(7.2.11),

$$\boxed{\begin{aligned} \frac{dh}{dt} &= -\lambda k h \\ \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} &= -\lambda\phi. \end{aligned}} \tag{7.2.13}$$

The eigenvalue $\lambda$ is determined by finding those values of $\lambda$ for which nontrivial solutions of (7.2.13) exist, subject to a homogeneous boundary condition on the entire boundary.

### 7.2.3 Summary

In situations described in this section the spatial part $\phi(x, y)$ or $\phi(x, y, z)$ of the solution of the partial differential equation satisfies the eigenvalue problem consisting of the partial differential equation,

$$\boxed{\nabla^2\phi = -\lambda\phi,} \tag{7.2.14}$$

with $\phi$ satisfying appropriate homogeneous boundary conditions, which may be of the form [see (1.5.2) and (4.5.5)]

$$\alpha\phi + \beta\nabla\phi \cdot \hat{n} = 0, \tag{7.2.15}$$

where $\alpha$ and $\beta$ can depend on $x, y$, and $z$. If $\beta = 0$, (7.2.15) is the fixed boundary condition. If $\alpha = 0$, (7.2.15) is the insulated or free boundary condition. If both $\alpha \neq 0$ and $\beta \neq 0$, then (7.2.15) is the higher-dimensional version of Newton's law of cooling or the elastic boundary condition. In Sec. 7.4 we will describe general results for this two- or three-dimensional eigenvalue problem, similar to our theorems concerning the general one-dimensional Sturm-Liouville eigenvalue problem. However, first we will describe the solution of a simple two-dimensional eigenvalue problem in a situation in which $\phi(x, y)$ may be further separated, producing two one-dimensional eigenvalue problems.

## EXERCISES 7.2

7.2.1. For a vibrating membrane of any shape that satisfies (7.2.1), show that (7.2.14) results after separating time.

7.2.2. For heat conduction in any two-dimensional region that satisfies (7.1.1), show that (7.2.14) results after separating time.

7 2.3. (a) Obtain product solutions, $\phi = f(x)g(y)$, of (7.2.14) that satisfy $\phi = 0$ on the four sides of a rectangle. (*Hint*: If necessary, see Sec. 7.3.)

(b) Using part (a), solve the initial value problem for a vibrating rectangular membrane (fixed on all sides).

(c) Using part (a), solve the initial value problem for the two-dimensional heat equation with zero temperature on all sides.

## 7.3 Vibrating Rectangular Membrane

In this section we analyze the vibrations of a rectangular membrane, as sketched in Fig. 7.3.1. The vertical displacement $u(x, y, t)$ of the membrane satisfies the two-dimensional wave equation,

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right).} \tag{7.3.1}$$

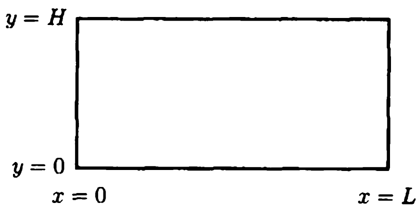


**Figure 7.3.1** Rectangular membrane.

We suppose that the boundary is given such that all four sides are fixed with zero displacement:

$$u(0, y, t) = 0 \qquad u(x, 0, t) = 0 \tag{7.3.2}$$

$$u(L, y, t) = 0 \qquad u(x, H, t) = 0. \tag{7.3.3}$$

We ask what is the displacement of the membrane at time $t$ if the initial position and velocity are given:

$$u(x, y, 0) = \alpha(x, y) \tag{7.3.4}$$

$$\frac{\partial u}{\partial t}(x, y, 0) = \beta(x, y). \tag{7.3.5}$$

As we indicated in Sec. 7.2.1, since the partial differential equation and the boundary conditions are linear and homogeneous, we apply the method of separation of variables. First, we separate only the time variable by seeking product solutions in the form

$$u(x, y, t) = h(t)\phi(x, y). \tag{7.3.6}$$