Supplement: Legendre's equation

$$[(1-x^2)\phi']' + \lambda\phi = 0 \quad , \quad -1 < x < 1$$

$s(x) = 1 - x^2$, $\rho(x) = q(x) = 1$

Eigensolutions with different λ's are orthogonal, since $s(\pm 1) = 0$.

So by series:
$$\phi(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$\phi'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$$
$$\phi''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute in: $(1-x^2)\phi'' - 2x\phi' + \lambda\phi = 0$

Need to relabel ϕ'' sum: let $n' = n-2$

$$\phi''(x) = \sum_{n'=-2}^{\infty} (n'+1)(n'+2)\, a_{n'+2}\, x^{n'}$$

Now drop the prime on n', and start sum at 0 since first two terms vanish anyways.

Combining the series, we have

$$(1-x^2)\phi'' - 2x\phi' + \lambda\phi$$

$$= \sum_{n=0}^{\infty} \left[(n+1)(n+2)a_{n+2} - n(n-1)a_n - 2na_n + \lambda a_n\right]x^n = 0$$

Since series (assuming convergent!) $= 0$, set coefficient of x^n to 0:

$$(n+1)(n+2)a_{n+2} - n(n-1)a_n - 2na_n + \lambda a_n = 0$$

$$a_{n+2} = \frac{n(n-1) + 2n - \lambda}{(n+1)(n+2)} a_n$$

$$\boxed{a_{n+2} = \frac{n(n+1) - \lambda}{(n+1)(n+2)} a_n}$$

Recurrence relation for coefficients

Two solutions: $a_0, a_2, a_4, \ldots$ (even)

$a_1, a_3, a_5, \ldots$ (odd)

But do these converge? Ratio test.

$$R_n = \left|\frac{a_{n+2}x^{n+2}}{a_n x^n}\right| = \left|\frac{n(n+1)-\lambda}{(n+1)(n+2)}\right| |x|^2$$

$$\lim_{n\to\infty} R_n = R = |x|^2.$$

So converges absolutely for $|x|<1$.

What about $|x|=1$? More difficult.

For n large, note that

$$a_{n+2} = \frac{n(n+1)-\lambda}{(n+1)(n+2)}\, a_n \approx \frac{n}{n+2}\, a_n$$

So $$a_{n+4} \approx \frac{n+2}{n+4}\, a_{n+2} = \frac{\cancel{n+2}}{n+4}\,\frac{n}{\cancel{n+2}}\, a_n = \frac{n}{n+4}\, a_n$$

$$a_{n+6} \approx \frac{n+4}{n+6}\,\frac{n}{n+4}\, a_n = \frac{n}{n+6}\, a_n$$

Easy to see: $$a_{n+2m} \approx \frac{n}{n+2m}\, a_n$$

So as $m\to\infty$, $$a_{n+2m} \sim \frac{1}{2m}$$ divergent

$\sum a_n x^n$ is thus a divergent series at $|x|=1$, since it behaves like harmonic series $\sum \frac{1}{n}$.

(Signs don't alternate at $x=-1$, since only even/odd powers.)

We conclude: for each λ we have two independent solutions (even, odd), but these diverge at $|x|=1$ (one or both).

(This is tied to the fact that $(1-x^2)\phi'' + \cdots$ has vanishing coeff for $|x|=1$.)

Hence, we cannot construct regular solutions for general λ.

BUT: if $\lambda = m(m+1)$, $m = 0, 1, 2, 3, \cdots$

then the series terminates when $n = m$!

$\Rightarrow$ Legendre polynomials.

When m is even, $a_0 + a_2 x^2 + \cdots + a_m x^m$ terminates.

When m is odd, $a_1 x + a_3 x^3 + \cdots + a_m x^m$ terminates

For instance, for $m=0$, we have $\phi_{even}(x) = 1$ as a solution.

The other solution is given by the odd series:

$$a_{n+2} = \frac{n}{n+2} a_n \quad , \quad a_1 = 1$$

$$a_{1+2m} = \frac{1}{1+2m} a_1 \Rightarrow a_{2m+1} = \frac{1}{2m+1}$$

$m=0$: $$\phi_{even}(x) = 1 \quad , \quad \phi_{odd}(x) = \sum_{m=0}^{\infty} \frac{x^{2m+1}}{2m+1}$$

So, for $m = 0, 1, 2, 3, \cdots$, two solutions:

$P_m(x)$ are Legendre polynomials (L. functions of first kind)

$Q_m(x)$ are Legendre functions of the second kind

In practical problems, we usually throw out $Q_m(x)$ since we demand regular (bounded) solution in $[-1, 1]$.

Why do some solutions of Legendre's equation diverge?

$$(1-x^2)\phi'' - 2x\phi' + \lambda\phi = 0$$

This is 0 at $x = \pm 1$. Allows $\phi'' \to \infty$, with $(1-x^2)\phi''$ finite

Let's examine the blowup at $x=1$. Let $y = 1-x$, or $x = 1-y$. Then

$$(2-y)y\,\phi'' + 2(1-y)\phi' + \lambda\phi = 0$$

If ϕ blows up as $y \to 0^+$, so do ϕ', ϕ''. In fact ϕ' blows up faster than ϕ.

Let's show this. First prove:

Lemma: If $f(x)$ is continuously differentiable in $[a, b)$, $b > a$, and $\lim_{x\to b^-} f(x) = \infty$, then $\lim_{x \to b^-} f'(x) = \infty$.

proof: Assume $f'(x) < M$, $x \in [a, b)$.

The mean value theorem says $f'(c) = \dfrac{f(x) - f(a)}{x - a}$, $a \leq x < b$,

for some $c \in [a, x]$. But $f'(c) < M$, so

$$\frac{f(x) - f(a)}{x - a} < M \implies f(x) < f(a) + M(x - a).$$

But this implies $f(x)$ is bounded as $x \to b^-$, which contradicts the assumption. $\boxed{/\!/\!/}$

Corollary: $\displaystyle\lim_{x \to b^-} \frac{f'}{f} = \infty$

proof: $\displaystyle\lim_{x \to b^-} \frac{f'}{f} = \lim_{x \to b^-} (\log f)' = \lim_{x \to b^-} \psi'$, $\psi = \log f$.

But ψ has $\displaystyle\lim_{x \to b^-} \psi = \infty$, since ϕ itself goes to ∞.

Then by the Lemma $\displaystyle\lim_{x \to b^-} \psi' = \infty$. $\boxed{/\!/\!/}$

The Corollary says that ϕ' goes to ∞ infinitely faster that ϕ.

In the same way, $\phi'' \to \infty$ faster than ϕ'!

Thus: $|\phi''| \gg |\phi'| \gg |\phi|$.

Now back to Legendre's equation in the coordinate $y = 1 - x$:

$$(2-y)y\,\phi'' + 2(1-y)\phi' + \lambda\phi = 0$$

If $\phi \to \infty$ as $y \to 0^+$, then so do ϕ', ϕ''.

The largest terms in the equation, as $y \to 0^+$, are

$$2y\phi'' + 2\phi' = 0, \quad y \to 0^+$$

(At fixed λ, there is no way $\lambda\phi$ is as large as ϕ'.)

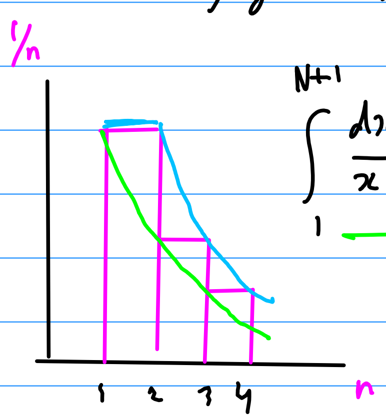
Hence: $\quad y\phi'' + \phi' = 0 \Rightarrow \phi' \sim \dfrac{1}{y}$

or $\quad \boxed{\phi(y) \sim \log y}$

Hence, the series solution diverges logarithmically as $y \to 0$ ($x \to \pm 1$).

This is consistent with the singularity being linked to the harmonic series, which diverges logarithmically:

$$\log(N+1) < \sum_{n=1}^{N} \frac{1}{n} \leq 1 + \log N$$



$$\int_1^{N+1} \frac{dx}{x} < \sum_{n=1}^{N} \frac{1}{n} \leq 1 + \int_2^{N+1} \frac{dx}{x-1}$$

Comes from this