The solution of this does not require (8.3.10):

$$a_n(t) = \begin{cases} a_n(0)e^{-n^2 t} & n \neq 3 \\ \frac{1}{8}e^{-t} + [a_3(0) - \frac{1}{8}]e^{-9t} & n = 3, \end{cases} \tag{8.3.12}$$

where

$$a_n(0) = \frac{2}{\pi}\int_0^\pi \left[f(x) - \frac{x}{\pi}\right] \sin nx \, dx. \tag{8.3.13}$$

The solution to the original nonhomogeneous problem is given by $u(x,t) = v(x,t) + x/\pi$, where v satisfies (8.3.11) with $a_n(t)$ determined from (8.3.12) and (8.3.13).

EXERCISES 8.3

8.3.1. Solve the initial value problem for the heat equation with time-dependent sources

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} + Q(x,t) \\ u(x,0) &= f(x) \end{aligned}$$

subject to the following boundary conditions:

(a) $u(0,t) = 0,$ $\quad \frac{\partial u}{\partial x}(L,t) = 0$

(b) $u(0,t) = 0,$ $\quad u(L,t) + 2\frac{\partial u}{\partial x}(L,t) = 0$

* (c) $u(0,t) = A(t),$ $\quad \frac{\partial u}{\partial x}(L,t) = 0$

(d) $u(0,t) = A \neq 0,$ $\quad u(L,t) = 0$

(e) $\frac{\partial u}{\partial x}(0,t) = A(t),$ $\quad \frac{\partial u}{\partial x}(L,t) = B(t)$

* (f) $\frac{\partial u}{\partial x}(0,t) = 0,$ $\quad \frac{\partial u}{\partial x}(L,t) = 0$

(g) Specialize part (f) to the case $Q(x,t) = Q(x)$ (independent of t) such that $\int_0^L Q(x)\,dx \neq 0$. In this case show that there arc no time-independent solutions. What happens to the time-dependent solution as $t \to \infty$? Briefly explain.

8.3.2. Consider the heat equation with a steady source

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x)$$

subject to the initial and boundary conditions described in this section:

$$u(0,t) = 0, u(L,t) = 0, \quad \text{and} \quad u(x,0) = f(x).$$

Obtain the solution by the method of eigenfunction expansion. Show that the solution approaches a steady-state solution.

*8.3.3. Solve the initial value problem

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + qu + f(x,t),$$

where c, ρ, K_0, and q are functions of x only, subject to the conditions

$$u(0,t) = 0, u(L,t) = 0, \quad \text{and} \quad u(x,0) = g(x).$$

Assume that the eigenfunctions are known. [*Hint*: let $L \equiv \frac{d}{dx}\left(K_0 \frac{d}{dx}\right) + q$.]

8.3.4. Consider

$$\frac{\partial u}{\partial t} = \frac{1}{\sigma(x)} \frac{\partial}{\partial x}\left[K_0(x) \frac{\partial u}{\partial x}\right] \qquad (K_0 > 0, \sigma > 0)$$

with the boundary conditions and initial conditions:

$$u(x,0) = g(x), u(0,t) = A, \quad \text{and} \quad u(L,t) = B.$$

*(a) Find a time-independent solution, $u_0(x)$.

(b) Show that $\lim_{t\to\infty} u(x,t) = f(x)$ independent of the initial conditions. [Show that $f(x) = u_0(x)$.]

*8.3.5. Solve

$$\frac{\partial u}{\partial t} = k\nabla^2 u + f(r,t)$$

inside the circle ($r < a$) with $u = 0$ at $r = a$ and initially $u = 0$.

8.3.6. Solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \sin 5x \; e^{-2t}$$

subject to $u(0,t) = 1, u(\pi,t) = 0$, and $u(x,0) = 0$.

*8.3.7. Solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

subject to $u(0,t) = 0, u(L,t) = t$, and $u(x,0) = 0$.

8.4 Method of Eigenfunction Expansion Using Green's Formula (With or Without Homogeneous Boundary Conditions)

In this section we reinvestigate problems that may have nonhomogeneous boundary conditions. We still use the method of eigenfunction expansion. For example, consider

$$\text{PDE:} \quad \boxed{\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t)} \tag{8.4.1}$$