as $w \to 0$. As $w \to \infty$, the behavior of $I_m(w)$ will be some linear combination of the two different asymptotic behaviors ($e^{\pm w}/w^{1/2}$). In general, it will be composed of both and hence is expected to exponentially grow as $w \to \infty$. In more advanced works, it is shown that

$$\boxed{I_m(w) \sim \sqrt{\frac{1}{2\pi w}}e^w,} \tag{7.9.44}$$

as $w \to \infty$. The most important facts about this function is that **$I_m(w)$ is well behaved at $w = 0$ but grows exponentially as $w \to \infty$.**

Some modified Bessel functions are sketched in Fig. 7.9.2. Although we have not proved it, note that both $I_m(w)$ and $K_m(w)$ are not zero for $w > 0$.

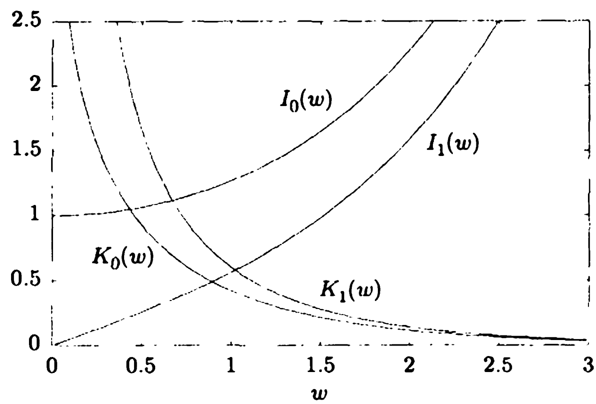

Figure 7.9.2 Various modified Bessel functions (from Abramowitz and Stegun [1974]).

EXERCISES 7.9

7.9.1. Solve Laplace's equation inside a circular cylinder subject to the boundary conditions

(a) $u(r,\theta,0) = \alpha(r,\theta)$, $\quad u(r,\theta,H) = 0$, $\quad u(a,\theta,z) = 0$

*(b) $u(r,\theta,0) = \alpha(r)\sin 7\theta$, $\quad u(r,\theta,H) = 0$, $\quad u(a,\theta,z) = 0$

(c) $u(r,\theta,0) = 0$, $\quad u(r,\theta,H) = \beta(r)\cos 3\theta$, $\quad \frac{\partial u}{\partial r}(a,\theta,z) = 0$

(d) $\frac{\partial u}{\partial z}(r,\theta,0) = \alpha(r)\sin 3\theta$, $\quad \frac{\partial u}{\partial z}(r,\theta,H) = 0$, $\quad \frac{\partial u}{\partial r}(a,\theta,z) = 0$

(e) $\frac{\partial u}{\partial z}(r,\theta,0) = \alpha(r,\theta)$, $\quad \frac{\partial u}{\partial z}(r,\theta,H) = 0$, $\quad \frac{\partial u}{\partial r}(a,\theta,z) = 0$

For (e) only, under what condition does a solution exist?

7.9.2. Solve Laplace's equation inside a semicircular cylinder, subject to the boundary conditions

(a) $u(r,\theta,0)=0,\quad u(r,\theta,H)=\alpha(r,\theta),\quad u(r,0,z)=0,$

$u(r,\pi,z)=0,\quad u(a,\theta,z)=0$

*(b) $u(r,\theta,0)=0,\quad \frac{\partial u}{\partial z}(r,\theta,H)=0,\quad u(r,0,z)=0,$

$u(r,\pi,z)=0,\quad u(a,\theta,z)=\beta(\theta,z)$

(c) $\frac{\partial}{\partial z}u(r,\theta,0)=0,\quad \frac{\partial}{\partial z}u(r,\theta,H)=0,\quad \frac{\partial u}{\partial \theta}(r,0,z)=0,$

$\frac{\partial u}{\partial \theta}(r,\pi,z)=0,\quad \frac{\partial u}{\partial r}(a,\theta,z)=\beta(\theta,z)$

For (c) only, under what condition does a solution exist?

(d) $u(r,\theta,0)=0,\quad u(r,0,z)=0,\quad u(a,\theta,z)=0,$

$u(r,\theta,H)=0,\quad \frac{\partial u}{\partial \theta}(r,\pi,z)=\alpha(r,z)$

7.9.3. Solve the heat equation

$$\frac{\partial u}{\partial t}=k\nabla^2 u$$

inside a quarter-circular cylinder ($0<\theta<\pi/2$ with radius a and height H) subject to the initial condition

$$u(r,\theta,z,0)=f(r,\theta,z)$$

Briefly explain what temperature distribution you expect to be approached as $t\to\infty$. Consider the following boundary conditions

(a) $u(r,\theta,0)=0,\quad u(r,\theta,H)=0,\quad u(r,0,z)=0,$

$u(r,\pi/2,z)=0,\quad u(a,\theta,z)=0$

*(b) $\frac{\partial u}{\partial z}(r,\theta,0)=0,\quad \frac{\partial u}{\partial z}(r,\theta,H)=0,\quad \frac{\partial u}{\partial \theta}(r,0,z)=0,$

$\frac{\partial u}{\partial \theta}(r,\pi/2,z)=0,\quad \frac{\partial u}{\partial r}(a,\theta,z)=0$

(c) $u(r,\theta,0)=0,\quad u(r,\theta,H)=0,\quad \frac{\partial u}{\partial \theta}(r,0,z)=0,$

$u(r,\pi/2,z)=0,\quad \frac{\partial u}{\partial r}(a,\theta,z)=0$

7.9.4. Solve the heat equation

$$\frac{\partial u}{\partial t}=k\nabla^2 u$$

inside a cylinder (of radius a and height H) subject to the initial condition,

$$u(r,\theta,z,0) = f(r,z),$$

independent of θ, if the boundary conditions are

*(a)	$u(r,\theta,0,t) = 0,$	$u(r,\theta,H,t) = 0,$	$u(a,\theta,z,t) = 0$
(b)	$\frac{\partial u}{\partial z}(r,\theta,0,t) = 0,$	$\frac{\partial u}{\partial z}(r,\theta,H,t) = 0,$	$\frac{\partial u}{\partial r}(a,\theta,z,t) = 0$
(c)	$u(r,\theta,0,t) = 0,$	$u(r,\theta,H,t) = 0,$	$\frac{\partial u}{\partial r}(a,\theta,z,t) = 0$

7.9.5. Determine the three ordinary differential equations obtained by separation of variables for Laplace's equation in spherical coordinates

$$0 = \frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right) + \frac{1}{\sin^2\phi}\frac{\partial^2 u}{\partial\theta^2}.$$

7.10 Spherical Problems and Legendre Polynomials

7.10.1 Introduction

Problems in a spherical geometry are of great interest in many applications. In the exercises, we consider the three-dimensional heat equation inside the spherical earth. Here, we consider the three-dimensional wave equation which describes the vibrations of the earth:

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u, \tag{7.10.1}$$

where u is a local displacement. In geophysics, the response of the real earth to point sources is of particular interest due to earthquakes and nuclear testing. Solid vibrations of the real earth are more complicated than (7.10.1). Compressional waves called P for primary are smaller than shear waves called S for secondary, arriving later because they propagate at a smaller velocity. There are also long (L) period surface waves, which are the most destructive in severe earthquakes because their energy is confined to a thin region near the surface. Real seismograms are more complicated because of scattering of waves due to the interior of the earth not being uniform. Measuring the vibrations is frequently used to determine the interior structure of the earth needed not only in seismology but also in mineral exploration, such as petroleum engineering. All displacements solve wave equations. Simple mathematical models are most valid for the destructive long waves, since the variations in the earth are averaged out for long waves. For more details, see Aki and Richards [1980], *Quantitative Seismology*. We use spherical coordinates (ρ, θ, ϕ), where ϕ is the angle from the pole and θ is the usual cylindrical angle. The