EXERCISES 7.10

7.10.1. Solve the initial value problem for the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u$ inside a sphere of radius a subject to the boundary condition $u(a,\theta,\phi,t) = 0$ and the initial conditions

(a) $u(\rho,\theta,\phi,0) = F(\rho,\theta,\phi)$ and $\frac{\partial u}{\partial t}(\rho,\theta,\phi,0) = 0$

(b) $u(\rho,\theta,\phi,0) = 0$ and $\frac{\partial u}{\partial t}(\rho,\theta,\phi,0) = G(\rho,\theta,\phi)$

(c) $u(\rho,\theta,\phi,0) = F(\rho,\phi)$ and $\frac{\partial u}{\partial t}(\rho,\theta,\phi,0) = 0$

(d) $u(\rho,\theta,\phi,0) = 0$ and $\frac{\partial u}{\partial t}(\rho,\theta,\phi,0) = G(\rho,\phi)$

(e) $u(\rho,\theta,\phi,0) = F(\rho,\phi)\cos 3\theta$ and $\frac{\partial u}{\partial t}(\rho,\theta,\phi,0) = 0$

(f) $u(\rho,\theta,\phi,0) = F(\rho)\sin 2\theta$ and $\frac{\partial u}{\partial t}(\rho,\theta,\phi,0) = 0$

(g) $u(\rho,\theta,\phi,0) = F(\rho)$ and $\frac{\partial u}{\partial t}(\rho,\theta,\phi,0) = 0$

(h) $u(\rho,\theta,\phi,0) = 0$ and $\frac{\partial u}{\partial t}(\rho,\theta,\phi,0) = G(\rho)$

7.10.2. Solve the initial value problem for the heat equation $\frac{\partial u}{\partial t} = k\nabla^2 u$ inside a sphere of radius a subject to the boundary condition $u(a,\theta,\phi,t) = 0$ and the initial conditions

(a) $u(\rho,\theta,\phi,0) = F(\rho,\theta,\phi)$

(b) $u(\rho,\theta,\phi,0) = F(\rho,\phi)$

(c) $u(\rho,\theta,\phi,0) = F(\rho,\phi)\cos\theta$

(d) $u(\rho,\theta,\phi,0) = F(\rho)$

7.10.3. Solve the initial value problem for the heat equation $\frac{\partial u}{\partial t} = k\nabla^2 u$ inside a sphere of radius a subject to the boundary condition $\frac{\partial u}{\partial \rho}(a,\theta,\phi,t) = 0$ and the initial conditions

(a) $u(\rho,\theta,\phi,0) = F(\rho,\theta,\phi)$

(b) $u(\rho,\theta,\phi,0) = F(\rho,\phi)$

(c) $u(\rho,\theta,\phi,0) = F(\rho,\phi)\sin 3\theta$

(d) $u(\rho,\theta,\phi,0) = F(\rho)$

7.10.4. Using the one-dimensional Rayleigh quotient, show that $\mu \geq 0$ (if $m \geq 0$) as defined by (7.10.11). Under what conditions does $\mu = 0$?

7.10.5. Using the one-dimensional Rayleigh quotient, show that $\mu \geq 0$ (if $m \geq 0$) as defined by (7.10.13). Under what conditions does $\mu = 0$?

7.10.6. Using the one-dimensional Rayleigh quotient, show that $\lambda \geq 0$ (if $n \geq 0$) as defined by (7.10.6) with the boundary condition $f(a) = 0$. Can $\lambda = 0$?

7.10.7. Using the three-dimensional Rayleigh quotient, show that $\lambda \geq 0$ as defined by (7.10.11) with $u(a,\theta,\phi,t) = 0$. Can $\lambda = 0$?

7.10.8. Differential equations related to Bessel's differential equation. Use this to show that

$$x^2\frac{d^2f}{dx^2}+x(1-2a-2bx)\frac{df}{dx}+[a^2-p^2+(2a-1)bx+(d^2+b^2)x^2]f=0 \quad (7.10.37)$$

has solutions $x^a e^{bx} Z_p(dx)$, where $Z_p(x)$ satisfies Bessel's differential equation (7.7.25). By comparing (7.10.21) and (7.10.37), we have $a=-\frac{1}{2}, b=0, \frac{1}{4}-p^2=-n(n+1)$, and $\mathrm{d}^2=\lambda$. We find that $p=(n+\frac{1}{2})$.

7.10.9. Solve Laplace's equation inside a sphere $\rho < a$ subject to the following boundary conditions on the sphere:

(a) $u(a,\theta,\phi)=F(\phi)\cos 4\theta$

(b) $u(a,\theta,\phi)=F(\phi)$

(c) $\frac{\partial u}{\partial \rho}(a,\theta,\phi)=F(\phi)\cos 4\theta$

(d) $\frac{\partial u}{\partial \rho}(a,\theta,\phi)=F(\phi)$ with $\int_0^\pi F(\phi)\sin\phi\, d\phi=0$

(e) $\frac{\partial u}{\partial \rho}(a,\theta,\phi)=F(\theta,\phi)$ with $\int_0^\pi\int_0^{2\pi} F(\theta,\phi)\sin\phi\, d\theta\, d\phi=0$

7.10.10. Solve Laplace's equation outside a sphere $\rho > a$ subject to the potential given on the sphere:

(a) $u(a,\theta,\phi)=F(\theta,\phi)$

(b) $u(a,\theta,\phi)=F(\phi)$, with cylindrical (azimuthal) symmetry

(c) $u(a,\theta,\phi)=V$ in the upper hemisphere, $-V$ in the lower hemisphere (do not simplify; do not evaluate definite integrals)

7.10.11. Solve Laplace's equation inside a sector of a sphere $\rho < a$ with $0<\theta<\frac{\pi}{2}$ subject to $u(\rho,0,\phi)=0$ and $u(\rho,\frac{\pi}{2},\phi)=0$ and the potential given on the sphere: $u(a,\theta,\phi)=F(\theta,\phi)$.

7.10.12. Solve Laplace's equation inside a hemisphere $\rho > a$ with $z>0$ subject to $u=0$ at $z=0$ and the potential given on the hemisphere: $u(a,\theta,\phi)=F(\theta,\phi)$ [*Hint*: Use symmetry and solve a different problem, a sphere with the antisymmetric potential on the lower hemisphere.]

7.10.13. Show that Rodrigues' formula agrees with the given Legendre polynomials for $n=0$, $n=1$, and $n=2$.

7.10.14. Show that Rodrigues' formula satisfies the differential equation for Legendre polynomials.

7.10.15. Derive (7.10.36) using (7.10.35), (7.10.18), and (7.10.25).