We have obtained two infinite families of product solutions

$$J_0\left(\sqrt{\lambda_n}r\right)\sin c\sqrt{\lambda_n}t \quad \text{and} \quad J_0\left(\sqrt{\lambda_n}r\right)\cos c\sqrt{\lambda_n}t.$$

According to the principle of superposition, we seek solutions to our original problem, (7.7.50)–(7.7.52) in the form

$$u(r,t)=\sum_{n=1}^{\infty}a_nJ_0\left(\sqrt{\lambda_n}r\right)\cos c\sqrt{\lambda_n}t+\sum_{n=1}^{\infty}b_nJ_0\left(\sqrt{\lambda_n}r\right)\sin c\sqrt{\lambda_n}t. \tag{7.7.64}$$

As before, we determine the coefficients a_n and b_n from the initial conditions. $u(r,0)=\alpha(r)$ implies that

$$\alpha(r)=\sum_{n=1}^{\infty}a_nJ_0\left(\sqrt{\lambda_n}r\right). \tag{7.7.65}$$

The coefficients a_n are thus the Fourier-Bessel coefficients (of order 0) of $\alpha(r)$. Since $J_0\left(\sqrt{\lambda_n}r\right)$ forms an orthogonal set with weight r, we can easily determine a_n,

$$a_n=\frac{\displaystyle\int_0^a \alpha(r)J_0\left(\sqrt{\lambda_n}r\right)r\,dr}{\displaystyle\int_0^a J_0^2\left(\sqrt{\lambda_n}r\right)r\,dr}. \tag{7.7.66}$$

In a similar manner, the initial condition $\partial/\partial t\ u(r,0)=\beta(r)$ determines b_n.

EXERCISES 7.7

*7.7.1. Solve as simply as possible:

$$\frac{\partial^2 u}{\partial t^2}=c^2\nabla^2 u$$

with $u(a,\theta,t)=0, u(r,\theta,0)=0$, and $\frac{\partial u}{\partial t}(r,\theta,0)=\alpha(r)\sin 3\theta$.

7.7.2. Solve as simply as possible:

$$\frac{\partial^2 u}{\partial t^2}=c^2\nabla^2 u \text{ subject to } \frac{\partial u}{\partial r}(a,\theta,t)=0$$

with initial conditions

(a) $u(r,\theta,0)=0$, $\quad\frac{\partial u}{\partial t}(r,\theta,0)=\beta(r)\cos 5\theta$

(b) $u(r,\theta,0)=0$, $\quad\frac{\partial u}{\partial t}(r,\theta,0)=\beta(r)$

(c) $u(r,\theta,0)=\alpha(r,\theta)$, $\quad\frac{\partial u}{\partial t}(r,\theta,0)=0$

*(d) $u(r,\theta,0)=0$, $\quad\frac{\partial u}{\partial t}(r,\theta,0)=\beta(r,\theta)$

7.7.3. Consider a vibrating quarter-circular membrane, $0 < r < a, 0 < \theta < \pi/2$, with $u = 0$ on the entire boundary.

*(a) Determine an expression for the frequencies of vibration.

(b) Solve the initial value problem if

$$u(r,\theta,0) = g(r,\theta), \qquad \frac{\partial u}{\partial t}(r,\theta,0) = 0.$$

7.7.4. Consider the displacement $u(r,\theta,t)$ of a "pie-shaped" membrane of radius a (and angle $\pi/3 = 60°$) that satisfies

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u.$$

Assume that $\lambda > 0$. Determine the natural frequencies of oscillation if the boundary conditions are

(a) $u(r,0,t) = 0, \quad u(r,\pi/3,t) = 0, \quad \frac{\partial u}{\partial r}(a,\theta,t) = 0$

(b) $u(r,0,t) = 0, \quad u(r,\pi/3,t) = 0, \quad u(a,\theta,t) = 0$

*7.7.5. Consider the displacement $u(r,\theta,t)$ of a membrane whose shape is a 90° sector of an annulus, $a < r < b, 0 < \theta < \pi/2$, with the conditions that $u = 0$ on the entire boundary. Determine the natural frequencies of vibration.

7.7.6. Consider the circular membrane satisfying

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u$$

subject to the boundary condition

$$u(a,\theta,t) = -\frac{\partial u}{\partial r}(a,\theta,t).$$

(a) Show that this membrane only oscillates.

(b) Obtain an expression that determines the natural frequencies.

(c) Solve the initial value problem if

$$u(r,\theta,0) = 0, \quad \frac{\partial u}{\partial t}(r,\theta,0) = \alpha(r)\sin 3\theta.$$

7.7.7. Solve the heat equation

$$\frac{\partial u}{\partial t} = k\nabla^2 u$$

inside a circle of radius a with zero temperature around the entire boundary, if initially

$$u(r,\theta,0) = f(r,\theta).$$

Briefly analyze $\lim_{t\to\infty} u(r,\theta,t)$. Compare this to what you expect to occur using physical reasoning as $t \to \infty$.

*7.7.8. Reconsider Exercise 7.7.7, but with the entire boundary insulated.

7.7.9. Solve the heat equation

$$\frac{\partial u}{\partial t} = k\nabla^2 u$$

inside a semicircle of radius a and briefly analyze the $\lim_{t\to\infty}$ if the initial conditions are

$$u(r,\theta,0) = f(r,\theta)$$

and the boundary conditions are

(a)	$u(r,0,t) = 0,$	$u(r,\pi,t) = 0,$	$\frac{\partial u}{\partial r}(a,\theta,t) = 0$
* (b)	$\frac{\partial u}{\partial \theta}(r,0,t) = 0,$	$\frac{\partial u}{\partial \theta}(r,\pi,t) = 0,$	$\frac{\partial u}{\partial r}(a,\theta,t) = 0$
(c)	$\frac{\partial u}{\partial \theta}(r,0,t) = 0,$	$\frac{\partial u}{\partial \theta}(r,\pi,t) = 0,$	$u(a,\theta,t) = 0$
(d)	$u(r,0,t) = 0,$	$u(r,\pi,t) = 0,$	$u(a,\theta,t) = 0$

*7.7.10. Solve for $u(r,t)$ if it satisfies the circularly symmetric heat equation

$$\frac{\partial u}{\partial t} = k\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$$

subject to the conditions

$$\begin{aligned} u(a,t) &= 0 \\ u(r,0) &= f(r). \end{aligned}$$

Briefly analyze the $\lim_{t\to\infty}$.

7.7.11. Reconsider Exercise 7.7.10 with the boundary condition

$$\frac{\partial u}{\partial r}(a,t) = 0.$$

7.7.12. For the following differential equations, what is the expected approximate behavior of all solutions *near $x = 0$*?

*(a) $x^2\frac{d^2y}{dx^2} + (x-6)y = 0$ (b) $x^2\frac{d^2y}{dx^2} + \left(x^2 + \frac{3}{16}\right)y = 0$

*(c) $x^2\frac{d^2y}{dx^2} + (x+x^2)\frac{dy}{dx} + 4y = 0$ (d) $x^2\frac{d^2y}{dx^2} + (x+x^2)\frac{dy}{dx} - 4y = 0$

*(e) $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + \left(6+x^3\right)y = 0$ (f) $x^2\frac{d^2y}{dx^2} + \left(x+\frac{1}{4}\right)y = 0$

7.7.13. Using the one-dimensional Rayleigh quotient, show that $\lambda > 0$ as defined by (7.7.18)–(7.7.20).