into the corresponding equidimensional equation that results by replacing $R(z)$ by $R(0)$ and $S(z)$ by $S(0)$. Thus

$$p(p-1)+R(0)p+S(0)=0$$

is the indicial equation. If the two values of p (the roots of the indicial equation) differ by a noninteger, then two independent solutions exist in the form (7.8.8). If the two roots of the indicial equation are identical, then only one solution is in the form (7.8.8) and the other solution is more complicated but always involves logarithms. If the roots differ by an integer, then sometimes both solutions exist in the form (7.8.8), while other times form (7.8.8) only exists corresponding to the larger root p and a series beginning with the smaller root p must be modified by the introduction of logarithms. Details of the method of Frobenius are presented in most elementary differential equations texts.

For Bessel's differential equation, we have shown that the indicial equation is

$$p(p-1)+p-m^2=0,$$

since $R(0) = 1$ and $S(0) = -m^2$. Its roots are $\pm m$. If $m = 0$, the roots are identical. Form (7.8.8) is valid for one solution, while logarithms must enter the second solution. For $m \neq 0$ the roots of the indicial equation differ by an integer. Detailed calculations also show that logarithms must enter. The following infinite series can be verified by substitution and are often considered as definitions of $J_m(z)$ and $Y_m(z)$:

$$J_m(z)=\sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{2k+m}}{k!(k+m)!} \tag{7.8.9}$$

$$\begin{aligned} Y_m(z) &= \frac{2}{\pi}\left[\left(\log\frac{z}{2}+\gamma\right)J_m(z)-\frac{1}{2}\sum_{k=0}^{m-1}\frac{(m-k-1)!(z/2)^{2k-m}}{k!}\right.\\ &+ \left.\frac{1}{2}\sum_{k=0}^{\infty}(-1)^{k+1}\left[\varphi(k)+\varphi(k+m)\right]\frac{(z/2)^{2k+m}}{k!(m+k)!}\right], \end{aligned} \tag{7.8.10}$$

where

(i) $\varphi(k)=1+\frac{1}{2}+\frac{1}{3}+\cdots+1/k,\ \phi(0)=0$

(ii) $\gamma=\lim_{k\to\infty}[\varphi(k)-\ln k]=0.5772157\ldots$, known as Euler's constant.

(iii) If $m=0,\ \sum_{k=0}^{m-1}\equiv 0$.

We have obtained these from the previously mentioned handbook edited by Abramowitz and Stegun.

EXERCISES 7.8

7.8.1. The boundary value problem for a vibrating annular membrane $1 < r < 2$ (fixed at the inner and outer radii) is

$$\frac{d}{dr}\left(r\frac{df}{dr}\right)+\left(\lambda r-\frac{m^2}{r}\right)f=0$$

with $f(1)=0$ and $f(2)=0$, where $m=0,1,2,\ldots$.

(a) Show that $\lambda > 0$.

*(b) Obtain an expression that determines the eigenvalues.

(c) For what value of m does the smallest eigenvalue occur?

*(d) Obtain an upper and lower bound for the smallest eigenvalue.

(e) Using a trial function, obtain an upper bound for the lowest eigenvalue.

(f) Compute approximately the lowest eigenvalue from part (b) using tables of Bessel functions. Compare to parts (d) and (e).

7.8.2. Consider the temperature $u(r, \theta, t)$ in a quarter-circle of radius a satisfying

$$\frac{\partial u}{\partial t} = k\nabla^2 u$$

subject to the conditions

$$\begin{aligned} u(r, 0, t) &= 0 \quad u(a, \theta, t) = 0 \\ u(r, \pi/2, t) &= 0 \quad u(r, \theta, 0) = G(r, \theta). \end{aligned}$$

(a) Show that the boundary value problem is

$$\frac{d}{dr}\left(r\frac{df}{dr}\right) + \left(\lambda r - \frac{\mu}{r}\right) f = 0$$

with $f(a) = 0$ and $f(0)$ bounded.

(b) Show that $\lambda > 0$ if $\mu \geq 0$.

(c) Show that for each μ, the eigenfunction corresponding to the smallest eigenvalue has no zeros for $0 < r < a$.

*(d) Solve the initial value problem.

7.8.3. Reconsider Exercise 7.8.2 with the boundary conditions

$$\frac{\partial u}{\partial \theta}(r, 0, t) = 0, \quad \frac{\partial u}{\partial \theta}\left(r, \frac{\pi}{2}, t\right) = 0, \quad u(a, \theta, t) = 0.$$

7.8.4. Consider the boundary value problem

$$\frac{d}{dr}\left(r\frac{df}{dr}\right) + \left(\lambda r - \frac{m^2}{r}\right) f = 0$$

with $f(a) = 0$ and $f(0)$ bounded. For each integral m, show that the nth eigenfunction has $n - 1$ zeros for $0 < r < a$.

7.8.5. Using the known asymptotic behavior as $z \to 0$ and as $z \to \infty$, roughly sketch for all $z > 0$

(a) $J_4(z)$	(b) $Y_1(z)$	(c) $Y_0(z)$
(d) $J_0(z)$	(e) $Y_5(z)$	(f) $J_2(z)$

7.8.6. Determine approximately the *large* frequencies of vibration of a circular membrane.

7.8.7. Consider Bessel's differential equation

$$z^2 \frac{d^2 f}{dz^2} + z \frac{df}{dz} + \left(z^2 - m^2\right) f = 0.$$

Let $f = y/z^{1/2}$. Derive that

$$\frac{d^2 y}{dz^2} + y\left(1 + \frac{1}{4} z^{-2} - m^2 z^{-2}\right) = 0.$$

*7.8.8. Using Exercise 7.8.7, determine exact expressions for $J_{1/2}(z)$ and $Y_{1/2}(z)$. Use and verify (7.8.3) and (7.7.33) in this case.

7.8.9. In this exercise use the result of Exercise 7.8.7. If z is large, verify as much as possible concerning (7.8.3).

7.8.10. In this exercise use the result of Exercise 7.8.7 in order to improve on (7.8.3):

(a) Substitute $y = e^{iz} w(z)$ and show that

$$\frac{d^2 w}{dz^2} + 2i \frac{dw}{dz} + \frac{\gamma}{z^2} w = 0, \text{ where } \gamma = \frac{1}{4} - m^2.$$

(b) Substitute $w = \sum_{n=0}^{\infty} \beta_n z^{-n}$. Determine the first few terms β_n (assuming that $\beta_0 = 1$).

(c) Use part (b) to obtain an improved asymptotic solution of Bessel's differential equation. For real solutions, take real and imaginary parts.

(d) Find a recurrence formula for β_n. Show that the series diverges. (Nonetheless, a finite series is very useful.)

7.8.11. In order to "understand" the behavior of Bessel's differential equation as $z \to \infty$, let $x = 1/z$. Show that $x = 0$ is a singular point, but an irregular singular point. [The asymptotic solution of a differential equation in the neighborhood of an irregular singular point is analyzed in an unmotivated way in Exercise 7.8.10. For a more systematic presentation, see advanced texts on asymptotic or perturbation methods (such as Bender and Orszag [1999].)]

7.8.12. The lowest eigenvalue for (7.7.34)–(7.7.36) for $m = 0$ is $\lambda = (z_{01}/a)^2$. Determine a reasonably accurate upper bound by using the Rayleigh quotient with a trial function. Compare to the exact answer.

7.8.13. Explain why the nodal circles in Fig. 7.8.3 are nearly equally spaced.