mp 2M + I = (M \mp I)^2 = 0 \Rightarrow M = \pm I + N, \quad N^2 = 0$$

$\begin{pmatrix} a - \tau/2 \\ c \end{pmatrix}$ is the eigenvector:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a-\tau/2 \\ c \end{pmatrix} = \begin{pmatrix} a(a-\frac{\tau}{2})+bc \\ c(a-\frac{\tau}{2})+cd \end{pmatrix} = \begin{pmatrix} a(a-\frac{\tau}{2}) + \overbrace{(bc-ad)}^{-1} + ad \\ c\underbrace{(a+d}_{\tau}-\tau/2) \end{pmatrix}$$

$$= \begin{pmatrix} a(\tau/2)-1 \\ c\tau/2 \end{pmatrix} = \frac{\tau}{2}\begin{pmatrix} a-\tau/2 \\ c \end{pmatrix} \qquad \text{since } 2/\tau = \tau/2$$

$(\tau/2 = \pm 1)$

Hence, the homotopy classes given by $\begin{pmatrix} a-\tau/2 \\ c \end{pmatrix}$ are invariant (or reverse direction) under M.

$\Rightarrow$ invariant curve (called reducible)

Let $R = \begin{pmatrix} 1 & a-\tau/2 \\ 0 & c \end{pmatrix}$. Then:

$$R^{-1}MR = \begin{pmatrix} \tau/2 & 0 \\ 1 & \tau/2 \end{pmatrix} \qquad \text{Jordan form}$$

other loops not invariant

Simplest type: $M = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, so $M\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $M\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Next time: case 3) $|\tau| > 2$!

Lecture 24: Anosov homeomorphisms 4/01/11

Recall $f: T^2 \to T^2$ homeomorphism (orientation-preserving)

$f_*: \pi_1(T^2) \to \pi_1(T^2)$ $\qquad f_* \in SL(2,\mathbb{Z})$

$f_* = M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc = 1$

$p(x) = x^2 - \tau x + 1$ characteristic polynomial with $\tau = a+d$ = trace

$x = \frac{1}{2}(\tau \pm \sqrt{\tau^2 - 4})$

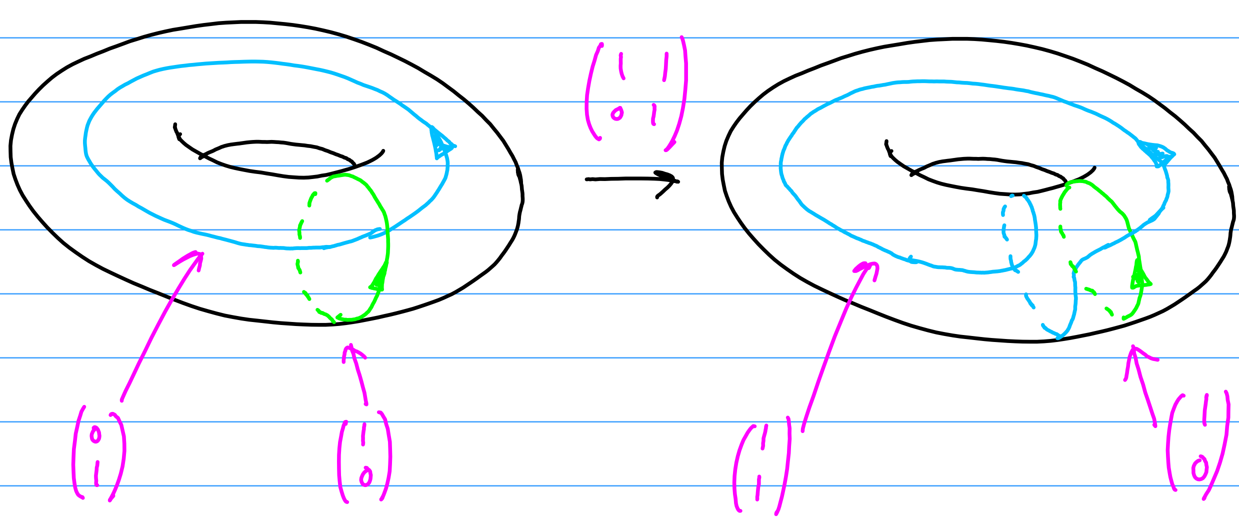
So far: 1) $|\tau| < 2 \Rightarrow M^{12} = I$ finite-order

2) $|\tau| = 2 \Rightarrow M = \pm I$ (also finite-order)

or $Mv = \pm v$ for some $v \in \pi_1(T^2)$

$\Rightarrow$ invariant loop (reducible)

example:



That is: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ leaves the loop $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ invariant.

Only one possibility left.

3) $\underline{|\tau| > 2}$: In that case we get two distinct real roots:

$$x_\pm = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2-4}\right), \quad \text{with} \quad x_+ x_- = 1$$

The roots are inverse of each other, and we define

$$\lambda = \max\left(|x_+|, |x_-|\right) > 1.$$

The eigenvectors of M are $\qquad \pm = \operatorname{sign}(\tau)$

$$u = \begin{pmatrix} \pm\lambda - d \\ c \end{pmatrix}, \quad s = \begin{pmatrix} \pm\lambda^{-1} - d \\ c \end{pmatrix}, \quad \lambda > 1$$

Check: (with $\tau > 2$)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \lambda - d \\ c \end{pmatrix} = \begin{pmatrix} a\lambda \underbrace{- ad + bc}_{-1} \\ c\lambda - \cancel{cd} + \cancel{cd} \end{pmatrix} = \begin{pmatrix} \underbrace{(-\lambda^2 + (a+d)\lambda - 1)}_{0} + \lambda^2 - \lambda d \\ c\lambda \end{pmatrix}$$

$$= \lambda \begin{pmatrix} \lambda - d \\ c \end{pmatrix}.$$

Claim: λ is irrational $\leftarrow$ There are easier ways, but this is nice...

Assume $\boxed{\tau > 2}$ (positive): $\leftarrow$ $\tau < -2$ of course has nearly identical proof.

$$\lambda = \frac{1}{2}\left(\tau + \sqrt{\tau^2-4}\right) = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots}}}$$

$$\lambda^2 - \tau\lambda + 1 = 0 \iff \lambda - \tau + \lambda^{-1} = 0$$

$$\lambda = \underbrace{\tau - 1}_{\substack{\text{Positive} \\ \text{integer}}} + \underbrace{(1 - \lambda^{-1})}_{\substack{0 < 1 - \lambda^{-1} < 1 \\ \text{Fractional part}}} = a_0 + (1 - \lambda^{-1})$$

$$\therefore \quad a_0 = \tau - 1$$

$$\lambda - a_0 = 1 - \lambda^{-1} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots}}}$$

$$\frac{1}{1-\lambda^{-1}} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots}}$$

ϕ = Golden ratio

$$1 - \lambda^{-1} = 1 - (\tau - \lambda) \quad \text{since} \quad \lambda^2 - \tau\lambda + 1 = 0$$
$$= (1-\tau) + \lambda$$

$\underbrace{\phi^2 !}$

Since $\lambda = \frac{1}{2}(\tau + \sqrt{\tau^2 - 4})$ is monotonic, $\min \lambda = \frac{1}{2}(3 + \sqrt{5})$ for $|\tau| = 3$,
$= 2.61803\ldots$

$\max \lambda^{-1} = (\min \lambda)^{-1}$
$= \frac{1}{2}(3 - \sqrt{5})$

Hence, $1 < \dfrac{1}{1-\lambda^{-1}} \leq \dfrac{1}{1 - \max \lambda^{-1}}$

$$= \frac{1}{\frac{1}{2}(\sqrt{5}-1)}$$

$$= \frac{2(\sqrt{5}+1)}{4} = \frac{1}{2}(1+\sqrt{5}) < 2.$$

$\therefore\ 1 < \dfrac{1}{1-\lambda^{-1}} < 2.$ Conclude: $a_1 = 1.$

$$\left(\frac{1}{1-\lambda^{-1}} - 1\right)^{-1} = a_2 + \cfrac{1}{a_3 + \cfrac{1}{\cdots}}$$

$$\left(\frac{1}{1-\lambda^{-1}} - 1\right)^{-1} = \left(\frac{\cancel{1} - (\cancel{1} - \lambda^{-1})}{1-\lambda^{-1}}\right)^{-1} = \left(\frac{1}{\lambda - 1}\right)^{-1} = \lambda - 1.$$

$$\lambda - 1 = a_2 + \cfrac{1}{a_3 + \cfrac{1}{\cdots}} \iff \lambda = \underbrace{(a_2+1)}_{a_0 = \tau - 1} + \cfrac{1}{a_3 + \cfrac{1}{\cdots}}$$

Same expression as when we started!

$\therefore\ a_2 = \tau - 2.$

One more time:

$$\lambda^2 - \tau\lambda + 1 = 0$$
$$\lambda - \tau + \lambda^{-1} = 0$$
$$\lambda - \tau + 2 = 2 - \lambda^{-1}$$

$$(\lambda - 1 - a_2)^{-1} = a_3 + \cfrac{1}{a_4 + \cfrac{1}{\cdots}}$$

$$(\lambda - 1 - a_2)^{-1} = (\lambda - \tau + 1)^{-1} = (1 - \lambda^{-1})^{-1}$$

$$\frac{1}{1 - \lambda^{-1}} = a_3 + \cfrac{1}{a_4 + \cfrac{1}{\cdots}} \quad \therefore \quad \begin{aligned} a_3 &= a_1 \\ a_4 &= a_2 \\ a_5 &= a_1 \\ a_6 &= a_2 \ \ldots \end{aligned}$$

We've seen this before!

Periodic!

Continued fraction representation:

$$\lambda = [\tau - 1; \underbrace{1, \tau - 2}_{\text{period-2}}, 1, \tau - 2, \ldots]$$

Of course, this shouldn't have surprised us: the irrational solutions of quadratic equations with integer coefficients are periodic. But this polynomial has a particularly simple form

More importantly, this shows that λ is irrational for any $|\tau| > 2$ (negative τ proceeds identically)

$[f]$ is the isotopy class of an Anosov homeomorphism with dilatation $\lambda > 1$.

Lecture 25: From the torus to the sphere 4/04/11

Recall: $f: T^2 \to T^2$ homeomorphism.

Specific map in an isotopy class: $f(x,y) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ mod 1

(Note: $f(0,0) = (0,0)$)

$ad - bc = 1$

If $|\text{trace}| > 2$, we have an Anosov homeomorphism.
These are "complex", in the sense that their action on $\pi_1(T^2)$ gives exponential growth in the number of twists.

Now we want to relate this to something more "physical": sphere (eventually disk)

Consider the map: $\iota: T^2 \to T^2$ $\quad \iota\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$ mod 1

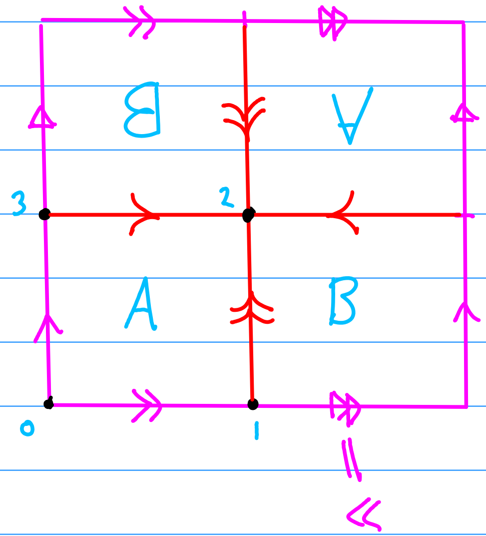
($\iota^2 = \text{id}$ involution)

Fixed points:
$$\begin{aligned} x &= -x \mod 1 \Rightarrow x = -x + n \Rightarrow 2x = n \Rightarrow x = 0, \tfrac{1}{2} \\ y &= -y \mod 1 \quad\;\; y = -y + m \quad\;\; 2y = m \quad\; y = 0, \tfrac{1}{2} \end{aligned}$$

Hence, ι has fixed points $(0,0)$, $(\tfrac{1}{2},0)$, $(0,\tfrac{1}{2})$, $(\tfrac{1}{2},\tfrac{1}{2})$

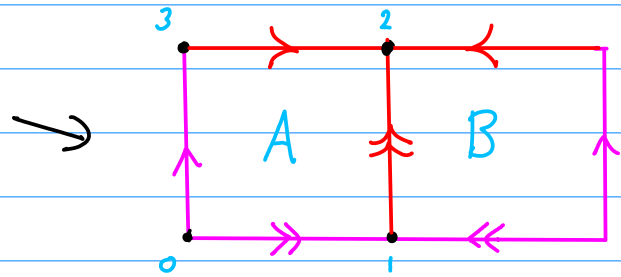
Claim: $T^2/\iota \cong S^2$ (each with 4 points removed)

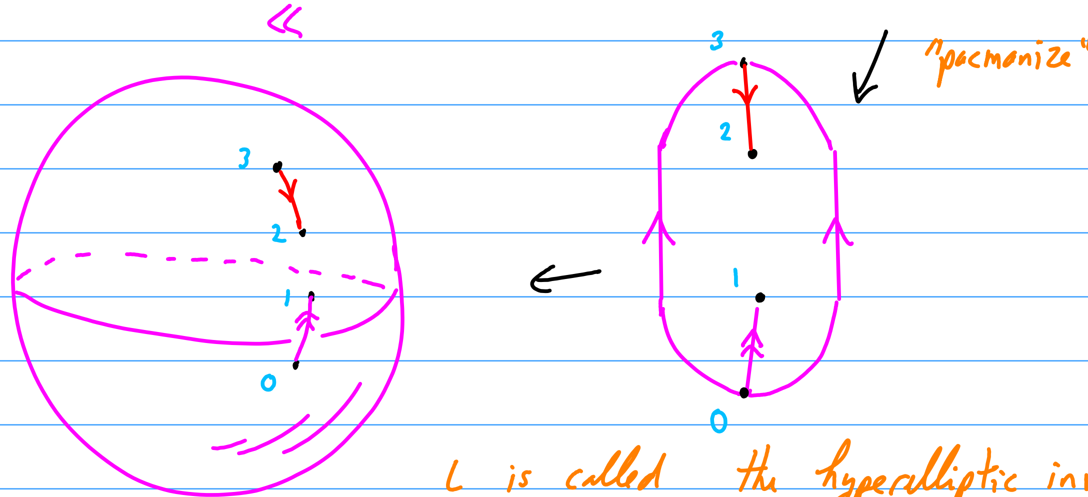
We show this by making the appropriate identifications.



The line $(x, \frac{1}{2}) \to (-x, -\frac{1}{2}) = (1-x, \frac{1}{2})$

The line $(\frac{1}{2}, y) \to (-\frac{1}{2}, -y) = (\frac{1}{2}, 1-y)$



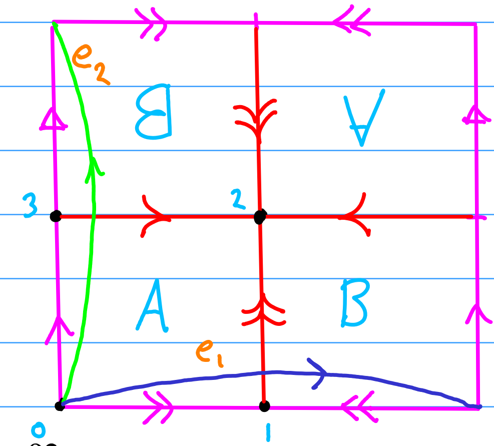


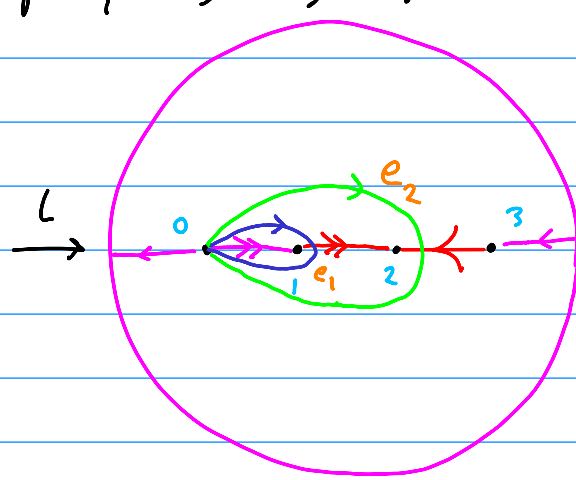
ι is called the hyperelliptic involution

The 4 points are the Weierstrass points.

We say that the torus (with 4 punctures) is a branched double cover of the sphere (with 4 punctures).

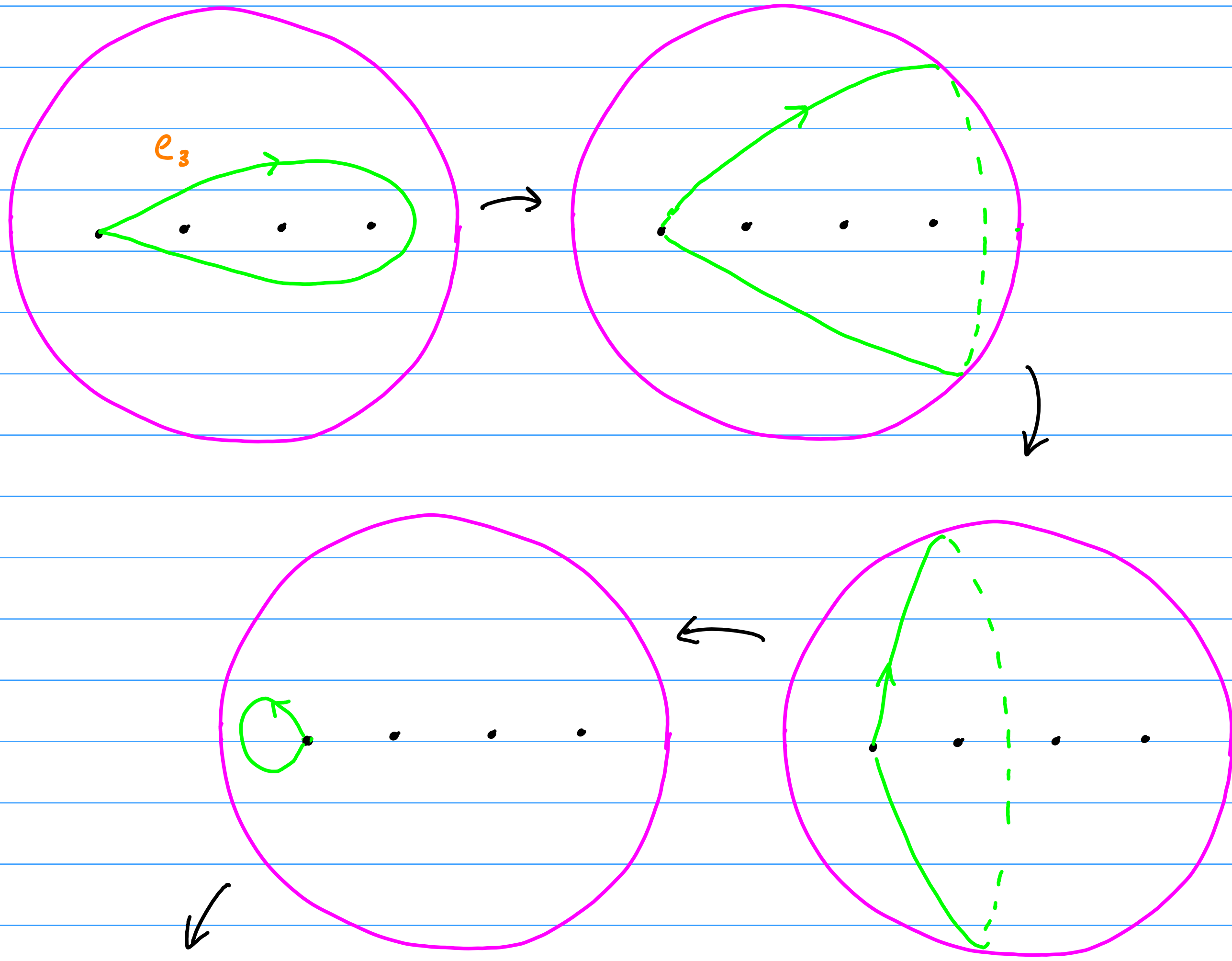
What does an element of $\pi_1(T^2, (0,0))$ map to?

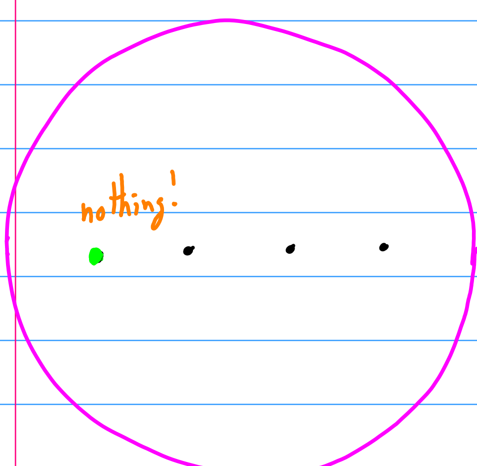




These are loops on the sphere!

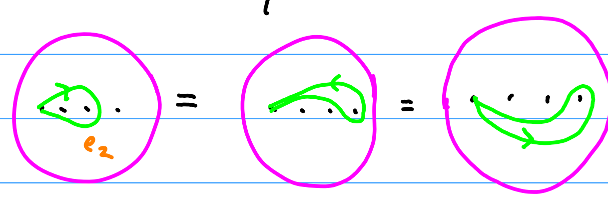
Are we missing a loop in $S^2 - 4$ punct., around puncture 3?



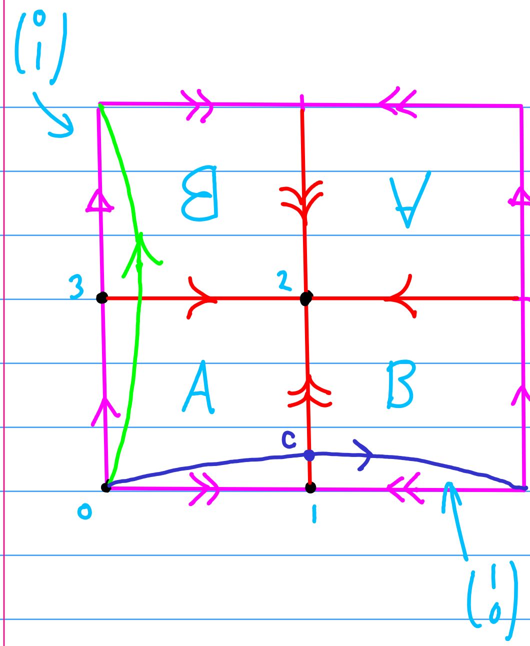


Hence, e_3 = trivial loop
= id in π_1.

Note also:

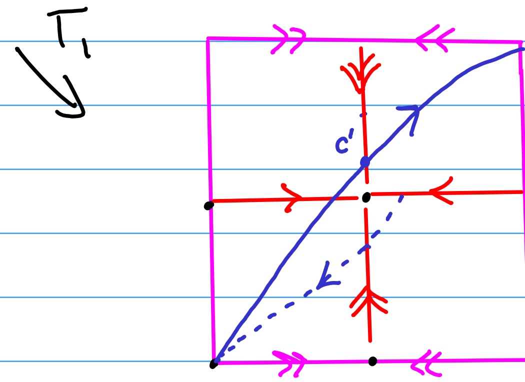


Now consider $T_1(x,y) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \bmod 1.$



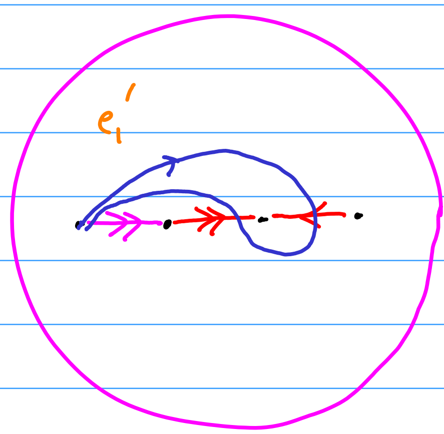
$T_1\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}, \quad T_1\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$

$c' = \begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1/2\\ \varepsilon\end{pmatrix}=\begin{pmatrix}1/2\\ 1/2+\varepsilon\end{pmatrix}$

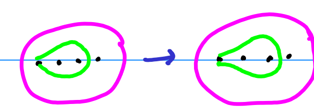


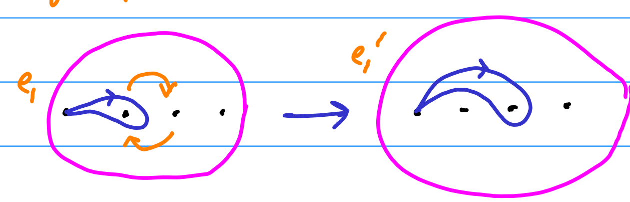
$$T_1\underset{1}{\begin{pmatrix}1/2\\0\end{pmatrix}}=\underset{2}{\begin{pmatrix}1/2\\1/2\end{pmatrix}}, \qquad T_1\underset{2}{\begin{pmatrix}1/2\\1/2\end{pmatrix}}=\begin{pmatrix}1/2\\1\end{pmatrix}=\underset{1}{\begin{pmatrix}1/2\\0\end{pmatrix}}, \qquad T_1\underset{3}{\begin{pmatrix}0\\1/2\end{pmatrix}}=\begin{pmatrix}0\\1/2\end{pmatrix}=\underset{3}{\begin{pmatrix}0\\1/2\end{pmatrix}}$$

T_1 fixes 0, 3, interchanges 1 and 2.



← image of e_1

(e_2 unchanged: )



Exactly like swapping 1 & 2 clockwise!

Note: Cannot write as e_1, e_2 by dragging, because of the punctures.

T_1^{-1} swaps counterclockwise.

(Can make this clearer by considering homotopy:

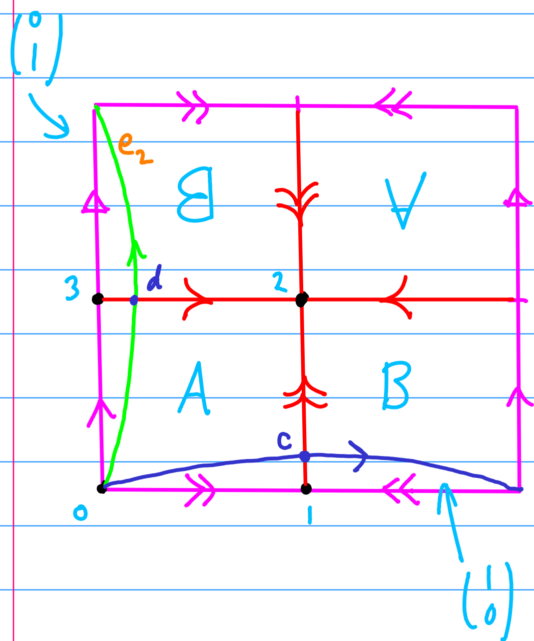
$$T_1(x,y,t)=\begin{pmatrix}1&t\\0&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} \bmod 1, \quad 0\le t\le 1)$$

Why doesn't it lead to isotopy?
→ not invertible on torus

Also define: $T_2(x,y) = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} \bmod 1.$

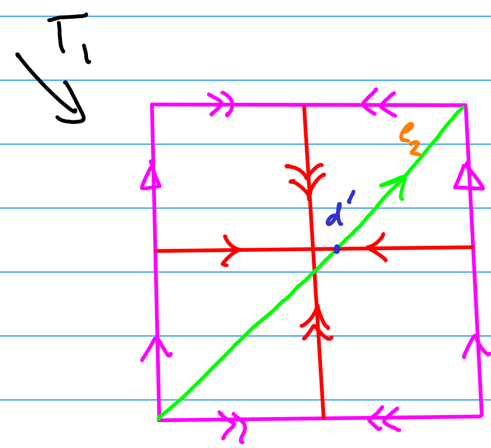
$$T_2\underset{1}{\begin{pmatrix}1/2\\ 0\end{pmatrix}} = \underset{1}{\begin{pmatrix}1/2\\ 0\end{pmatrix}}, \quad T_2\underset{2}{\begin{pmatrix}1/2\\ 1/2\end{pmatrix}} = \begin{pmatrix}1\\ 1/2\end{pmatrix} = \underset{3}{\begin{pmatrix}0\\ 1/2\end{pmatrix}}, \quad T_2\underset{3}{\begin{pmatrix}0\\ 1/2\end{pmatrix}} = \underset{2}{\begin{pmatrix}1/2\\ 1/2\end{pmatrix}}$$

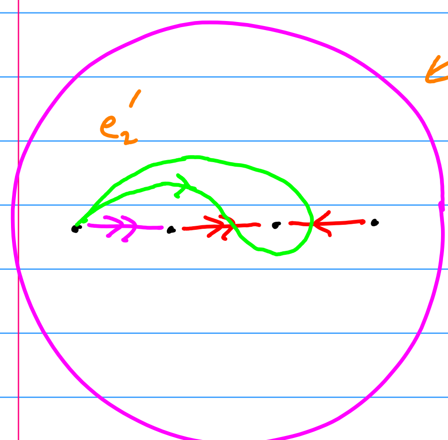
T_2 fixes 0, 1, interchanges 2 and 3.



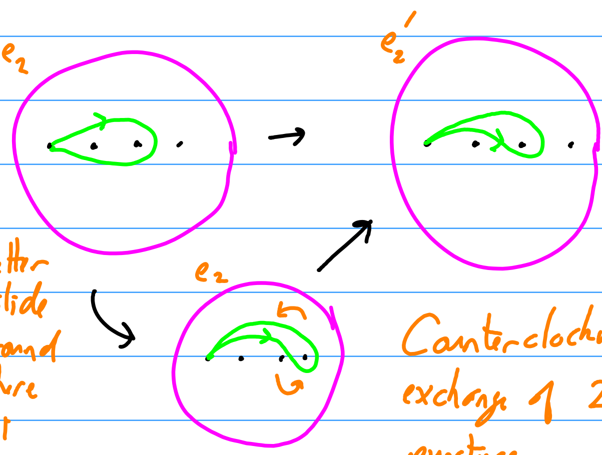
$T_2\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}, \quad T_1\begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}1\\ 1\end{pmatrix}$

$d' = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}\varepsilon\\ 1/2\end{pmatrix} = \begin{pmatrix}\varepsilon\\ 1/2+\varepsilon\end{pmatrix}$



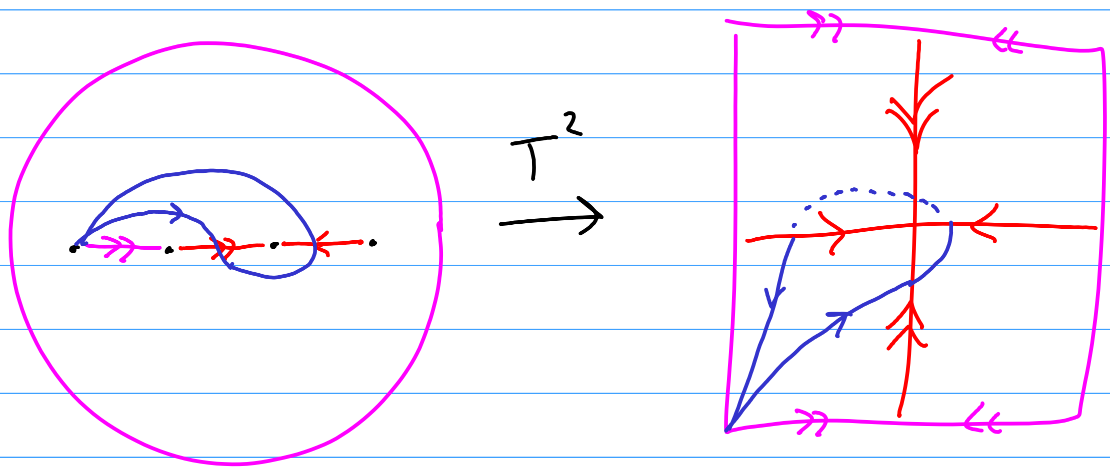


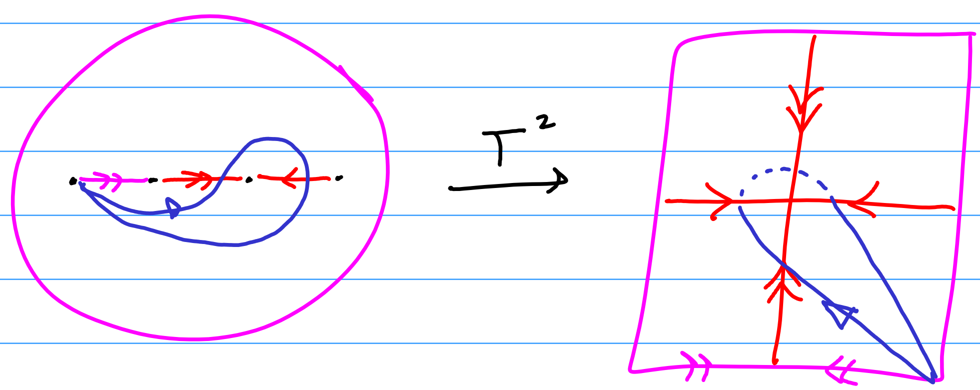
← image of e_2 (e_1 unchanged) under T_2



better slide around sphere first

Counterclockwise exchange of 2nd & 3rd puncture.


T²


T²

Lecture 26: Topological stirring

4/06/11

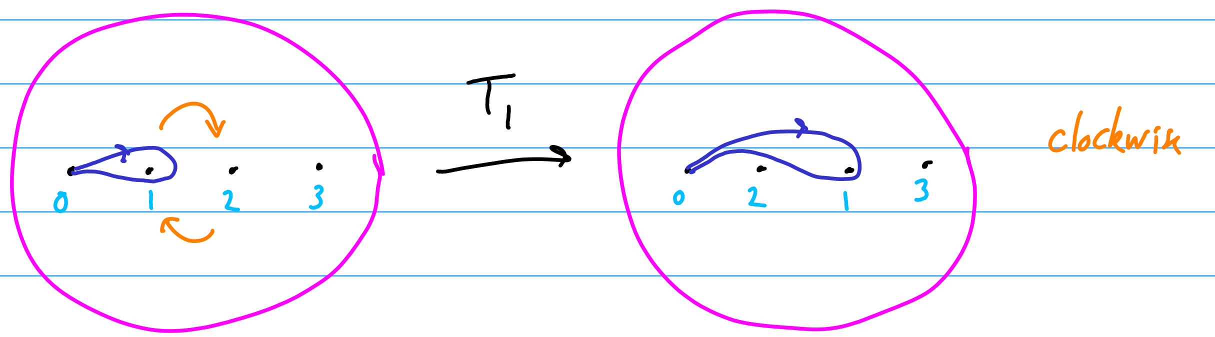
Last time: $T^2/\iota \cong S^2$ ι = hyperelliptic involution (4 branch points)

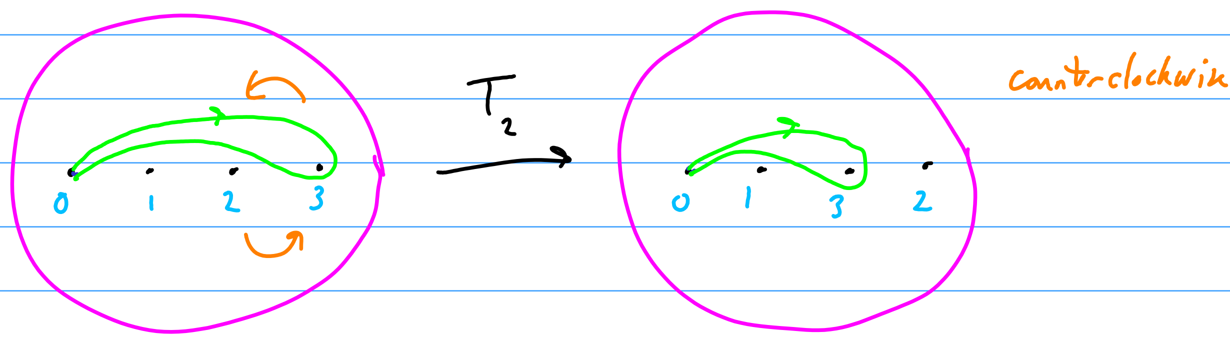
Two special mappings:

$$T_1\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} \bmod 1 \qquad T^2\to T^2$$

$$T_2\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} \bmod 1$$

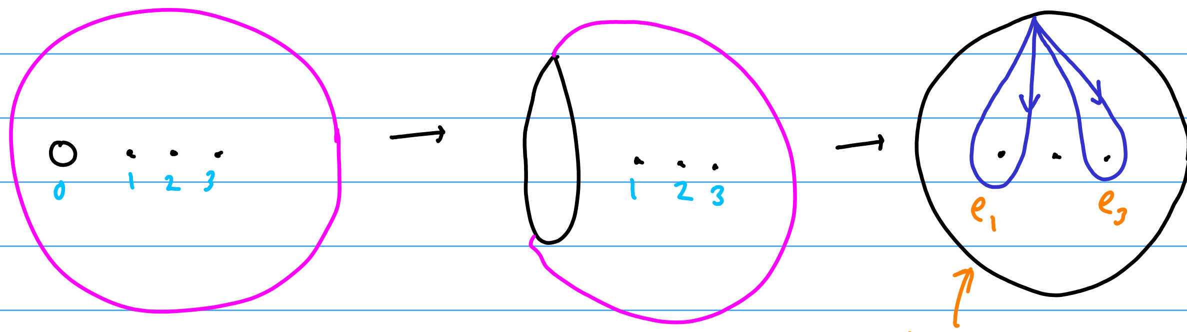
When projected down to the sphere, these can be interpreted as maps that "exchange" punctures $1\leftrightarrow 2$ or $2\leftrightarrow 3$.





T_1 and T_2 generate the mapping class group of the sphere with 4 punctures, with one puncture fixed.

Now take out a disk at puncture 0:



boundary of a disk with 3 punctures, not sphere.

Now we can connect homeos on T^2 to motion of points (or rods) in a two-dimensional domain.

$\Rightarrow$ stirring a fluid. [movie]

Movie 1: (boyland 1)

$$T_1 T_2^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

Trace < 2, so expect some power = id. (recall classification)

$$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}^4 = \text{id}$$

Movie 2: (boyland 2)

$$T_1^{-1} T_2^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

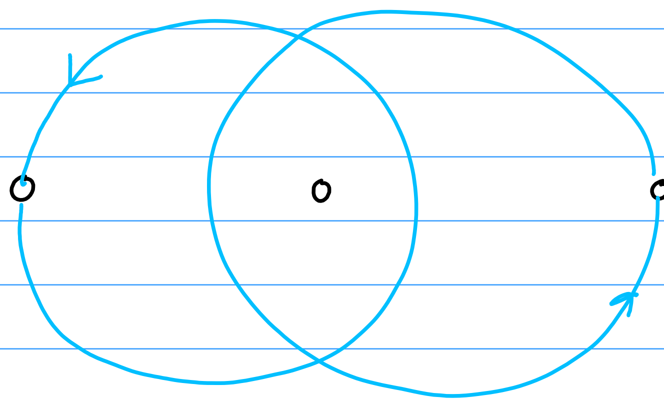
Trace = 3 $\Rightarrow$ Anosov on the torus (pseudo-Anosov on disk)

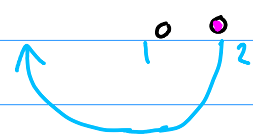
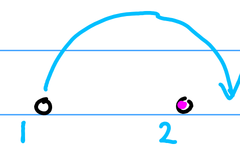
Dilatation $\lambda = \dfrac{\tau + \sqrt{\tau^2 - 4}}{2} = \dfrac{3+\sqrt{5}}{2} = \varphi^2$

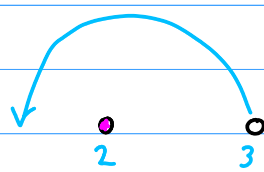
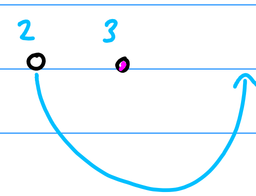
$\varphi = \frac{1}{2}(1+\sqrt{5})$
= Golden ratio

This tells you something about how "entangled" the fluid motion is.

Taffy puller:



Look at movie:  T_1 then  T_1 again

Other side:  T_2 then  T_2 again

Hence, after all the rods return to same initial configuration, we've done

$$T_1^2 T_2^2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^2 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$$

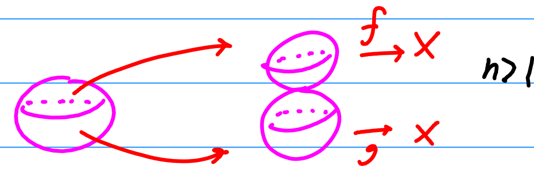
$$\lambda = \frac{1}{2}\left(\tau + \sqrt{\tau^2 - 4}\right) = \frac{1}{2}\left(6 + \sqrt{32}\right)$$

$$\lambda = 3 + 2\sqrt{2} = (1+\sqrt{2})^2$$ Silver ratio!

Lecture 27: Path connectivity and π_0 — 4/08/11

$\pi_0(X) = [S^0, X]$ Equivalence classes of mappings from $S^0 \to X$.

Recall: $f \# g$ defined by gluing S^n:

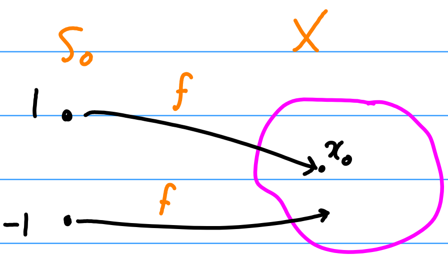


However, no way to do this for $S^0 = \{-1, 1\}$

Hence, π_0 does not have a group structure (except maybe a trivial one).

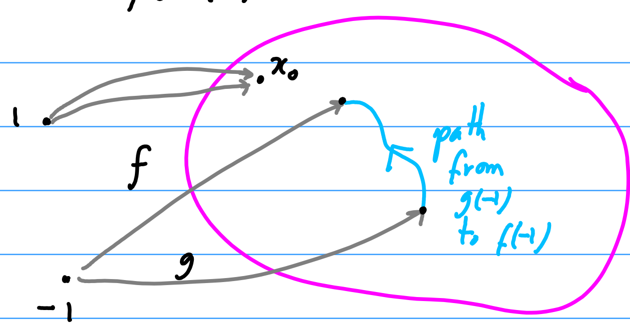
$f : S^0 \to X$:

↓
pointed



Only get to choose $f(-1)$, since $f(1) = x_0$ = base pt.

For $g : S^0 \to X$ to be homotopic to f, we need to be able to displace $g(-1)$ to $f(-1)$.



X is path-connected if, given any $x, y \in X$, there is a continuous map $p : [0,1] \to X$ s.t. $p(0) = x$ and $p(1) = y$.

Lemma: X is path connected iff $\pi_0(X)$ has only one element.

Proof: If $f, g: S^0 \to X$ pointed maps, they are homotopic if there is a path from $f(-1)$ to $g(-1)$. Always true if path connected.
Conversely, if $\pi_0(X) = 1$, all f, g homotopic. For any x, y, define f by $f(-1) = x$, g by $g(-1) = y$. Homotopy defines a path. $\blacksquare$

Define equivalence relation: $x \sim y$ if there is a path from x to y.

$$X/\sim \;\cong\; \pi_0(X)$$

Stronger that connectivity:

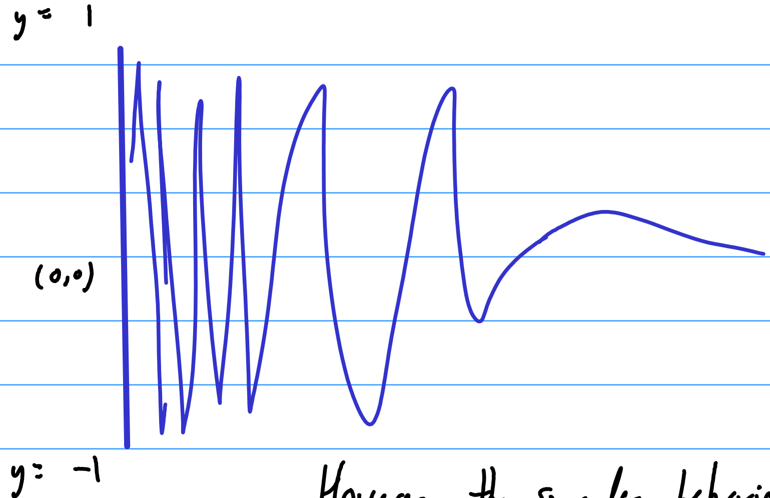
Prop: If X is path connected, then it is connected.

Proof: Suppose X disconnected: $X = U \cup V$, $U \cap V = \phi$, $U \neq \phi$, $V \neq \phi$. Since X is path connected, we have $p: [0,1] \to X$ s.t. $p(0) = x \in U$, $p(1) = y \in V$. U & V open $\Rightarrow$ $p^{-1}(U)$ and $p^{-1}(V)$ open in $[0,1]$. Neither is ϕ, since one contains 0 and the other 1. $U \cup V = X \Rightarrow p^{-1}(U) \cup p^{-1}(V) = [0,1]$. Hence, $[0,1]$ is disconnected! Not so $\Rightarrow$ contradiction $\blacksquare$

The converse is not true: consider $X \subset \mathbb{R}^2$

$$X = \{(x,y) : x = 0, -1 \leq y \leq 1\} \cup \{(x,y) : 0 < x \leq 1,\ y = \sin(1/x)\}$$

(subspace topo)



This is connected, because even though there is a problem near 0 there is no way to choose U, V open.

(see Crossley)

However, the singular behavior does prevent making a continuous path from $(0,0)$ to, say, $\sin(1)$.

So X is connected but not path connected.

Prop: $n > 0$ and X a pointed topol. space. Then $\pi_n(X) = \pi_n(X_0)$, where X_0 is path component that contains the base point.

This is because the image of a path connected space is path connected.

Constructing $\pi_1(X)$ when X is a "union" of spaces:

Theorem: (Van Kampen or Seifert-Van Kampen)

Suppose $X = U \cup V$, U and V open subsets of X such that $U \cap V$ is path connected and contains the base point of X.

Then every element $\alpha \in \pi_1(X)$ can be written as

$$\alpha = \beta_1 + \beta_2 + \cdots + \beta_n$$ (recall: "+" not necessarily Abelian!)

where $\beta_i \in j_*(\pi_1(U))$ or $\beta_i \in k_*(\pi_1(V))$

$j_*: \pi_1(U) \to \pi_1(X)$ induced homo. by inclusions $j: U \to X$
$k_*: \pi_1(V) \to \pi_1(X)$ $k: V \to X$

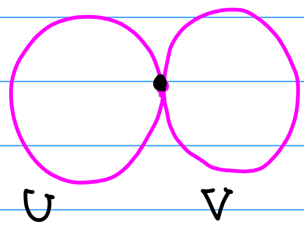
In other words, we can use $\pi_1(U)$ and $\pi_1(V)$ to "generate" $\pi_1(X)$. (This is called the free product of $\pi_1(U)$ and $\pi_1(V)$)

What this "simplified" form of S-VK leaves out is the actual group structure of $\pi_1(X)$, that is, it could be that

$$\alpha = \beta_1 + \beta_2 + \cdots + \beta_n = 0$$ even though we couldn't deduce this from $\pi_1(U)$, $\pi_1(U \cap V)$

Thus, in general we need to understand $\pi_1(U \cap V)$ as well.

Example:



$U \cap V = x_0$ intersection is a point, so simply connected.

In that case $\pi_1(X) = \pi_1(U) * \pi_1(V)$

"free product"

Write $\pi_1(U) = \{\alpha^m \mid m \in \mathbb{Z}\}$

$\pi_2(V) = \{\beta^m \mid m \in \mathbb{Z}\}$

really should use inclusion maps

Then $\pi_1(X)$ has elements such as $\alpha\beta\alpha^2\beta^{-1}\alpha^{-3}\cdots$

Lecture 28: Simplicial homology modulo 2 4/11/11

Similar goal to homotopy groups (probe spaces!), but easier to compute.

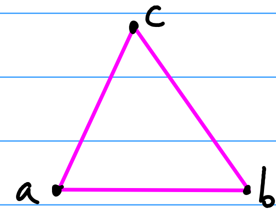
Also more intuitive: isn't it a bit weird that $\pi_n(S^m)$, $n > m$, is so hard to calculate?

Recall: simplices. k-simplex $[v_0, \dots, v_k]$ has $k+1$ faces.
Each face is a k-simplex.

Convention: $\wedge$ (hat) means an element is omitted from a list.

Example: faces of 2-simplex $[a,b,c]$ are $[\hat{a},b,c] = [b,c]$
$[a,\hat{b},c] = [a,c]$
$[a,b,\hat{c}] = [a,b]$

Boundary is union of faces.



Given a simplicial complex K, $S_n(K)$ is set of all n-simplices in K.

Boundary of each element in $S_n(K)$ is a list of elements of $S_{n-1}(K)$.

"boundary operator" is not a map $S_n(K) \to S_{n-1}(K)$, since returns a set.

So let $C_n(K)$ be the collection of all subsets of $S_n(K)$.

Boundary of n-simplex $\in C_n(K)$.

Another way: $C_n(K)$ is the $\mathbb{Z}_2 = \mathbb{Z}/2$ vector space spanned by $S_n(K)$:

n-chain $\quad \lambda_1 \sigma_1 + \cdots + \lambda_k \sigma_k \;, \quad \lambda_i \in \underbrace{\mathbb{Z}_2}_{\{0,1\}}$ for $1 \leq i \leq k$

where σ_i is a n-simplex in K.

Can add n-chains together, but recall $1+1=0$ in $\mathbb{Z}_2$.
$(\sigma + \sigma = 2\sigma = 0)$

We now have our boundary function
$$d_n : S_n(K) \to C_{n-1}(K)$$

but can extend it to $\delta_n : C_n(K) \to C_{n-1}(K)$ by $\quad$ boundary operator, δ_n

$$\delta_n(\lambda_1 \sigma_1 + \cdots + \lambda_n \sigma_n) = \lambda_1 d_n(\sigma_1) + \cdots + \lambda_k d_n(\sigma_k)$$

example: Given $d_n(\sigma_1) = s_1 + s_2$, $d_n(\sigma_2) = s_2 + s_3$ $\quad (s_i \in C_{n-1}(K))$

$$\Rightarrow \delta_n(\sigma_1 + \sigma_2) = (s_1 + s_2) + (s_2 + s_3) = s_1 + s_3$$

example: $\delta_1[a,b] = a + b$, $\quad \delta_2[a,b,c] = [b,c] + [a,c] + [a,b]$

Generally,
$$\delta_n[v_0, \ldots, v_n] = \sum_{i=0}^{n} [v_0, \ldots, \hat{v}_i, \ldots, v_n]$$

Composition:
$$\begin{aligned}\delta_1(\delta_2[a,b,c]) &= \delta_1([b,c] + [a,c] + [a,b]) \\ &= (b+c) + (a+c) + (a+b) = 2a + 2b + 2c = 0\end{aligned}$$

But something much stronger holds:

Lemma: For every $n \geq 1$, $\delta_n \circ \delta_{n+1} : C_{n+1}(K) \to C_{n-1}(K)$ is zero.

Proof: Linear, so use basis for $C_{n+1}(K) \Rightarrow S_{n+1}(K)$.

Let $\sigma \in S_{n+1}(K)$, $\sigma = [v_0, \cdots, v_{n+1}]$.

$$\delta_{n+1}\sigma = \sum_{i=0}^{n+1} [v_0, \dots, \hat{v}_i, \dots, v_{n+1}]$$

$$\delta_n \delta_{n+1} \sigma = \sum_{\substack{j=0 \\ j \neq i}}^{n+1} \sum_{i=0}^{n+1} [v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_{n+1}]$$

$\leftarrow$ since element i no longer exists

Every summand occurs twice, so $\delta_n \delta_{n+1} \sigma = 0$. $\blacksquare$

We thus have the sequence

$$\cdots \longrightarrow C_n(K) \xrightarrow{\delta_n} C_{n-1}(K) \xrightarrow{\delta_{n-1}} C_{n-2}(K) \longrightarrow \cdots \longrightarrow C_1(K) \xrightarrow{\delta_1} C_0(K)$$

where the composite of any two transformations is 0. **chain complex** $(C_*(K), \delta_*)$

boundary of $(n+1)$ simplex $\in \operatorname{Ker} \delta_n$.

"a boundary has no boundary"

$Z_n(K)$ Let $\underline{Z_n = \operatorname{Ker} \delta_n}$ ($Z_0 = C_0$ by convention)

Elements of Z_n are **cycles**

for all $n \geq 0$

Boundaries are cycles, but are all cycles boundaries?

$B_n(K)$ $\underline{B_n = \operatorname{Im} \delta_{n+1}}$

Elements of B_n are **boundaries**

So discard the elements of Z_n we understand $\to$ the boundaries.

Take quotient Z_n / B_n: equivalence classes of cycles under the relation $z_1 \sim z_2$ if $z_1 - z_2 \in B_n$.

Two chains are homologous if $z_1 \sim z_2$.

$$H_n(K) = \frac{Z_n(K)}{B_n(K)} = \begin{cases} \operatorname{Ker} \delta_n / \operatorname{Im} \delta_{n+1}, & n > 0 \\ C_0 / \operatorname{Im} \delta_1, & n = 0 \end{cases}$$

n^{th} homology group of simplicial complex K. actually a vector space

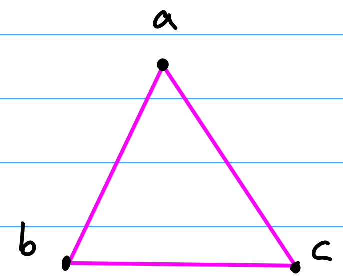
The homology of K is $\{ H_0(K), H_1(K), H_2(K), \ldots \}$

Lecture 29: Computing simplicial homology modulo 2 4/13/11

$$H_n(K) = Z_n(K) / B_n(K)$$

Z_n cycles $\delta_n \sigma = 0$
B_n boundaries $\sigma = \delta_{n+1} \tilde{\sigma}$

example: simplicial circle



a, b, c 0-simplices
$[a,b], [a,c], [b,c]$ 1-simpl.

Chain complex:

C_0 & C_1 have dim 3
$C_i = 0, \ i > 1.$

$$C_1 \xrightarrow{\delta_1} C_0$$

Let $\sigma = \lambda_1 [a,b] + \lambda_2 [b,c] + \lambda_3 [a,c] \in C_1$ $\quad \lambda_i \in \mathbb{Z}/2$

$$\delta_1 \sigma = \lambda_1 (a+b) + \lambda_2 (b+c) + \lambda_3 (a+c)$$
$$= (\lambda_1 + \lambda_3) a + (\lambda_1 + \lambda_2) b + (\lambda_2 + \lambda_3) c$$

$\delta_1 \sigma = 0$ requires
$$\begin{aligned} \lambda_1 + \lambda_3 &= 0 \\ \lambda_1 + \lambda_2 &= 0 \\ \lambda_2 + \lambda_3 &= 0 \end{aligned} \Rightarrow \begin{aligned} \lambda_1 &= \lambda_3 \\ \lambda_1 &= \lambda_2 \\ \lambda_2 &= \lambda_3 \end{aligned} \Rightarrow \lambda_1 = \lambda_2 = \lambda_3$$

Hence, $\sigma = \lambda ([a,b] + [b,c] + [a,c]) \in Z_1 = \mathrm{Ker}\, \delta_1$

$\dim Z_1 = 1$. $\quad \mathrm{Im}\, \delta_2 = 0$, so $B_1 = \mathrm{Im}\, \delta_2 = 0$.

Hence, $H_1 = Z_1 / B_1 = Z_1 = \mathbb{Z}/2$

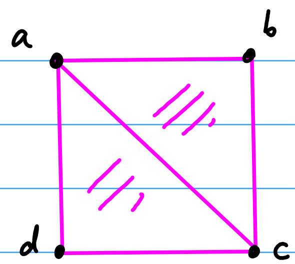
Also: $\dim \mathrm{Im}\, \delta_1 = \dim C_1 - \dim \mathrm{Ker}\, \delta_1 = 3 - 1 = 2$

Hence $\dim H_0 = \dim(C_0/B_0) = \dim C_0 - \dim B_0 = 3-2=1$

So $H_0 = \mathbb{Z}/2$, since it is a one-dimensional vector space.

Also $H_i = 0$, $i > 1$.

example:



simplicial square

$$C_2 \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0$$

$\dim C_0 = 4$
$\dim C_1 = 5$
$\dim C_2 = 2$

$$\sigma = \lambda_1[a,b] + \lambda_2[a,c] + \lambda_3[a,d] + \lambda_4[b,c] + \lambda_5[c,d]$$

$$\delta_1\sigma = \lambda_1(a+b) + \lambda_2(a+c) + \lambda_3(a+d) + \lambda_4(b+c) + \lambda_5(c+d)$$
$$= (\lambda_1+\lambda_2+\lambda_3)a + (\lambda_1+\lambda_4)b + (\lambda_2+\lambda_4+\lambda_5)c + (\lambda_3+\lambda_5)d$$

Kernel: $\delta_1\sigma = 0$. Require $\lambda_1 = \lambda_4$, $\lambda_3 = \lambda_5$, $\lambda_1+\lambda_2+\lambda_3 = 0$ $\Rightarrow \lambda_2 = \lambda_1 + \lambda_3$

So $\sigma = \lambda_1[a,b] + (\lambda_1+\lambda_3)[a,c] + \lambda_3[a,d] + \lambda_1[b,c] + \lambda_3[c,d]$

$$= \lambda_1\big([a,b]+[a,c]+[b,c]\big) + \lambda_3\big([a,c]+[a,d]+[c,d]\big)$$ 2-dim

Hence, $\dim Z_1 = \dim \operatorname{Ker}\delta_1 = 2$

So $\dim B_0 = \dim \operatorname{Im}\delta_1 = \dim C_1 - \dim \operatorname{Ker}\delta_1 = 5-2=3$

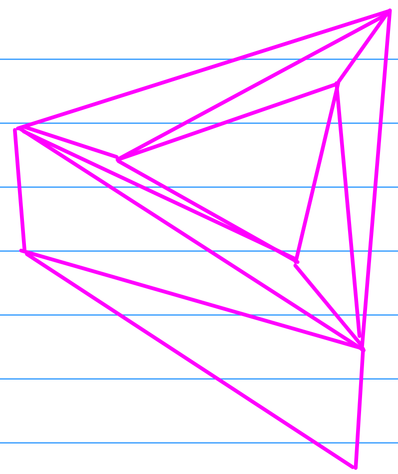
Hence, $\dim H_0 = \dim C_0 - \dim B_0 = 4-3=1$

Also: $\dim \operatorname{Im}\delta_2 = 2$ since the 2-simplices have different boundaries

Hence, $\dim H_1 = \dim \operatorname{Ker} \delta_1 - \dim \operatorname{Im} \delta_2 = 2 - 2 = 0$

Conclude: $H_0 = \mathbb{Z}/2$, $H_i = 0$, $i > 0$. acyclic

example: simplicial torus (example 7.8, page 121)



$\dim C_0 = 9$
$\dim C_1 = 27$
$\dim C_2 = 18$

$C_2 \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0$

could not be bounding otherwise!

Im of δ_1 = sum of even # of vertices

$\dim \operatorname{Im} \delta_1 = \dim C_0 - 1 = 8$
$\Rightarrow \dim H_0 = 1$

So we must have $\dim \operatorname{Ker} \delta_1 = 27 - 8 = 19$

All the 2-simplices together have $\delta_2 = 0$, since share all edges.
so $\in \operatorname{Ker} \delta_2$.
Easy to see this is the only element of Z_2.

$\Rightarrow \dim \operatorname{Ker} \delta_2 = 1$, so $\dim \operatorname{Im} \delta_2 = 18 - 1 = 17$.

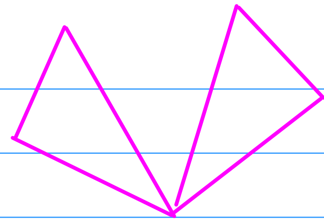
Hence, $\dim H_1 = \dim \operatorname{Ker} \delta_1 - \dim \operatorname{Im} \delta_2 = 19 - 17 = 2$

$\dim H_2 = \dim \operatorname{Ker} \delta_2 - \dim \underbrace{\operatorname{Im} \delta_3}_{0} = 1 - 0 = 1$

$\therefore H_0 = \mathbb{Z}/2$, $H_1 = \mathbb{Z}/2 \oplus \mathbb{Z}/2$, $H_2 = \mathbb{Z}/2$, $H_i = 0$, $i > 2$.

$$C_1 \xrightarrow{\delta_1} C_0$$

<u>example</u>: rabbit ears



$\dim C_0 = 5$

$\dim C_1 = 6$

$$\dim \operatorname{Im} \delta_1 = \dim C_0 - 1 = 4$$

$$\Rightarrow \dim H_0 = 1, \quad \dim H_1 = 2.$$

$H_0 = \mathbb{Z}/2$

$H_1 = \mathbb{Z}/2 \oplus \mathbb{Z}/2$

$H_i = 0, \quad i > 1$

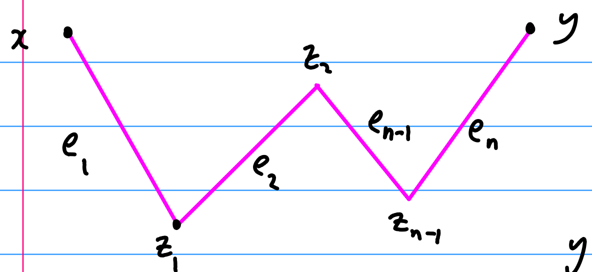
Lecture 30: Integral simplicial homology 4/15/11

Recall $H_0(K) = \mathbb{Z}/2$ for all examples so far.

Prop.: $\dim H_0(K) = \#$ of path components of K
$= \#$ of elements of $\pi_0(K)$

Proof: $\pi_0(K)$ contains pointed maps $f: S^0 \to K$ up to homotopy. Determined by $f(-1)$, and equivalent to classes under relation $x \sim y$ iff path between x and y.

Each $x \in K$ is in the interior of exactly one simplex, and joined to vertices of simplex by path. So $\pi_0(K)$ is same as 0-simplices (S_0) up to path connectivity.



$$\delta([x, z_1] + [z_1, z_2] + \cdots + [z_{n-1}, y])$$
$$= x + y$$
$$= y - x \quad (\text{mod } 2)$$

$y - x \in B_0(K) \to 0$ element in $H_0(K)$

Hence, for any y, x in the same component, $y \sim x$.

(See Crossley for rest of proof.) ▨

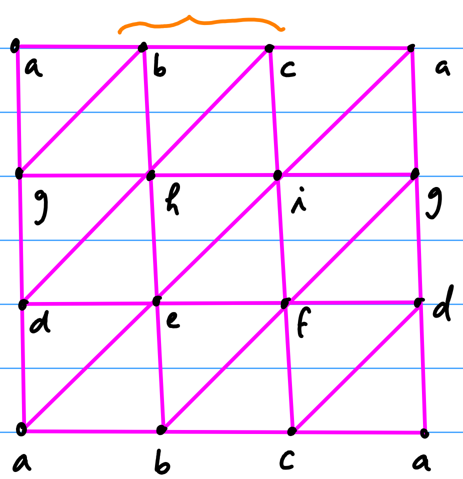
Prop: $\chi(K) = \sum_{n \geq 0} (-1)^n \underbrace{\dim H_n(K)}_{\text{Betti numbers } b_n}$

Proof: Recall $\chi(K) = \sum_{n \geq 0} (-1)^n \dim C_n(K)$

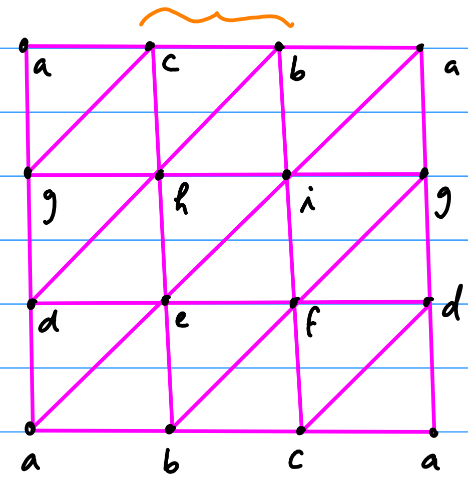
$$\begin{aligned}\dim H_n &= \dim \operatorname{Ker} \delta_n - \dim \operatorname{Im} \delta_{n+1} \\ &= \dim C_n \underbrace{- \dim \operatorname{Im} \delta_n + \dim \operatorname{Im} \delta_{n+1}}_{\text{these cancel in the alternating sum.}}\end{aligned}$$

Limitations of homology mod 2:

Torus



Klein bottle

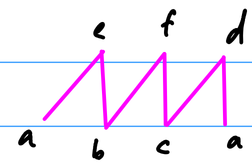


Bottom row: $[a,b] + [b,c] + [c,a] \in C_1$

Add boundaries of 2-simplices:
$\rightarrow$ get same element in H_1

$$\begin{aligned}\delta_2[a,b,e] &= [a,b] + [b,e] + [a,e] \\ \delta_2[b,f,c] &= [b,c] + [c,f] + [b,f] \\ \delta_2[c,d,a] &= [c,a] + [a,d] + [c,d]\end{aligned}$$

$$[a,b]+[b,c]+[c,a] + \delta_2\big([a,b,e]+[b,f,c]+[c,d,a]\big) = [a,e] + [e,b] + [b,f] + [f,c] + [c,d] + [d,a]$$



Keep going, adding δ_2 (2-simplices). Eventually get to

$[a,b] + [b,c] + [c,a] \sim [a,b] + [b,c] + [c,a]$ (top row – torus)

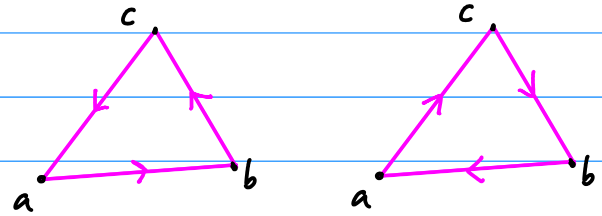
or

$\sim [a,c] + [c,b] + [b,a]$ (top row – Klein)

But these are the same! No orientation info in homology mod 2, so cannot distinguish.

Need a way to say $[a,b] = -[b,a]$.

Oriented simplices: $[a,b,c] = -[b,a,c]$, etc., whenever two interchanged.

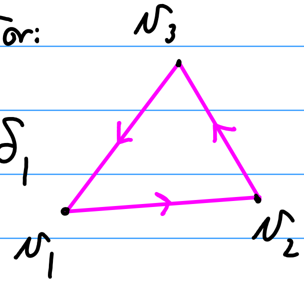


even # of swaps – same orientation
odd # of swaps – opposite

Elements of $C_n(K)$ now look like $2\sigma_1 + 3\sigma_2 - 2\sigma_4$ (coeffs in $\mathbb{Z}$)

n-chains

Boundary operator:

δ_1 

$= [v_1, v_2] + [v_2, v_3] + [v_3, v_1]$

$= [v_1, v_2] + [v_2, v_3] - [v_1, v_3]$

In general: $\delta_n : C_n \to C_{n-1}$

$$\delta_n [v_0, \ldots, v_n] = \sum_{i=0}^{n} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_n]$$

With this def'n, $\quad \delta_n \circ \delta_{n+1} = 0.$

Can define homology groups, chain complex, etc., as before.

example: K = simplicial torus $\Rightarrow$ $H_0(K) = \mathbb{Z}$

$H_1(K) = \mathbb{Z} \oplus \mathbb{Z} \qquad H_i(K) = 0$

$H_2(K) = \mathbb{Z} \qquad\qquad\qquad i \geq 2$

example. K = Klein bottle. How does it differ from torus?

First difference: $H_2(K) = 0$, since sum of all 2-simplices is 0.

(This is because Klein bottle has no "interior")

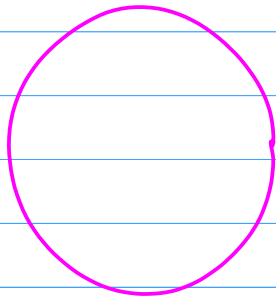
Can also show: $H_1(K) = \mathbb{Z} \oplus \mathbb{Z}/2$ "torsion"

Lecture 31: Knot theory

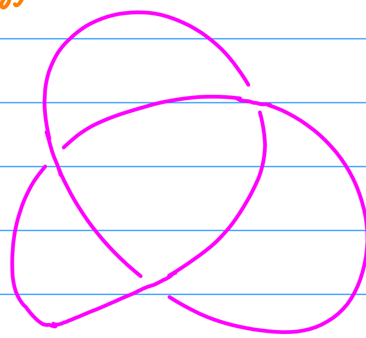
4/18/11

Knot: continuous injective map $S^1 \to \mathbb{R}^3$, up to isotopy
(embedding)

Textbook: Adams, "The Knot Book"

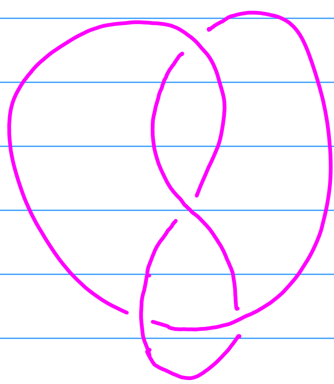


unknot
(trivial knot)

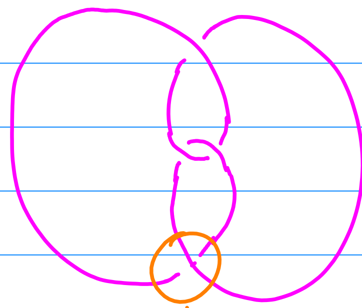


trefoil knot

Three projections of the figure 8 knot:

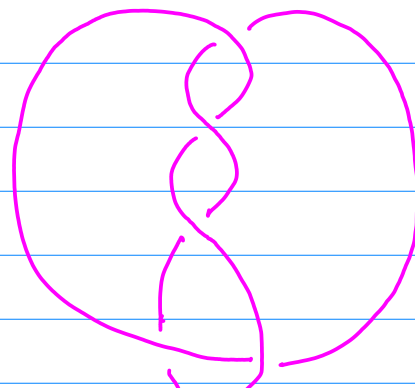


4 crossings



crossings

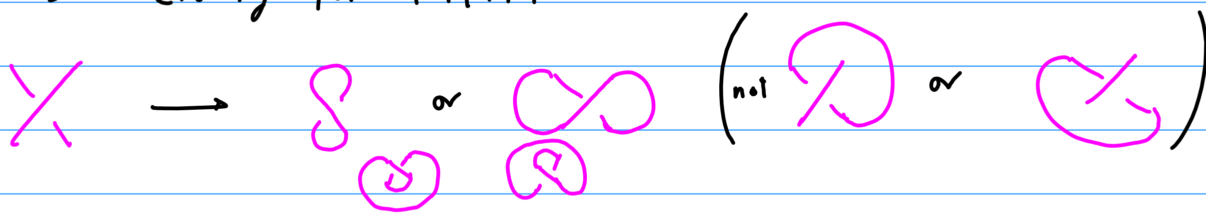
4 crossings



5 crossings

The fig-8 knot is a 4-crossing knot. (min #)

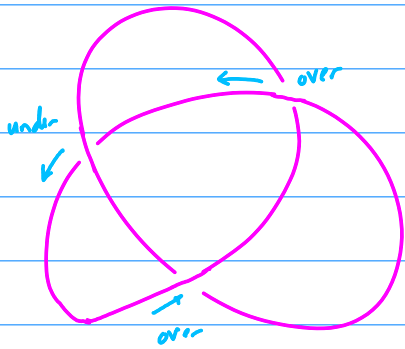
Need > 1 crossing for nontrivial knot!



Given a projection, is it the unknot? Tough!

Haken's algorithm → no implementation!

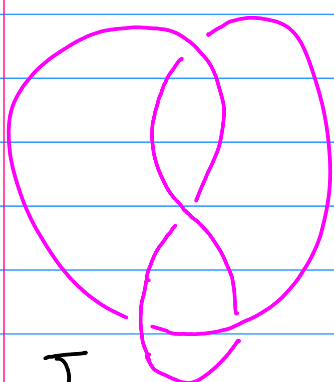
An alternating knot alternates over/under crossings as you traverse it.



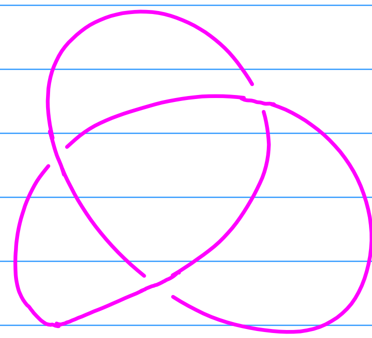
← trefoil is alternating, as is fig-8 in first and middle pic in previous page.

(Two fun problems: 1.6, 1.7)

Composition of knots:

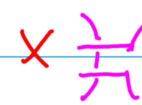


J



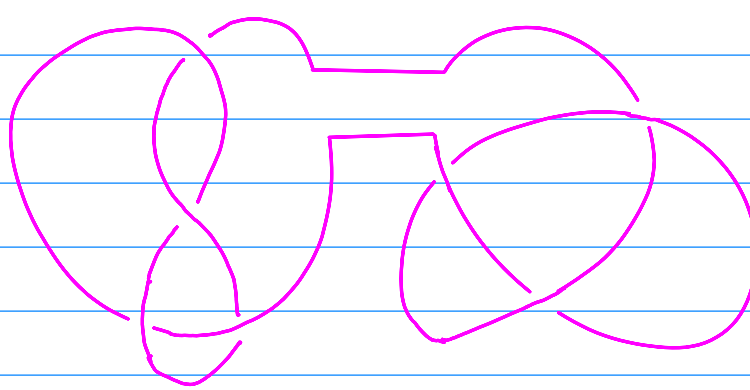
K

remove small arc in each projection, then connect.
(Do not create new crossings!)



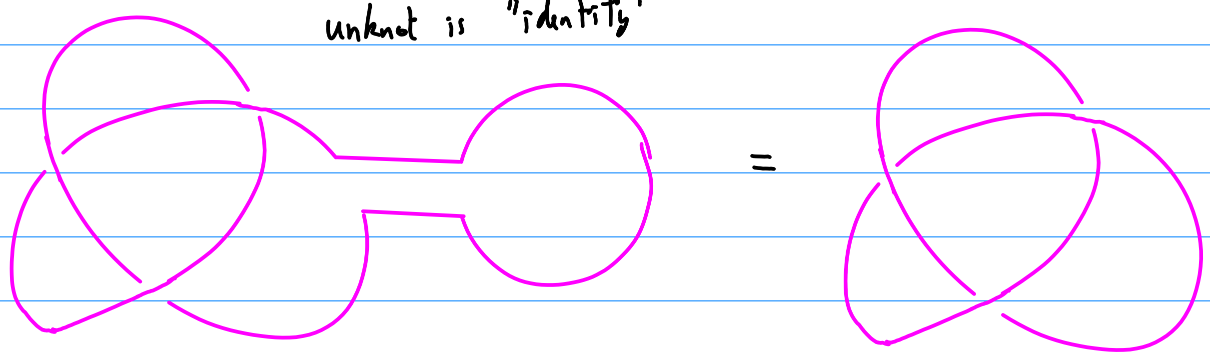
J#K is composite knot if neither J or K is unknot

J, K are factor knots



J#K

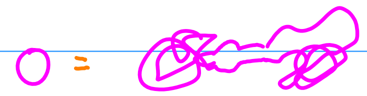
unknot is "identity"



$K \# \text{unknot} = K$

A prime knot cannot be decomposed into two nontrivial knots.

i.e., somehow



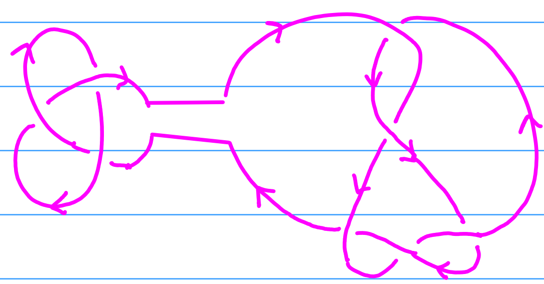
Is the unknot composite? If that were so, every knot would be composite! Because we could compose with $U = J \# K$, and therefore get a decomposition. Fortunately not the case! (Hopefully we'll have time to show this later, and uniqueness of decomp.)

So, one goal of knot theory is to enumerate all prime knots.

In general, $\#$ depends on the choice of "bridge". (i.e., where we remove the arc)

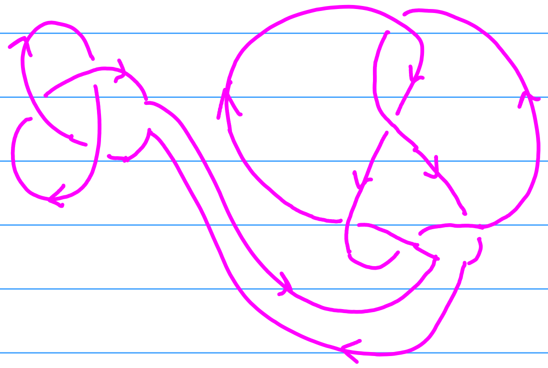
Oriented knot: a knot with an arrow!

(a)



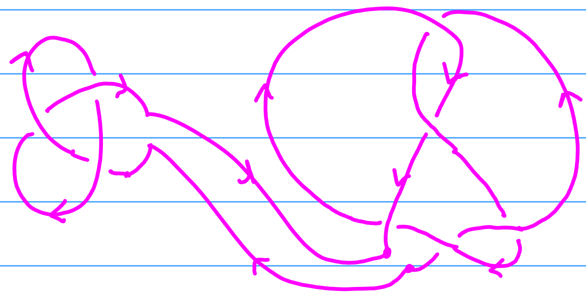
same orientation (match)

(b)



same orientation (match)

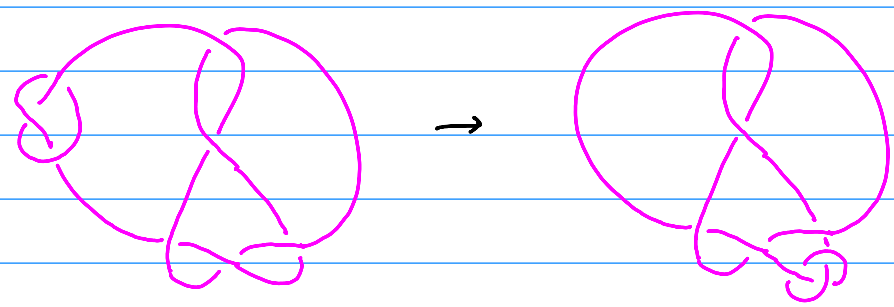
(c)



orientations differ!

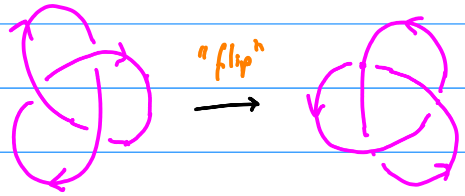
Claim: • all composite knots with matching orientation are the same
• " " " " non-matching " " " "

To convince yourself of (a) = (b): slide trefoil down



However, in this particular example (a) = (b) = (c), because one of the factors is invertible. (= to its reverse-oriented counterpart)

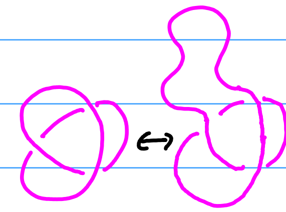
Hard to determine!
But actually many "simple" knots are invertible. Simplest non-invertible: 8_{17}



Lecture 32: Reidemeister moves and links 4/20/11

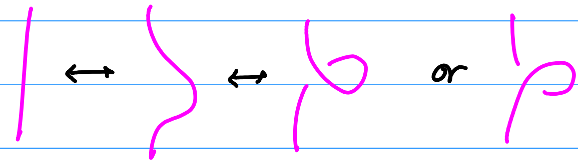
Rearranging of strings: ambient isotopy. (in $\mathbb{R}^3$)

Knot projection: planar isotopy (in $\mathbb{R}^2$ as projection of $\mathbb{R}^3$)

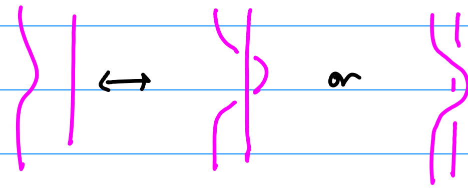


Ambient isotopies that modify crossings can be written as series of "moves"

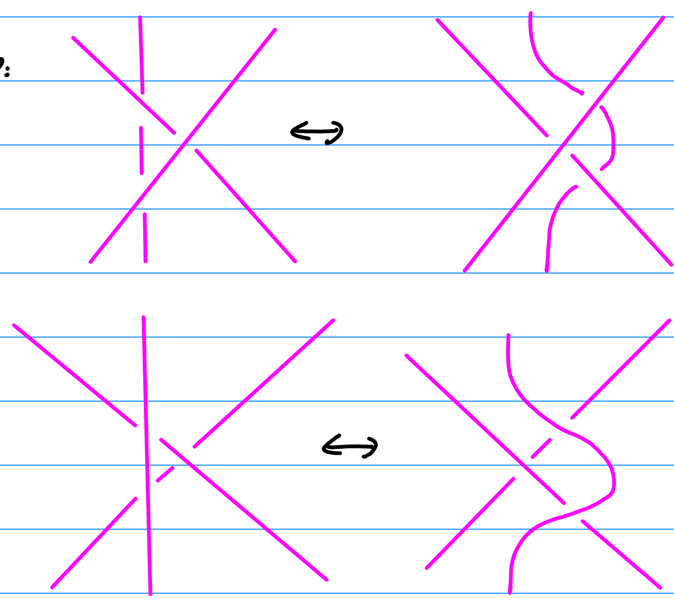
Type I Reidemeister move:



Type II Reidemeister move:



Type III Reidemeister move:



Reidemeister proved that this is all we need to get from one projection to any other

In principle, showing two knots are equal simply involves finding a sequence of Reidemeister moves. In practice this is hard: crossing number can increase before decreasing again

Example: fig-8 equivalent to its mirror image

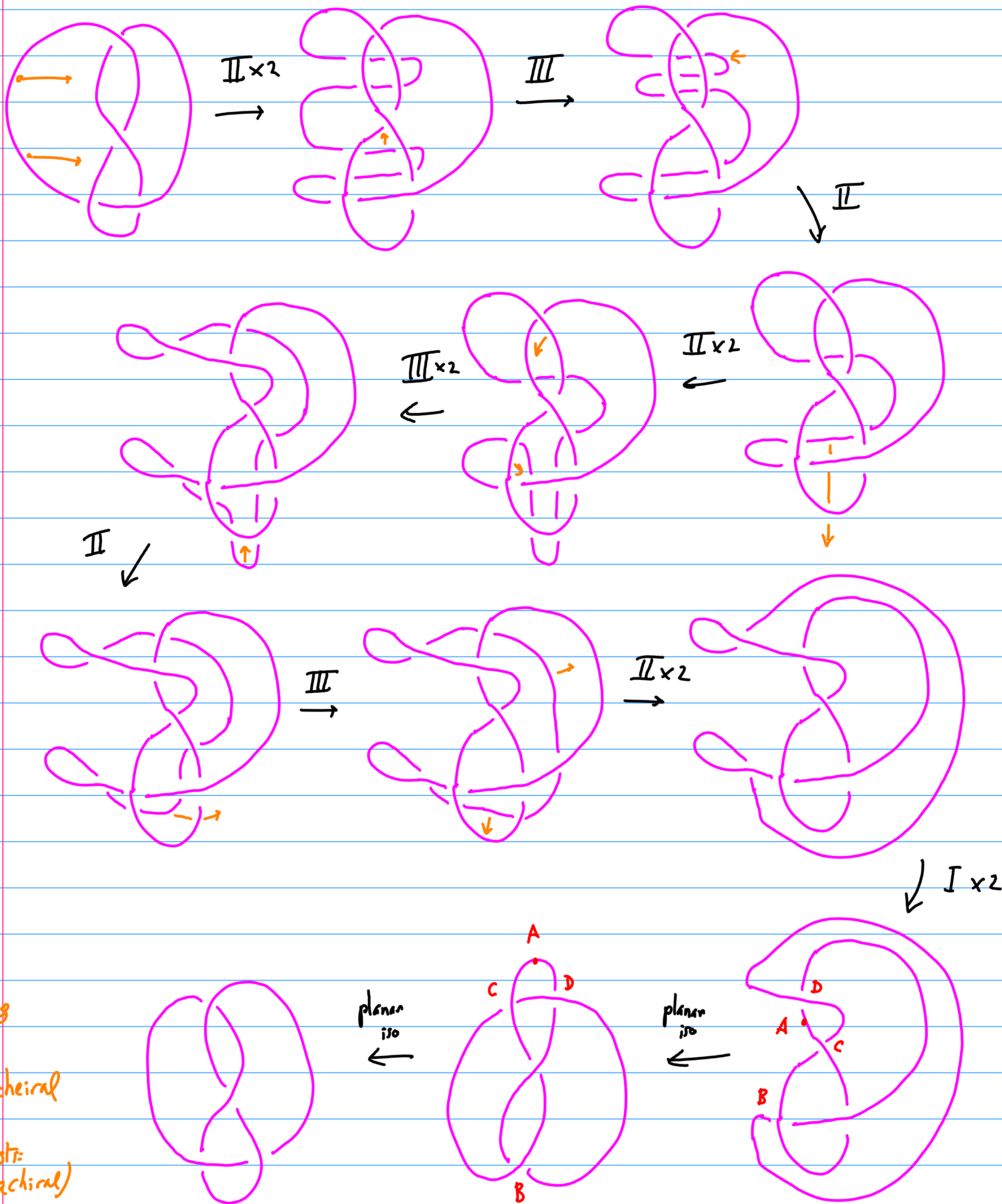
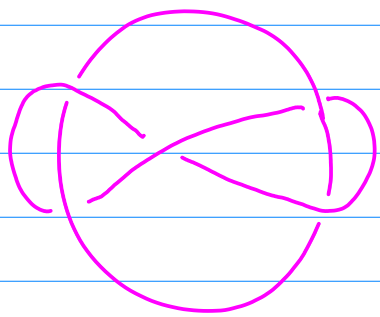


fig 8 is amphicheiral

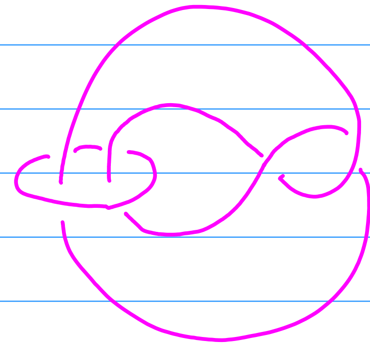
(chemists: achiral)

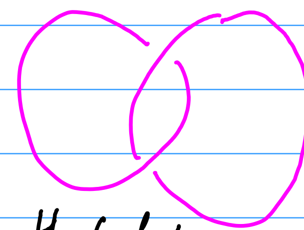
(NO PROOF THAT # OF INTERMEDIATE CROSSINGS BOUNDED BY SIMPLE FUNCTION OF # OF CROSSINGS)

Links: "multiknots"

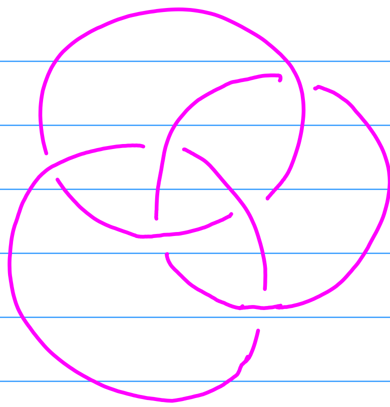


Whitehead link





Hopf link

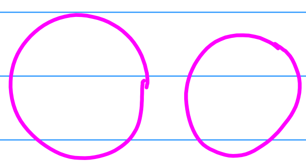


Borremean rings

A knot is a link with one component

Can use Reidemeister moves to show equivalence

A link is splittable if it can be deformed in two pieces with a nonintersecting plane in between. (not always obvious!)

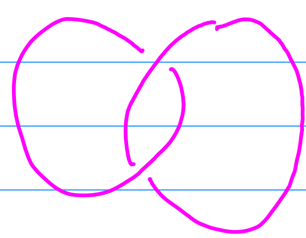


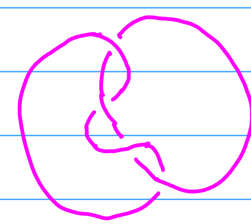
(linking number 0)

unlink with two components (or trivial link)

Two equivalent links must have the same number of components.

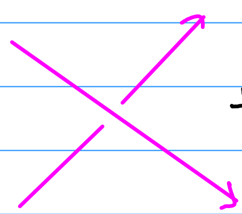
Linking number 1:



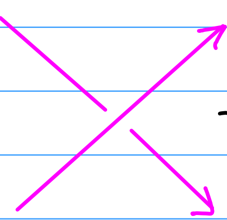


linking number = 2

How do we measure the linking number? First orient the two components.

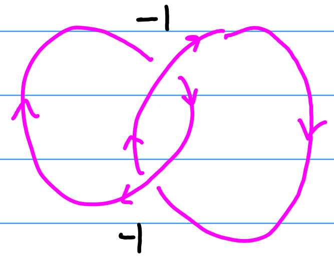


$+1$

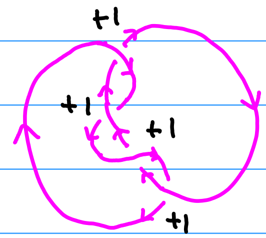


-1

Then sum the signs of crossings and divide by 2.



$$\frac{(-1)+(-1)}{2} = -1$$

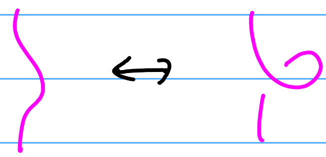


$$\frac{4\cdot(1)}{2} = 2$$

The sign depends on how the components are oriented

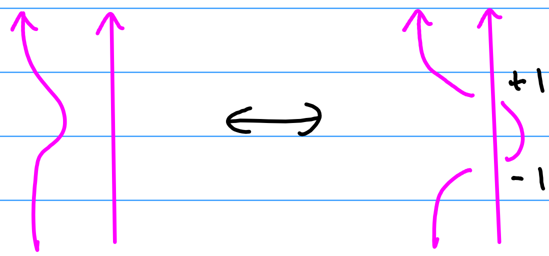
How do we know this doesn't depend on projection?
Show that Reidemeister moves do not change linking #.

Type I:

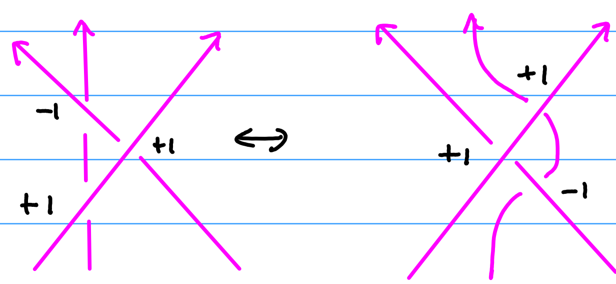


Changes self-crossing, but not crossing between two components

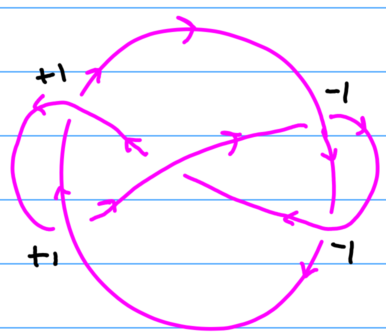
Type II:



Type III:



Linking number is an <u>invariant</u> of the oriented link



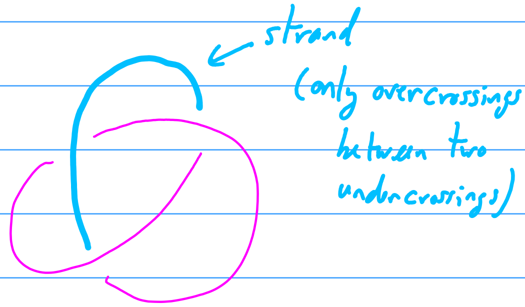
Whitehead link has linking # = 0, but it is not the unlink!

Borromean rings: remove any ring → unlink!

So linking number can be used to determine if links are <u>not</u> the same, but can't show equivalence.

Lecture 33: Tricolorability 4/22/11

So far: we have not shown that there is any other knot beside the unknot... We will now prove that unknot $\neq$ trefoil.

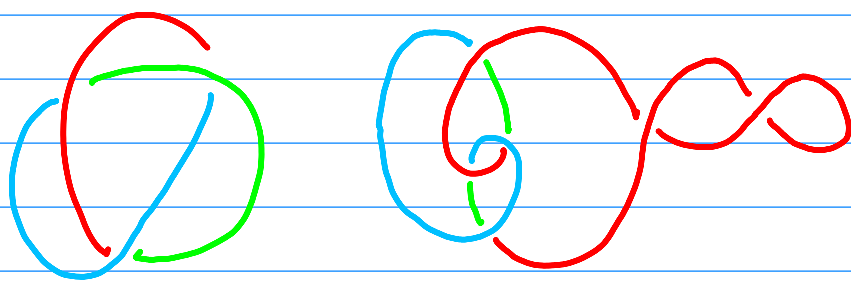


Projection is tricolorable if each strand in projection can be colored in one of 3 colors such that, at each crossing, either

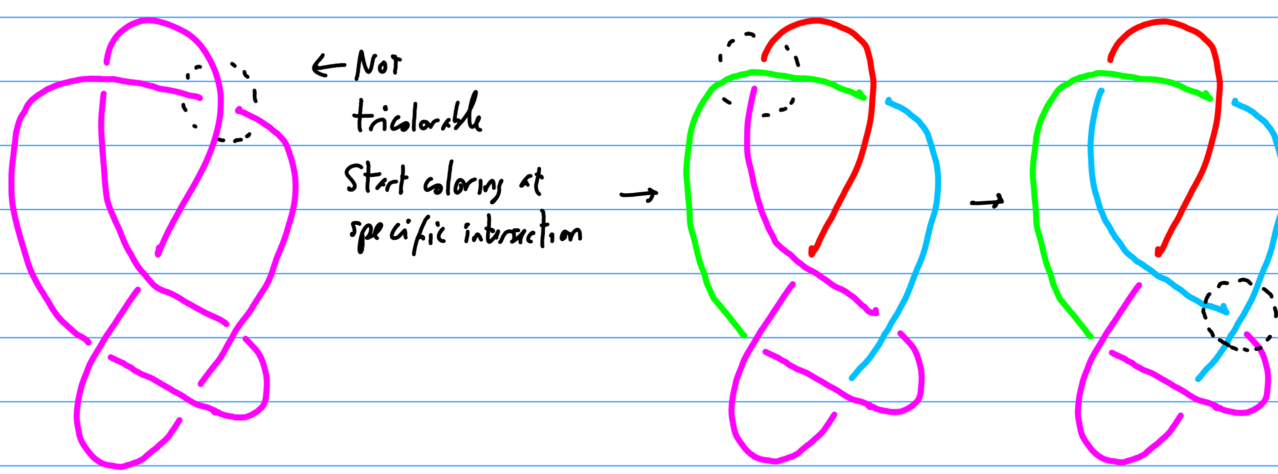
- all 3 colors come together
- 3 of the same color come together

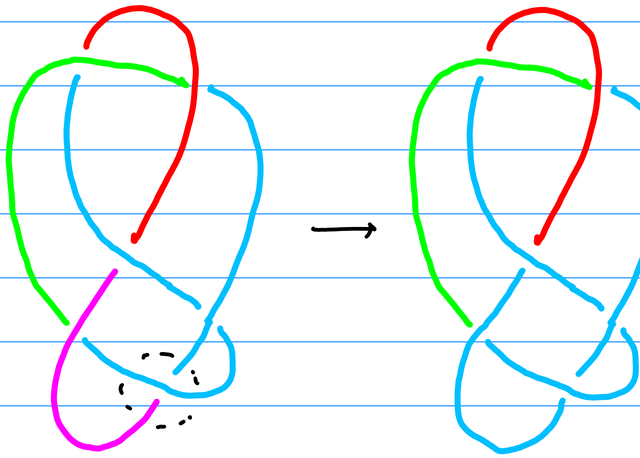
Need to use at least 2 colors!

Trefoil:



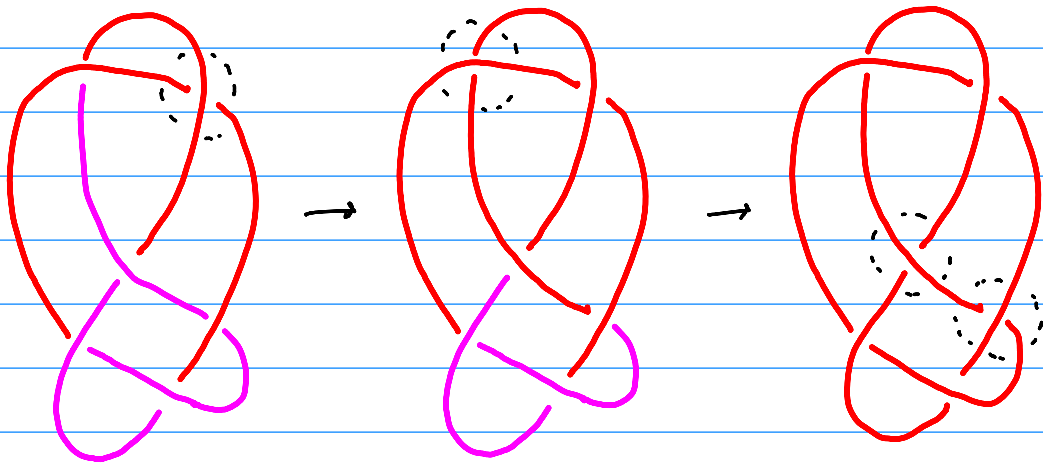
Let's show that the knot 6_2 is not tricolorable:





oops!

Try same color:

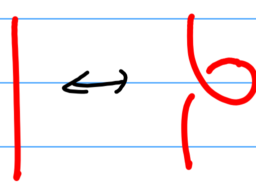


Only one color → not allowed!

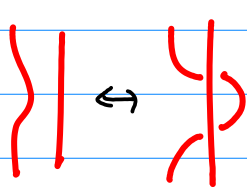
Hence, 6_2 is not tricolorable.

Reidemeister moves preserve tricolorability.

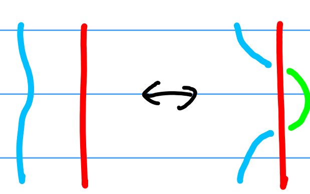
Type I:



Type II:

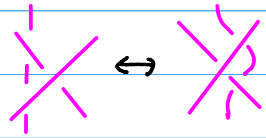


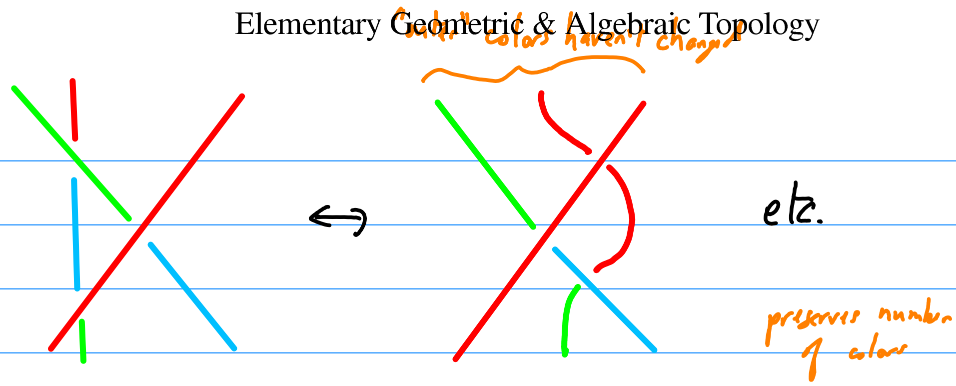
or



increases number of colors

Type III:





etc.

So the number of colors never decreases, hence still tricolorable after R. move.

<u>Conclude:</u> tricolorability is independent of projection

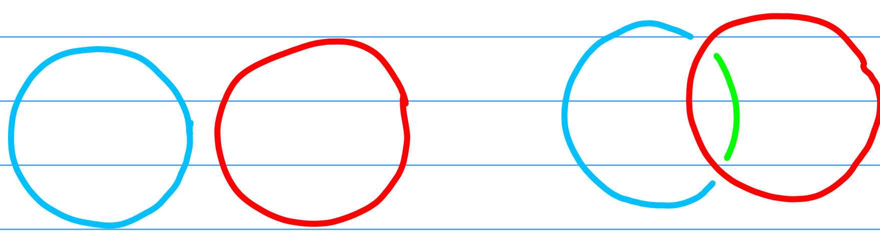
So either every projection is tricolorable, or none is.

Hence, since trefoil is tricolorable but not unknot, they must be distinct.

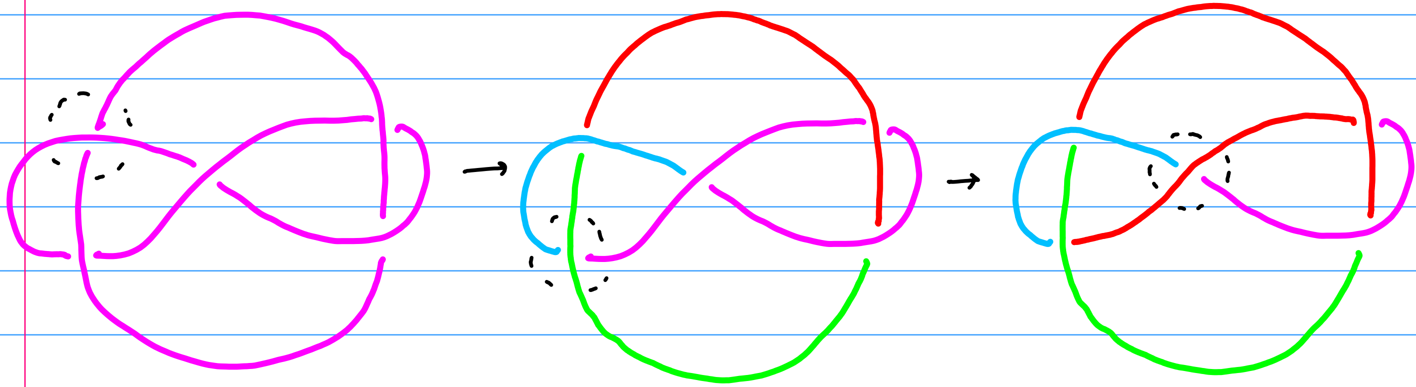
A more mathematical interpretation: colors are integers 0, 1, 2, and at every intersection we require $x + y + z = 0 \bmod 3$.

Also works: $x, y, z = 0, 1, 2, 3, 4$ with $x + y - 2z = 0 \bmod 5$ (z = over-strand)
This allows us to distinguish more knots.

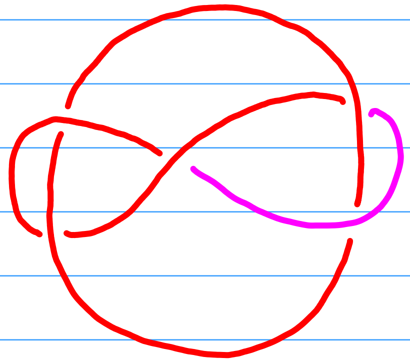
<u>Links:</u> Tricolorability works the same way, except that the unlink itself <u>is</u> tricolorable:



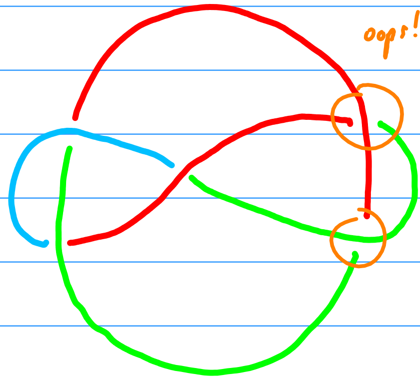
We can use this to show, for example, that the Whitehead link is not the unlink (linking # = 0 for both, BTW):



OK try one color at intersection:



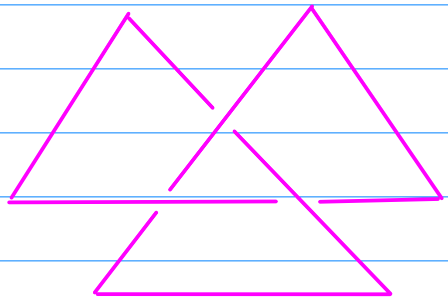
...but then has to be all one color,



Hence, the Whitehead link is not tricolorable, but the unlink is, so they are not the same.

Lecture 34: Tabulating knots 4/25/11

Stick #: minimum number of sticks to build a knot $s(K)$



Trefoil has stick # of 6.

$$s(J \# K) \leq s(J) + s(K) - 1$$

Relevance to chemistry!

See Adams, 30-34

Tabulation: early history: Gauss, Kelvin, Kirkman
↳ projection

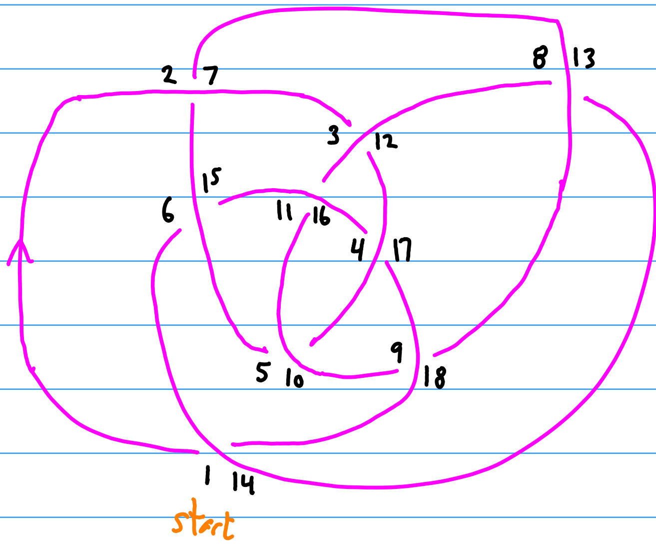
19th	Tait	Alternating up to 10 crossings
1899	Little	43 non-alternating of 10 crossings (1974: Perko pair — two are same!)
1917	Haseman	Amphichiral knots of 12 crossings (except a few)
1927	Alexander & Briggs	Proof that knots up to 9 crossings are distinct (Alexander polynomial)
1932	Reidemeister	Rigorous classification up to 9 crossings
1969	Conway	Prime knots ≤ 11 crossings (fixed by Caudron 1978)
		Prime nonsplittable links ≤ 10 crossings
1981	Dowker, Thistlethwaite	Prime knots ≤ 12 crossings (≤ 13 in 1982)
1998	Hoste, Thistlethwaite, Weeks	≤ 16 crossings ("The first 1,701,936 knots.")

#cross	3	4	5	6	7	8	9	10	11	12	13	14	15	16
#prime*	1	1	2	3	7	21	49	165	552	2176	9988	46,972	253,293	1,388,705

 * not counting mirror image

17 knots, as they say in math, will require "new ideas".

The Dowker notation for alternating knots:



- Pick an orientation
- Start from a crossing; label 1
- Traverse knot and label each xing
- Each crossing gets two numbers.

Notice that we get even/odd pairs! (why?)

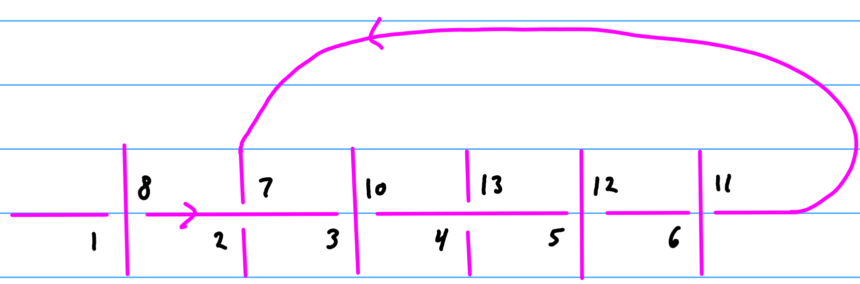
Pairings:

1	3	5	7	9	11	13	15	17
14	12	10	2	18	16	8	6	4

← shorthand: just use lower row

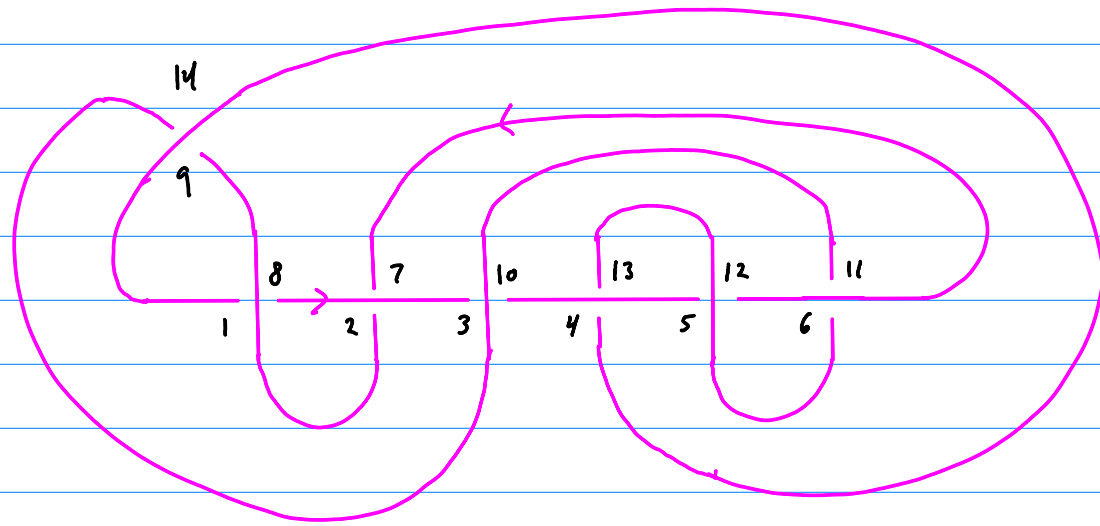
How do we go the other way? Take 8 10 12 2 14 6 4:

1	3	5	7	9	11	13
8	10	12	2	14	6	4



- First draw alternating crossings until encounter number we've seen, 7 here
- Then start joining (ignore direction for now)

Create new alternating crossing if we haven't a number before: (here 9)



Note that only the first crossing direction is ambiguous; after that we're stuck!

Composite knots:

1	3	5	7	9	11
4	6	2	10	12	8

Can be broken up into subgroups (like reducible permutations)

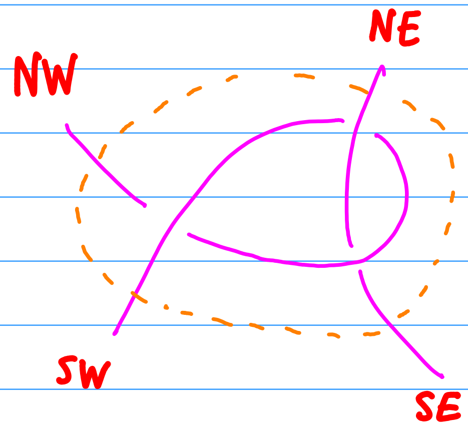
If we disallow composite knots, then the ambiguity above results in a knot or its mirror image.

Note that many number sequences do not correspond to knots! (This is why it always seems to work "just right")

For non-alternating knots, use signed even numbers:

>0 overcrossing
<0 undercrossing
} w.r.t. when the number is assigned

Lecture 35: Tangles & Conway's notation   4/27/11

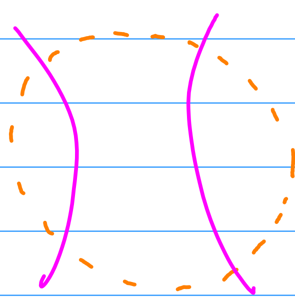


A tangle is a region in the projection plane surrounded by a circle such that the knot or link crosses the circle exactly four times.
(always labeled by NW, NE, SE, SW)

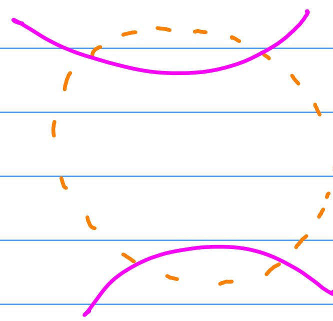
Note that if we close tangle, then equiv. iff tangle is

Two tangles are equivalent if we can relate them by Reidemeister moves while the four endpoints remain fixed and strings never leave the circle.

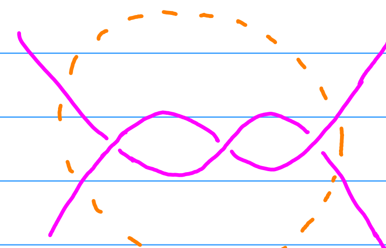
Special tangles:



∞ tangle



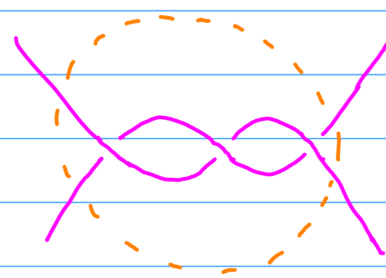
0 tangle



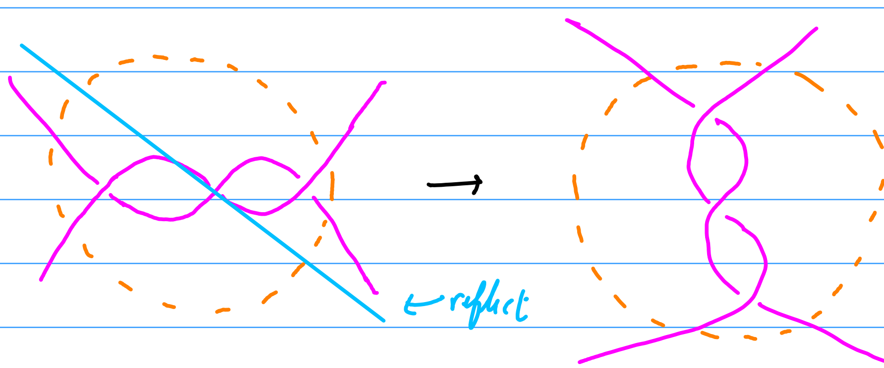
3 tangle

overstrand has positive slope

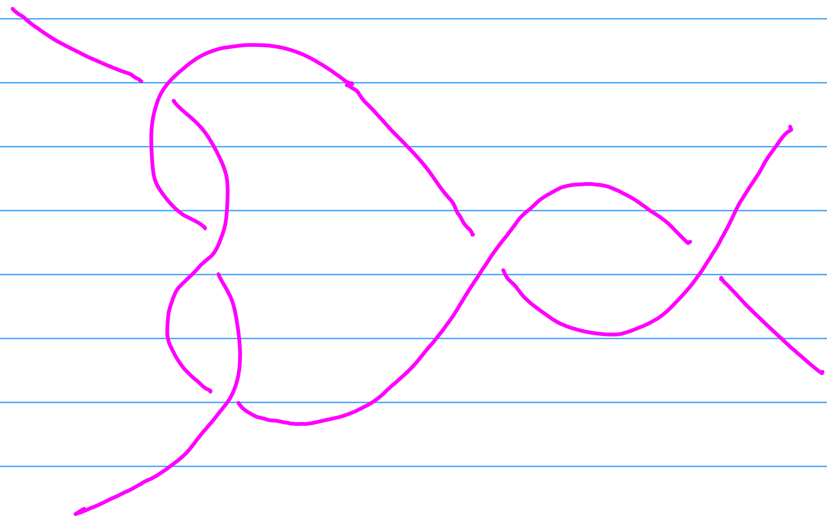
-3 tangle



Now let's make a more complicated tangle.



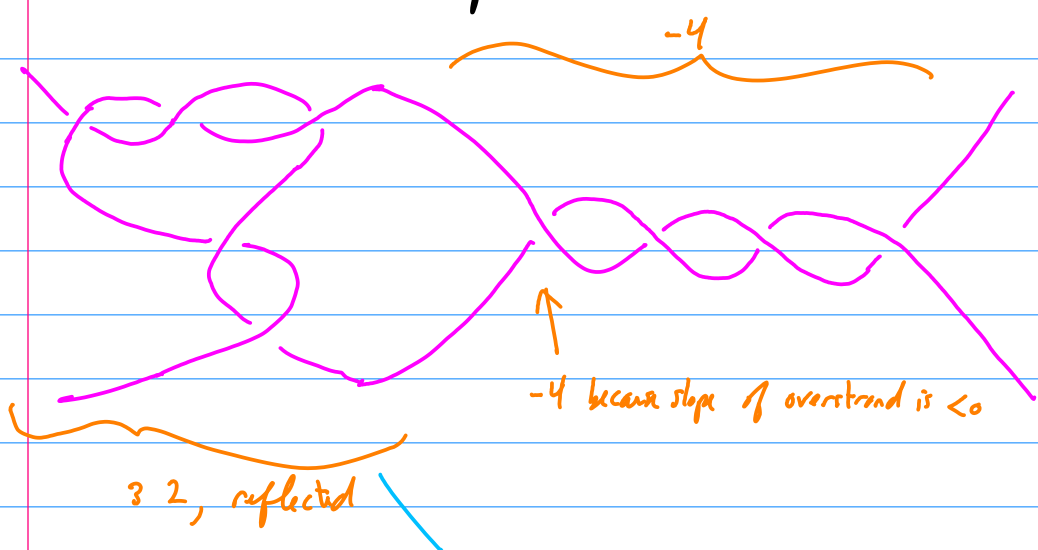
Then join with with 2 tangle:



Denote this 3 2

Reflection necessary otherwise we would get 5.

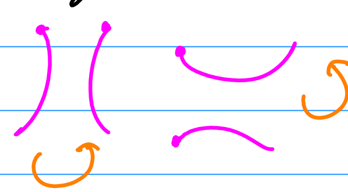
We could continue: reflect 3 2 then add a −4:



Denoted 3 2 −4

This construction gives <u>rational tangles</u>

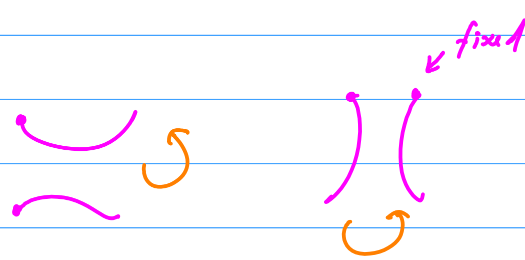
Create by alternating twists.



even number of integers

A positive twist always gives the overstrand a positive slope.

odd number of integers:

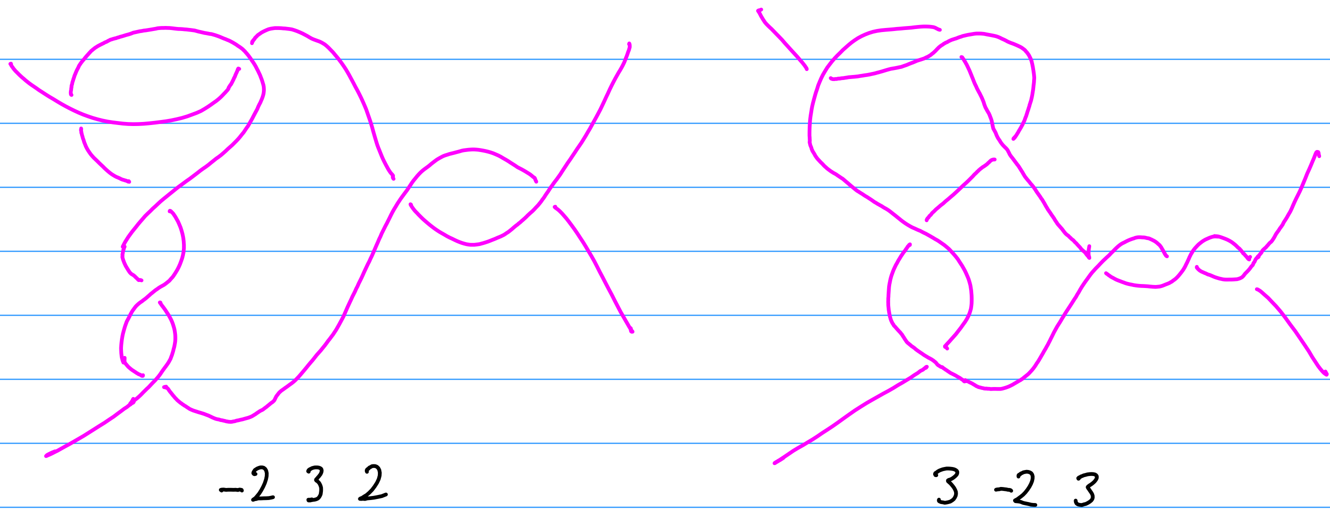


Why? Just a convention to agree with reflection picture above.

Here is the awesome thing: recall from earlier continued fractions:

example: Write $-2\ 3\ 2$ as $2 + \cfrac{1}{3 + \cfrac{1}{-2}} = \frac{12}{5}$

Write $3\ -2\ 3$ as $3 + \cfrac{1}{-2 + \cfrac{1}{3}} = \frac{12}{5}$



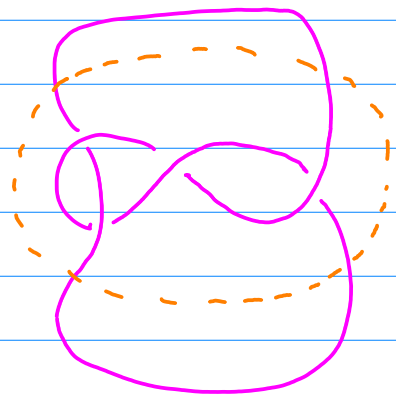
In fact these are equivalent tangles! This is true in general:

Two rational tangles are equivalent iff their continued fractions are equal.

Unfortunately, the proof of this is difficult. (See refs in Adams)

If we close up the tangle above and below we get a rational link

example: the figure-8 knot is a rational link formed from the tangle 2 2:



This is Conway's notation for rat. links.

this knot is denoted 2 2

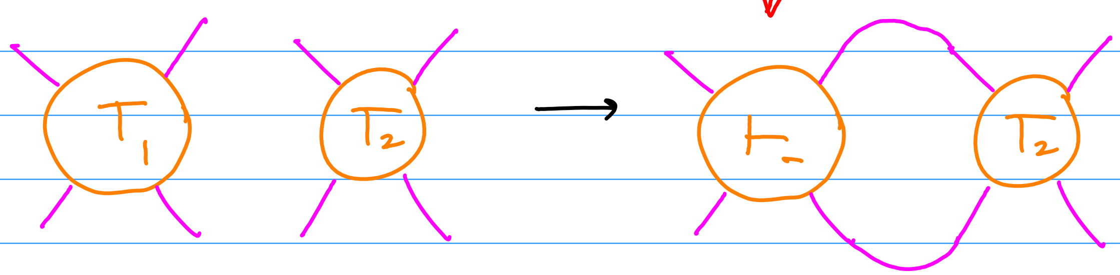
Rational links have either one or two components.

when are there two?

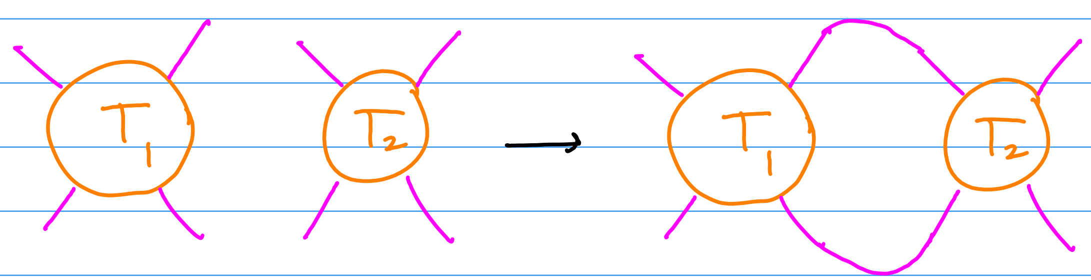
All rational links are alternating.

Can also "multiply" tangles

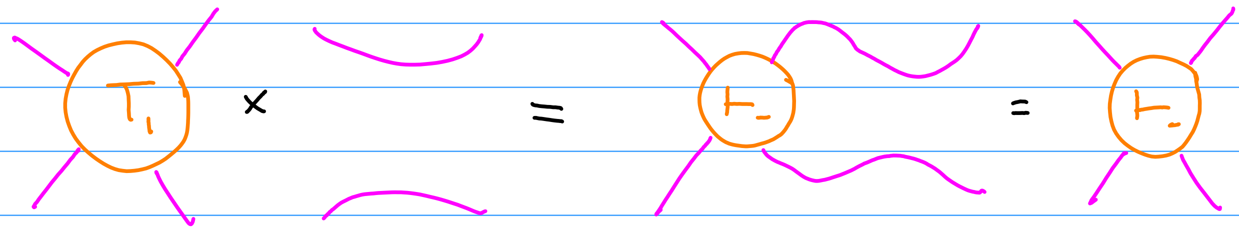
reflection makes it compatible with earlier construction



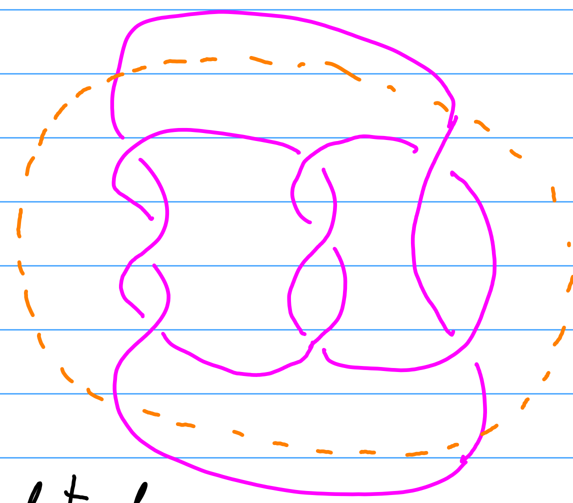
or "add" them:



Note that multiplication by 0 tangle is reflection:



The 8_5 knot, for example, can be written $3\,0 + 3\,0 + 2\,0$:



The standard notation for this is to place a comma:

$3\;0 + 3\;0 + 2\;0 = 3,3,2.$

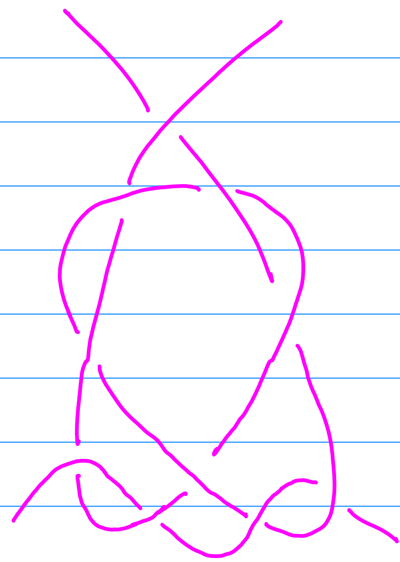
Tangles obtained by addition and multiplication of rational tangles are called <u>algebraic tangles</u>.

Lecture 36: Polynomial invariants 4/29/11

Recall: algebraic tangles obtained from addition and multiplication of rational tangles.

Closing them give algebraic links, such as 8_5.

However, not all tangles are algebraic!



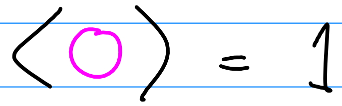
big jump in Adams (p. 149)

Now on to polynomials. We start by constructing the bracket polynomial. Let K be a knot, and associate with it a polynomial $\langle K \rangle$.

(Here we mean Laurent polynomials, which may include negative powers: $t^{-3} + 2t^{-1} + 1 + t^3$, for example.)

We will construct $\langle K \rangle$ to be a knot invariant.

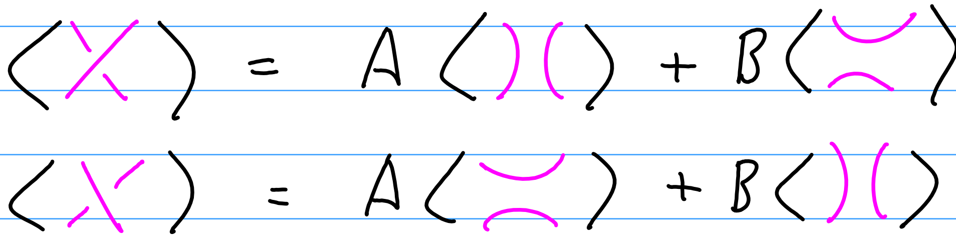
Start with the unknot and define:



Rule 1

Now at a crossing, we demand something like this:

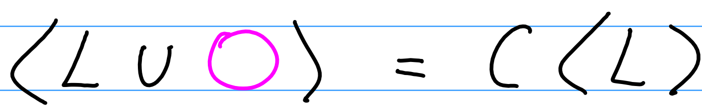
Rule 2



this is the same, but rotated

This is called a skein relation. A, B will be determined later.

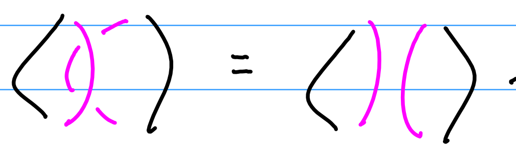
Finally, let's look at disjoint union with the unknot: if L is a link, let



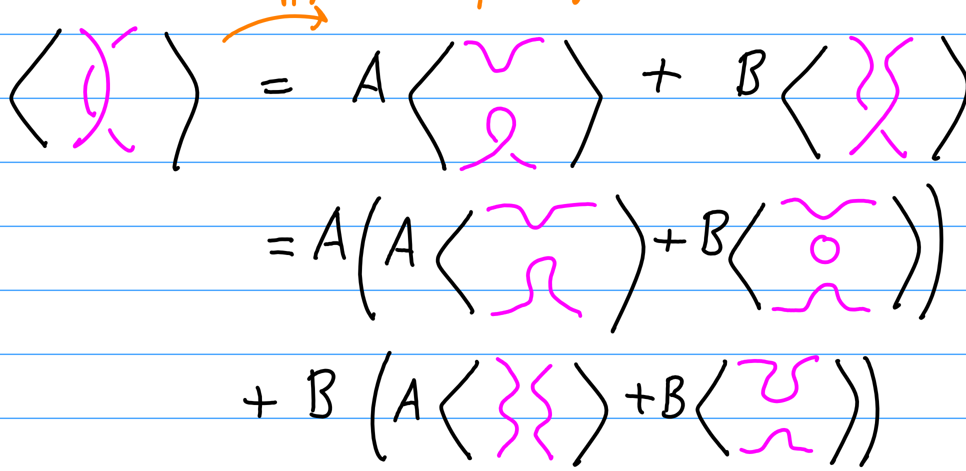
Rule 3

The real test of these rules is whether we can choose A, B, C such that the polynomial is invariant under Reidemeister moves.

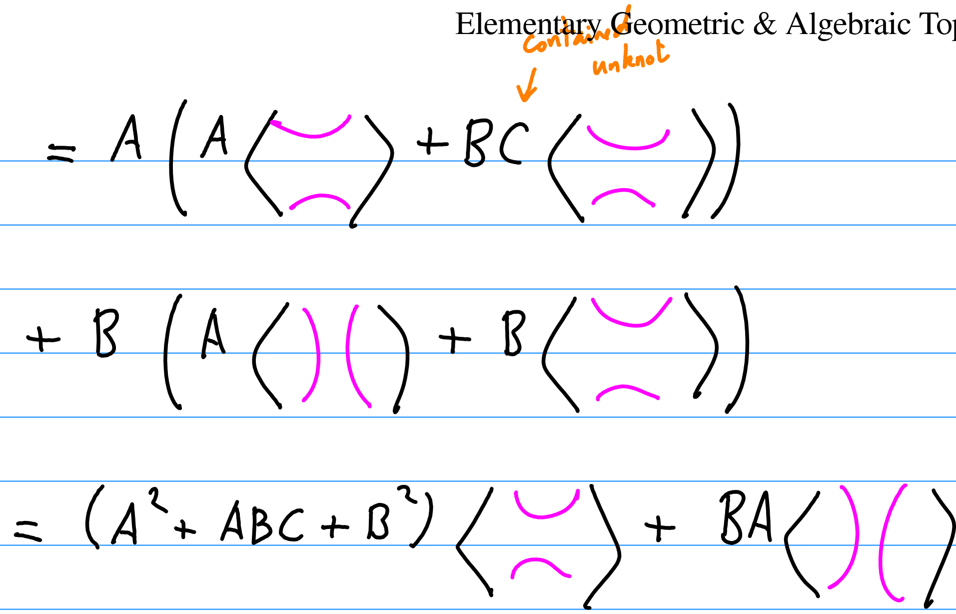
Start with type II,

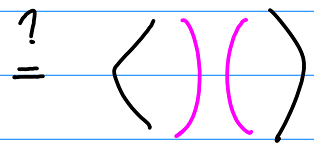


apply rule 2 to top crossing



now apply rule 2 to bottom crossing



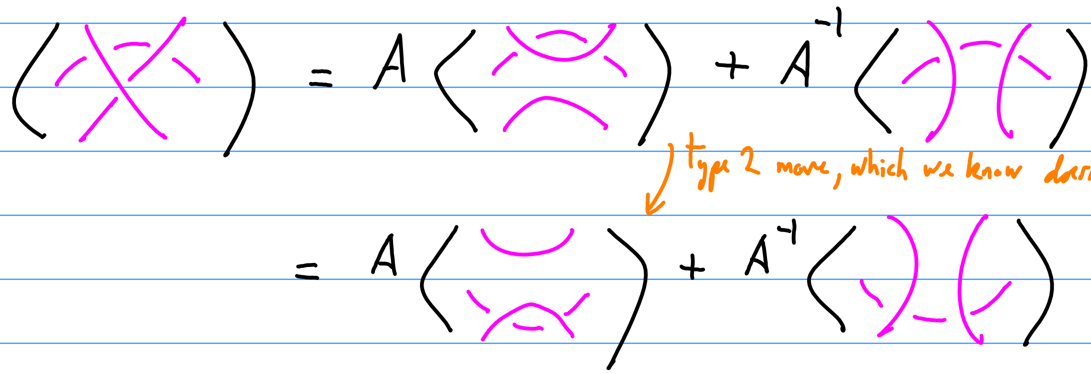


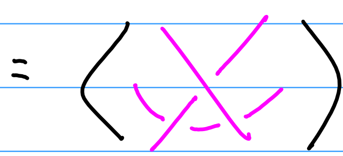
Hence, $B = A^{-1}$, and $A^2 + C + A^{-2} = 0$

$C = -A^2 - A^{-2}$

Only one variable left, C!

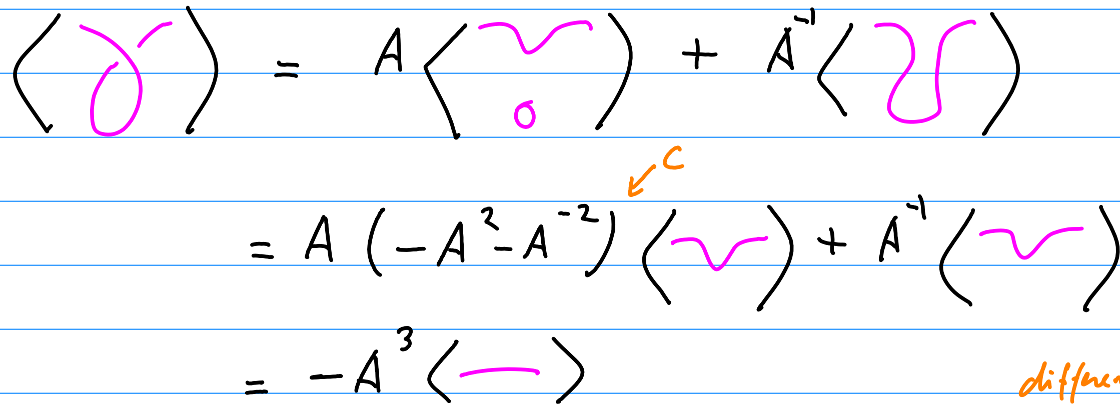
Type II:





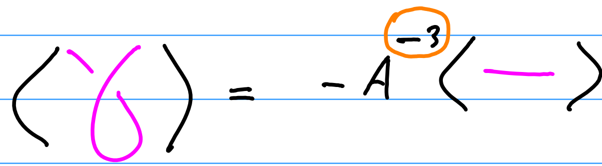
Get type III for free, given that type II works!

Now for type I.



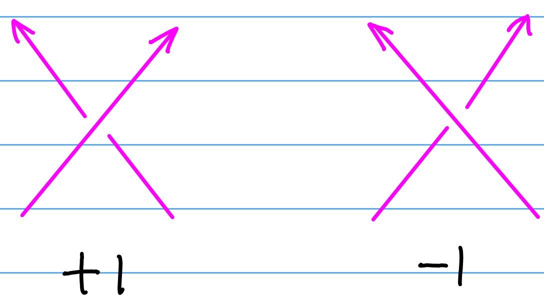
different sign!

Looks good! Except that

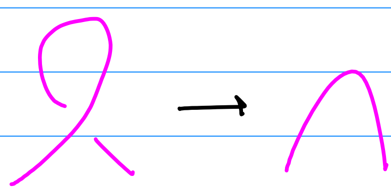


Ooops! But we can fix this.

Recall our linking number, defined by adding signs of crossings:



We can define the same thing using every crossing *(earlier we only used crossings between different components)*, and sum them.
This is called the <u>writhe</u> of an oriented link projection.



Type I moves change writhe by ±1.

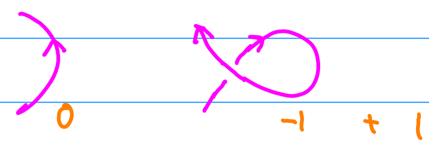
NOT A KNOT INVARIANT!

So define a new polynomial

$$X(L) = (-A^3)^{-w(L)} \langle L \rangle$$

$w(L)$ and $\langle L \rangle$ both unaffected by type II, III moves.

After type I move: $w(L') = w(L) + 1$



$$-A^{-3}\langle L' \rangle = \langle L \rangle \qquad \text{(earlier)}$$

$$X(L') = (-A^3)^{-w(L')}\langle L' \rangle = (-A^3)^{-w(L)-1}(-A^3)\langle L \rangle$$

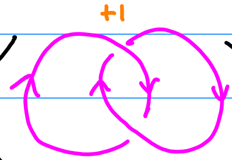
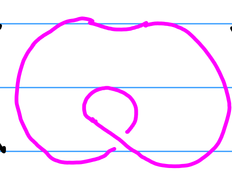
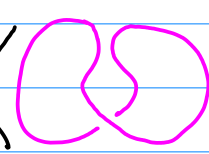
$$= (-A^3)^{-w(L)}\langle L \rangle$$

$$= X(L) \; !$$

polynomial in $t^{1/2}$

Replacing A by $t^{-1/4}$ in X gives us the Jones polynomial.

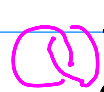
example:

$$\Big\langle$$  $$\Big\rangle = A \Big\langle$$  $$\Big\rangle + A^{-1}\Big\langle$$  $$\Big\rangle$$

Adams 151

$w = 2$

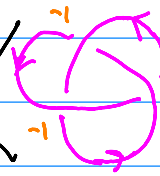
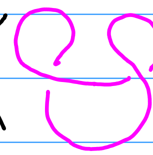
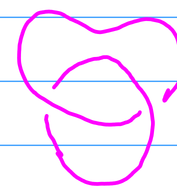
$$= A(-A^3)\langle \bigcirc \rangle + A^{-1}(-A^{-3})\langle \bigcirc \rangle = -A^4 - A^{-4}$$

$$X = (-A^3)^{-w}\Big\langle$$  $$\Big\rangle = (-A^3)^{-2}(-A^4 - A^{-4}) = -A^{-6}(A^4 + A^{-4})$$

$(A \to t^{-1/4})$

The Jones polynomial is then $-t^{3/2}(t+t^{-1}) = -t^{1/2}(1+t^2)$

example:

$\Big\langle$  $\Big\rangle = A \Big\langle$  $\Big\rangle + A^{-1} \Big\langle$  $\Big\rangle$

$wr = -3$

$$= A(-A^3)(-A^3)\langle \bigcirc \rangle + A^{-1}(-A^4 - A^{-4})$$

$$= A^7 - A^{-1}(A^4 + A^{-4}) = A^7 - A^3 - A^{-5}$$

Adams p.158

$$X = (-A^3)^3 (A^7 - A^3 - A^{-5}) = -A^9(A^7 - A^3 - A^{-5})$$

$$= A^4 + A^{12} - A^{16}$$

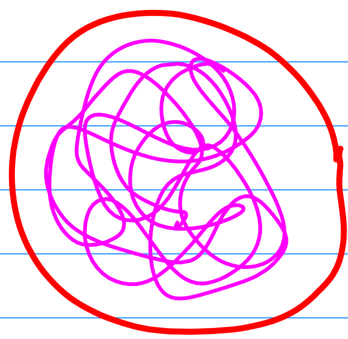
Jones polynomial is then $t + t^3 - t^4$

Lecture 37: DNA and knot theory

5/02/11

Adams 7.1

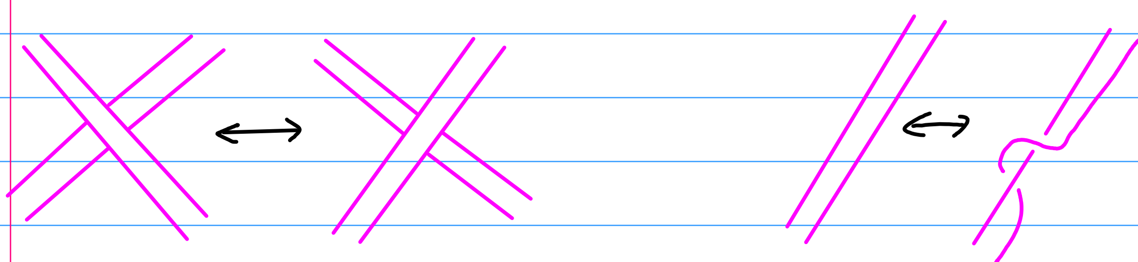
Everyone is familiar with the "double-helix" structure of DNA.

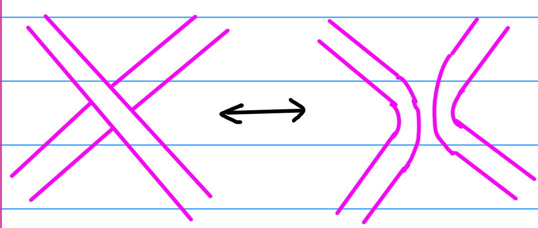


Extremely long DNA molecules are packed into cell nuclei

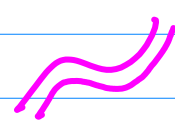
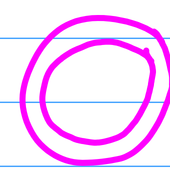
When it comes time to duplicate DNA, something has to deal with this mess!

Class of enzymes: Topoisomerases





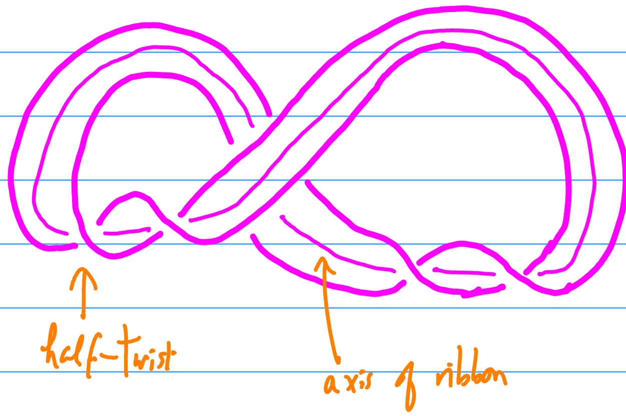
These enzymes locally change the topology of DNA.

Now most DNA is linear:  But some is cyclic: 

If one wants to study the role of topoisomerases, better to work with cyclic DNA, since then we can "detect" topological changes.

Thus we consider "cyclic duplex DNA".

This is modelled as a ribbon:



Because of the chemistry of the ends of DNA strands, there must be an <u>even number</u> of <u>half-twists</u>.

For now treat this as a rigid structure in space.

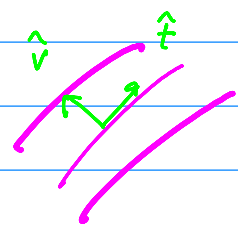
Let's define the <u>twist</u>, which measures how much the ribbon twists around its axis,

$$Tw(R)$$

If axis is flat in plane, the $Tw(R)$ is $\frac{1}{2}$(sum of half-twists)

+1 or −1

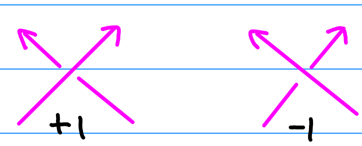
Otherwise use an integral definition.



$$Tw(R) = \frac{1}{2\pi}\int_{\text{axis}} (\hat{t} \times \hat{v}) \cdot \frac{d\hat{v}}{ds}\, ds$$

Next, the <u>writhe</u> $Wr(R)$:

The <u>signed crossover number</u> is the sum of signed crossovers of the axis with itself.



The writhe is the average of the signed crossover number over all possible projections. Amazingly enough, this is obtained by an integral formula due to Gauss:

$$Wr(K) = \frac{1}{4\pi} \iint_{\text{axis}\ \text{axis}} \frac{(dp \times dq) \cdot (p-q)}{|p-q|^3}$$

Finally, the linking number $Lk(R)$ is the linking number (as defined previously) of the two boundaries of the ribbon.

In fact this is almost the same integral as Wr but with different domain:

$$Lk(K) = \frac{1}{4\pi} \iint_{b_1\ b_2} \frac{(dp \times dq) \cdot (p-q)}{|p-q|^3}$$

$\leftarrow$ the two boundaries

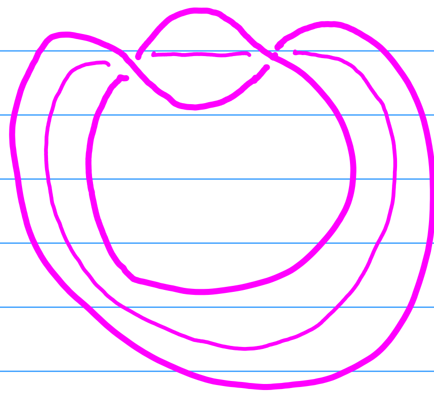
Tw, Wr, and Lk depend on the "rigid structure" but not the way it is projected. (Lk better than that)

Now for the great formula:

$$\boxed{Lk(R) = Tw(R) + Wr(R)}$$

If we isotope our ribbon (relax rigidity), then Lk can't change, so changes in Tw and Wr must cancel out.

examples:

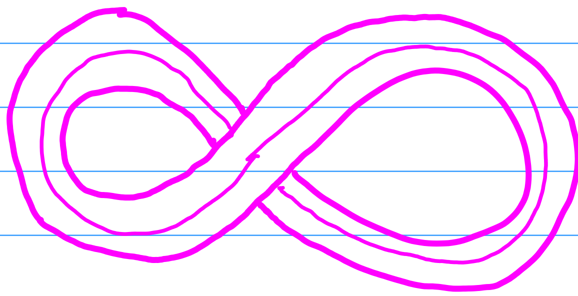


$Lk(R) = +1$

$Tw(R) = +1$

$Wr(R) = 0$

There are the same ribbon!

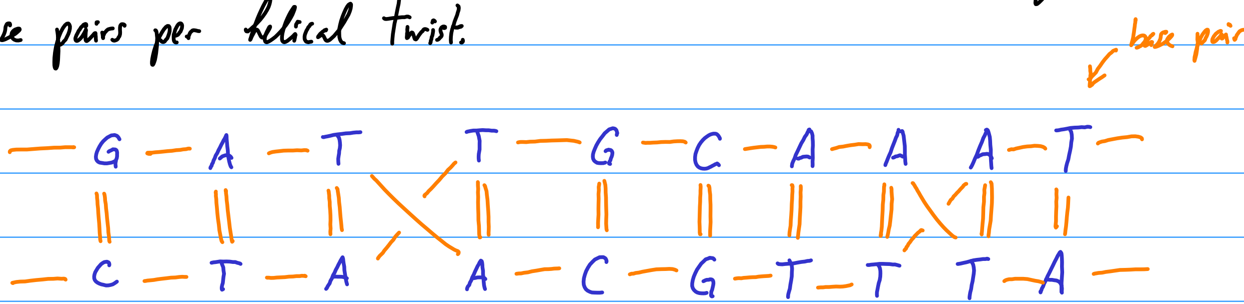


$Lk(R) = +1$

$Tw(R) = 0$

$Wr(R) = +1$

DNA in its relaxed state twists around its axis at a rate of 10.5 base pairs per helical twist.



So if we lie flat in the plane a piece of cyclic duplex DNA with 105 base pairs, it will have

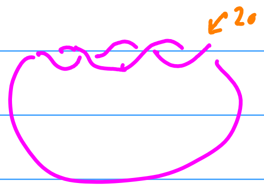
$$Tw = 10, \quad Lk = 10, \quad Wr = 0$$

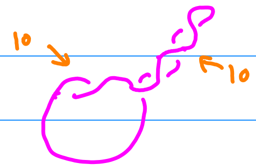
But now assume it is not relaxed: in the plane with $Wr = 0$, say we increase Tw to 20, Lk to 20. (5.25 base pairs per twist)

Then the DNA will want to relax to its natural state of 10.5 base pairs per twist.

That means Tw decreases to 10, but Wr must increase by 10!

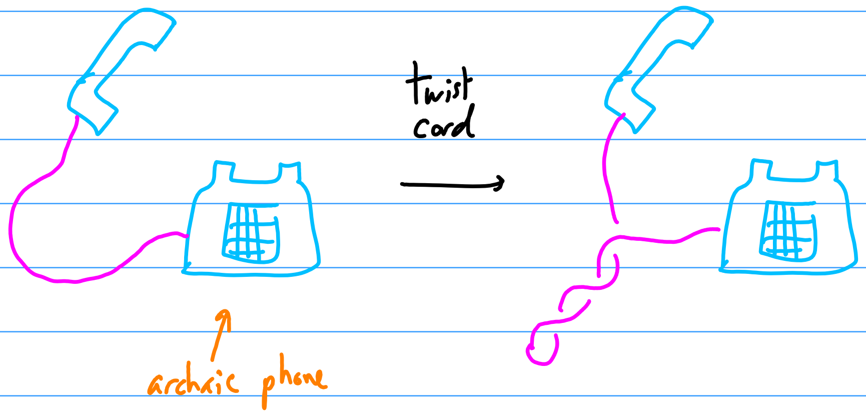
$20 = 20 + 0 \implies 20 = 10 + 10$



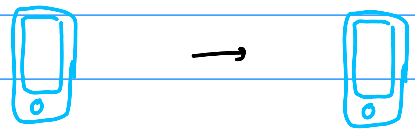


SUPERCOILING

This is the well-known "telephone cord" effect:



Modern version:



(not as helpful)