In general, the partial differential equation for $v(x,t)$ is of the same type as for $u(x,t)$, but with a different nonhomogeneous term, since $r(x,t)$ usually does not satisfy the homogeneous heat equation. The initial condition is also usually altered:

$$v(x,0) = f(x) - r(x,0) = f(x) - A(0) - \frac{x}{L}[B(0) - A(0)] \equiv g(x). \qquad (8.2.29)$$

It can be seen that in general only the boundary conditions have been made homogeneous. In Sec. 8.3 we will develop a method to analyze nonhomogeneous problems with homogeneous boundary conditions.

## EXERCISES 8.2

8.2.1. Solve the heat equation with time-independent sources and boundary conditions

$$\begin{aligned}\frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} + Q(x)\\ u(x,0) &= f(x)\end{aligned}$$

if an equilibrium solution exists. Analyze the limits as $t \to \infty$. If no equilibrium exists, explain why and reduce the problem to one with homogeneous boundary conditions (but do not solve). Assume

| | | | |
|---|---|---|---|
| * (a) | $Q(x) = 0,$ | $u(0,t) = A,$ | $\frac{\partial u}{\partial x}(L,t) = B$ |
| (b) | $Q(x) = 0,$ | $\frac{\partial u}{\partial x}(0,t) = 0,$ | $\frac{\partial u}{\partial x}(L,t) = B \neq 0$ |
| (c) | $Q(x) = 0,$ | $\frac{\partial u}{\partial x}(0,t) = A \neq 0,$ | $\frac{\partial u}{\partial x}(L,t) = A$ |
| * (d) | $Q(x) = k,$ | $u(0,t) = A,$ | $u(L,t) = B$ |
| (e) | $Q(x) = k,$ | $\frac{\partial u}{\partial x}(0,t) = 0,$ | $\frac{\partial u}{\partial x}(L,t) = 0$ |
| (f) | $Q(x) = \sin\frac{2\pi x}{L},$ | $\frac{\partial u}{\partial x}(0,t) = 0,$ | $\frac{\partial u}{\partial x}(L,t) = 0$ |

8.2.2. Consider the heat equation with time-dependent sources and boundary conditions:

$$\begin{aligned}\frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} + Q(x,t)\\ u(x,0) &= f(x).\end{aligned}$$

Reduce the problem to one with homogeneous boundary conditions if

| | | | |
|---|---|---|---|
| * (a) | $\frac{\partial u}{\partial x}(0,t) = A(t)$ | and | $\frac{\partial u}{\partial x}(L,t) = B(t)$ |
| (b) | $u(0,t) = A(t)$ | and | $\frac{\partial u}{\partial x}(L,t) = B(t)$ |
| * (c) | $\frac{\partial u}{\partial x}(0,t) = A(t)$ | and | $u(L,t) = B(t)$ |
| (d) | $u(0,t) = 0$ | and | $\frac{\partial u}{\partial x}(L,t) + h(u(L,t) - B(t)) = 0$ |
| (e) | $\frac{\partial u}{\partial x}(0,t) = 0$ | and | $\frac{\partial u}{\partial x}(L,t) + h(u(L,t) - B(t)) = 0$ |

8.2.3. Solve the two-dimensional heat equation with circularly symmetric time-independent sources, boundary conditions, and initial conditions (inside a circle):

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + Q(r)$$

with

$$u(r,0) = f(r) \quad \text{and} \quad u(a,t) = T.$$

8.2.4. Solve the two-dimensional heat equation with time-independent boundary conditions:

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

subject to the boundary conditions

$$\begin{aligned} u(0,y,t) &= 0 & \tfrac{\partial}{\partial y}u(x,0,t) &= 0 \\ u(L,y,t) &= 0 & u(x,H,t) &= g(x) \end{aligned}$$

and the initial condition

$$u(x,y,0) = f(x,y).$$

Analyze the limit as $t \to \infty$.

8.2.5. Solve the initial value problem for a two-dimensional heat equation inside a circle (of radius $a$) with time-independent boundary conditions:

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\nabla^2 u \\ u(a,\theta,t) &= g(\theta) \\ u(r,\theta,0) &= f(r,\theta). \end{aligned}$$

8.2.6. Solve the wave equation with time-independent sources,

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2\frac{\partial^2 u}{\partial x^2} + Q(x) \\ u(x,0) &= f(x) \\ \tfrac{\partial}{\partial t}u(x,0) &= g(x), \end{aligned}$$

if an "equilibrium" solution exists. Analyze the behavior for large $t$. If no equilibrium exists, explain why and reduce the problem to one with homogeneous boundary conditions. Assume that

* (a) $Q(x) = 0$, $u(0,t) = A$, $u(L,t) = B$
(b) $Q(x) = 1$, $u(0,t) = 0$, $u(L,t) = 0$
(c) $Q(x) = 1$, $u(0,t) = A$, $u(L,t) = B$
[*Hint*: Add problems (a) and (b).]
* (d) $Q(x) = \sin\frac{\pi x}{L}$, $u(0,t) = 0$, $u(L,t) = 0$