Evaluating this at the endpoints yields

$$\left.\frac{dG(x,x_0)}{dx_0}\right|_{x_0=L} = \frac{x}{L} \quad \text{and} \quad \left.\frac{dG(x,x_0)}{dx_0}\right|_{x_0=0} = -\left(1-\frac{x}{L}\right).$$

Consequently,

$$u(x) = \int_0^L f(x_0)G(x,x_0)\,dx_0 + \beta\frac{x}{L} + \alpha\left(1-\frac{x}{L}\right). \tag{9.3.52}$$

The solution is the sum of a particular solution of (9.3.42) satisfying homogeneous boundary conditions obtained earlier, $\int_0^L f(x_0)G(x,x_0)\,dx_0$, and a homogeneous solution satisfying the two required nonhomogeneous boundary conditions, $\beta(x/L)+\alpha(1-x/L)$.

9.3.6 Summary

We have described three fundamental methods to obtain Green's functions:

1. Variation of parameters
2. Method of eigenfunction expansion
3. Using the defining differential equation for the Green's function

In addition, steady-state Green's functions can be obtained as the limit as $t \to \infty$ of the solution with steady sources. To obtain Green's functions for partial differential equations, we will discuss one important additional method. It will be described in Sec. 9.5.

EXERCISES 9.3

9.3.1. The Green's function for (9.3.1) is given explicitly by (9.3.16). The method of eigenfunction expansion yields (9.3.6). Show that the Fourier sine series of (9.3.16) yields (9.3.6).

9.3.2. (a) Derive (9.3.17).

(b) Integrate (9.3.17) by parts to derive (9.3.16).

(c) Instead of part (b), simplify the double integral in (9.3.17) by interchanging the orders of integration. Derive (9.3.16) this way.

9.3.3. Consider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t)$$

subject to $u(0,t)=0$, $\frac{\partial u}{\partial x}(L,t)=0$, and $u(x,0)=g(x)$.

(a) Solve by the method of eigenfunction expansion.

(b) Determine the Green's function for this time-dependent problem.

(c) If $Q(x,t) = Q(x)$, take the limit as $t \to \infty$ of part (b) in order to determine the Green's function for

$$\frac{d^2u}{dx^2} = f(x) \quad \text{with} \quad u(0) = 0 \quad \text{and} \quad \frac{du}{dx}(L) = 0.$$

9.3.4. (a) Derive (9.3.29) from (9.3.28) [*Hint*: Let $f(x) = 1$.]

(b) Show that (9.3.33) satisfies (9.3.31).

(c) Derive (9.3.30) [*Hint*: Show for any continuous $f(x)$ that

$$\int_{-\infty}^{\infty} f(x_0)\delta(x - x_0)\,dx_0 = \int_{-\infty}^{\infty} f(x_0)\delta(x_0 - x)\,dx_0$$

by letting $x_0 - x = s$ in the integral on the right.]

(d) Derive (9.3.34) [*Hint*: Evaluate $\int_{-\infty}^{\infty} f(x)\delta[c(x - x_0)]\,dx$ by making the change of variables $y = c(x - x_0)$.]

9.3.5. Consider

$$\frac{d^2u}{dx^2} = f(x) \quad \text{with} \quad u(0) = 0 \quad \text{and} \quad \frac{du}{dx}(L) = 0.$$

*(a) Solve by direct integration.

*(b) Solve by the method of variation of parameters.

*(c) Determine $G(x, x_0)$ so that (9.3.15) is valid.

(d) Solve by the method of eigenfunction expansion. Show that $G(x, x_0)$ is given by (9.3.23).

9.3.6. Consider

$$\frac{d^2G}{dx^2} = \delta(x - x_0) \quad \text{with} \quad G(0, x_0) = 0 \quad \text{and} \quad \frac{dG}{dx}(L, x_0) = 0.$$

*(a) Solve directly.

*(b) Graphically illustrate $G(x, x_0) = G(x_0, x)$.

(c) Compare to Exercise 9.3.5.

9.3.7. Redo Exercise 9.3.5 with the following change: $\frac{du}{dx}(L) + hu(L) = 0,\ h > 0.$

9.3.8. Redo Exercise 9.3.6 with the following change: $\frac{dG}{dx}(L) + hG(L) = 0,\ h > 0.$

9.3.9. Consider

$$\frac{d^2u}{dx^2} + u = f(x) \quad \text{with} \quad u(0) = 0 \quad \text{and} \quad u(L) = 0.$$

Assume that $(n\pi/L)^2 \neq 1$ (i.e., $L \neq n\pi$ for any n).

(a) Solve by the method of variation of parameters.

*(b) Determine the Green's function so that $u(x)$ may be represented in terms of it [see (9.3.15)].

9.3.10. Solve the problem of Exercise 9.3.9 using the method of eigenfunction expansion.

9.3.11. Consider

$$\frac{d^2G}{dx^2} + G = \delta(x - x_0) \quad \text{with} \quad G(0, x_0) = 0 \quad \text{and} \quad G(L, x_0) = 0.$$

*(a) Solve for this Green's function directly. Why is it necessary to assume that $L \neq n\pi$?

(b) Show that $G(x, x_0) = G(x_0, x)$.

9.3.12. For the following problems, determine a representation of the solution in terms of the Green's function. Show that the nonhomogeneous boundary conditions can also be understood using homogeneous solutions of the differential equation:

(a) $\frac{d^2u}{dx^2} = f(x)$, $u(0) = A$, $\frac{du}{dx}(L) = B$. (See Exercise 9.3.6.)

(b) $\frac{d^2u}{dx^2} + u = f(x)$, $u(0) = A$, $u(L) = B$. Assume $L \neq n\pi$. (See Exercise 9.3.11.)

(c) $\frac{d^2u}{dx^2} = f(x)$, $u(0) = A$, $\frac{du}{dx}(L) + hu(L) = 0$. (See Exercise 9.3.8.)

9.3.13. Consider the one-dimensional infinite space wave equation with a periodic source of frequency ω:

$$\frac{\partial^2\phi}{\partial t^2} = c^2\frac{\partial^2\phi}{\partial x^2} + g(x)e^{-i\omega t}. \tag{9.3.53}$$

(a) Show that a particular solution $\phi = u(x)e^{-i\omega t}$ of (9.3.53) is obtained if u satisfies a nonhomogeneous Helmholtz equation

$$\frac{d^2u}{dx^2} + k^2u = f(x).$$

*(b) The Green's function $G(x, x_0)$ satisfies

$$\frac{d^2G}{dx^2} + k^2G = \delta(x - x_0).$$

Determine this infinite space Green's function so that the corresponding $\phi(x, t)$ is an outward-propagating wave.

(c) Determine a particular solution of (9.3.53).

9.3.14. Consider $L(u) = f(x)$ with $L = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q$. Assume that the appropriate Green's function exists. Determine the representation of $u(x)$ in terms of the Green's function if the boundary conditions are nonhomogeneous:

(a) $u(0) = \alpha$ and $u(L) = \beta$

(b) $\frac{du}{dx}(0) = \alpha$ and $\frac{du}{dx}(L) = \beta$

(c) $u(0) = \alpha$ and $\frac{du}{dx}(L) = \beta$

*(d) $u(0) = \alpha$ and $\frac{du}{dx}(L) + hu(L) = \beta$

9.3.15. Consider $L(G) = \delta(x - x_0)$ with $L = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q$ subject to the boundary conditions $G(0, x_0) = 0$ and $G(L, x_0) = 0$. Introduce for all x two homogeneous solutions, y_1 and y_2, such that each solves one of the homogeneous boundary conditions:

$$\begin{array}{ll} L(y_1) = 0 & L(y_2) = 0 \\ y_1(0) = 0 & y_2(L) = 0 \\ \frac{dy_1}{dx}(0) = 1 & \frac{dy_2}{dx}(L) = 1. \end{array}$$

Even if y_1 and y_2 cannot be explicitly obtained, they can be easily calculated numerically on a computer as two *initial value problems*. Any homogeneous solution must be a linear combination of the two.

*(a) Solve for $G(x, x_0)$ in terms of $y_1(x)$ and $y_2(x)$. You may assume that $y_1(x) \neq cy_2(x)$.

(b) What goes wrong if $y_1(x) = cy_2(x)$ for all x and why?

9.3.16. Reconsider (9.3.41), whose solution we have obtained, (9.3.46). For (9.3.41), what is y_1 and y_2 in Exercise 9.3.15? Show that $G(x, x_0)$ obtained in Exercise 9.3.15 reduces to (9.3.46) for (9.3.41).

9.3.17. Consider

$$L(u) = f(x) \qquad \text{with} \qquad L = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q$$

$$u(0) = 0 \qquad \text{and} \qquad u(L) = 0.$$

Introduce two homogeneous solutions y_1 and y_2, as in Exercise 9.3.15.

(a) Determine $u(x)$ using the method of variation of parameters.

(b) Determine the Green's function from part (a).

(c) Compare to Exercise 9.3.15.

9.3.18. Reconsider Exercise 9.3.17. Determine $u(x)$ by the method of eigenfunction expansion. Show that the Green's function satisfies (9.3.23).

9.3.19. (a) If a concentrated source is placed at a node of some mode (eigenfunction), show that the amplitude of the response of that mode is zero. [*Hint:* Use the result of the method of eigenfunction expansion and recall that a node x^* of an eigenfunction means anyplace where $\phi_n(x^*) = 0$.]

(b) If the eigenfunctions are $\sin n\pi x/L$ and the source is located in the middle, $x_0 = L/2$, show that the response will have no even harmonics.

9.3.20. Derive the eigenfunction expansion of the Green's function (9.3.23) directly from the defining differential equation (9.3.41) by letting

$$G(x, x_0) = \sum_{n=1}^{\infty} a_n \phi_n(x).$$

Assume that term-by-term differentiation is justified.

*9.3.21. Solve

$$\frac{dG}{dx} = \delta(x - x_0) \quad \text{with} \quad G(0, x_0) = 0.$$

Show that $G(x, x_0)$ is not symmetric even though $\delta(x - x_0)$ is.

9.3.22. Solve

$$\frac{dG}{dx} + G = \delta(x - x_0) \quad \text{with} \quad G(0, x_0) = 0.$$

Show that $G(x, x_0)$ is not symmetric even though $\delta(x - x_0)$ is.

9.3.23. Solve

$$\frac{d^4G}{dx^4} = \delta(x - x_0)$$

$$G(0, x_0) = 0 \qquad G(L, x_0) = 0$$

$$\frac{dG}{dx}(0, x_0) = 0 \qquad \frac{d^2G}{dx^2}(L, x_0) = 0.$$

9.3.24. Use Exercise 9.3.23 to solve

$$\frac{d^4u}{dx^4} = f(x)$$

$$u(0) = 0 \qquad u(L) = 0$$

$$\frac{du}{dx}(0) = 0 \qquad \frac{d^2u}{dx^2}(L) = 0.$$

(*Hint:* Exercise 5.5.8 is helpful.)

9.3.25. Use the convolution theorem for Laplace transforms to obtain particular solutions of

(a) $\dfrac{d^2u}{dx^2} = f(x)$ (See Exercise 9.3.5.)

*(b) $\dfrac{d^4u}{dx^4} = f(x)$ (See Exercise 9.3.24.)

9.3.26 Determine the Green's function satisfying $\frac{d^2G}{dx^2} - G = \delta(x - x_0)$:

(a) Directly on the interval $0 < x < L$ with $G(0, x_0) = 0$ and $G(L, x_0) = 0$
(b) Directly on the interval $0 < x < L$ with $G(0, x_0) = 0$ and $\frac{dG}{dx}(L, x_0) = 0$
(c) Directly on the interval $0 < x < L$ with $\frac{dG}{dx}(0, x_0) = 0$ and $\frac{dG}{dx}(L, x_0) = 0$
(d) Directly on the interval $0 < x < \infty$ with $G(0, x_0) = 0$
(e) Directly on the interval $0 < x < \infty$ with $\frac{dG}{dx}(0, x_0) = 0$
(f) Directly on the interval $-\infty < x < \infty$

Appendix to 9.3: Establishing Green's Formula with Dirac Delta Functions

Green's formula is very important when analyzing Green's functions. However, our derivation of Green's formula requires integration by parts. Here we will show that Green's formula,

$$\int_a^b [uL(v) - vL(u)]\,dx = p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b, \text{ where } L = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q \quad (9.3.54)$$

is valid even if v is a Green's function,

$$L(v) = \delta(x - x_0). \quad (9.3.55)$$

We will derive (9.3.54). We calculate the left-hand side of(9.3.54). Since there is a singularity at $x = x_0$, we are not guaranteed that (9.3.54) is valid. Instead, we divide the region into three parts:

$$\int_a^b = \int_a^{x_0-} + \int_{x_0-}^{x_0+} + \int_{x_0+}^b.$$

In the regions that exclude the singularity, $a \le x \le x_{0-}$ and $x_{0+} \le x \le b$, Green's formula can be used. In addition, due to the property of the Dirac delta function,

$$\int_{x_0-}^{x_0+} [uL(v) - vL(u)]\ dx = \int_{x_0-}^{x_0+} [u\delta(x - x_0) - vL(u)]\ dx = u(x_0),$$

since $\int_{x_0-}^{x_0+} vL(u)\ dx = 0$. Thus, we obtain

$$\begin{aligned}\int_a^b [uL(v) - vL(u)]\ dx &= p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^{x_0-} + p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_{x_0+}^b + u(x_0)\\ &= p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b + p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_{x_0+}^{x_0-} + u(x_0).\end{aligned} \quad (9.3.56)$$