Since the coefficient $\sigma(x) = 1$ in (5.8.41), the eigenfunctions $\phi_n(x)$ are orthogonal with weight 1. Thus, we know that the generalized Fourier coefficients of the initial condition $f(x)$ are

$$a_n = \frac{\int_0^L f(x)\phi_n(x)\,dx}{\int_0^L \phi_n^2\,dx} = \begin{cases} \int_0^L f(x)\sinh\sqrt{s_1}x\,dx \Big/ \int_0^L \sinh^2\sqrt{s_1}x\,dx & n = 1 \\ \int_0^L f(x)\sin\sqrt{\lambda_n}x\,dx \Big/ \int_0^L \sin^2\sqrt{\lambda_n}x\,dx & n \geq 2. \end{cases}$$

In particular, we could show $\int_0^L \sin^2\sqrt{\lambda_n}x\,dx \neq L/2$. Perhaps we should emphasize one additional point. We have utilized the theorem that states that eigenfunctions corresponding to different eigenvalues are orthogonal; it is guaranteed that $\int_0^L \sin\sqrt{\lambda_n}x \sin\sqrt{\lambda_m}x\,dx = 0 (n \neq m)$ and $\int_0^L \sin\sqrt{\lambda_n}x \sinh\sqrt{s_1}x\,dx = 0$. We do not need to verify these by integration (although it can be done).

Other problems with boundary conditions of the third kind appear in the Exercises.

EXERCISES 5.8

5.8.1. Consider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

subject to $u(0,t) = 0$, $\frac{\partial u}{\partial x}(L,t) = -hu(L,t)$, and $u(x,0) = f(x)$.

(a) Solve if $hL > -1$.

(b) Solve if $hL = -1$.

5.8.2. Consider the eigenvalue problem (5.8.8)–(5.8.10). Show that the nth eigenfunction has $n-1$ zeros in the interior if

(a) $h > 0$

(b) $h = 0$

*(c) $-1 < hL < 0$

(d) $hL = -1$

5.8.3. Consider the eigenvalue problem

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0,$$

subject to $\frac{d\phi}{dx}(0) = 0$ and $\frac{d\phi}{dx}(L) + h\phi(L) = 0$ with $h > 0$.

(a) Prove that $\lambda > 0$ (without solving the differential equation).

*(b) Determine all eigenvalues graphically. Obtain upper and lower bounds. Estimate the large eigenvalues.

(c) Show that the nth eigenfunction has $n-1$ zeros in the interior.

5.8.4. Redo Exercise 5.8.3 parts (b) and (c) only if $h < 0$.

5.8.5. Consider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

with $\frac{\partial u}{\partial x}(0,t) = 0$, $\frac{\partial u}{\partial x}(L,t) = -hu(L,t)$, and $u(x,0) = f(x)$.

(a) Solve if $h > 0$.

(b) Solve if $h < 0$.

5.8.6. Consider (with $h > 0$)

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial x}(0,t) - hu(0,t) = 0 \qquad u(x,0) = f(x)$$

$$\frac{\partial u}{\partial x}(L,t) = 0 \qquad \frac{\partial u}{\partial t}(x,0) = g(x).$$

(a) Show that there are an infinite number of different frequencies of oscillation.

(b) Estimate the large frequencies of oscillation.

(c) Solve the initial value problem.

*5.8.7. Consider the eigenvalue problem

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \text{ subject to } \phi(0) = 0 \text{ and } \phi(\pi) - 2\frac{d\phi}{dx}(0) = 0.$$

(a) Show that usually

$$\int_0^\pi \left(u\frac{d^2v}{dx^2} - v\frac{d^2u}{dx^2}\right)\,dx \neq 0$$

for any two functions u and v satisfying these homogeneous boundary conditions.

(b) Determine all positive eigenvalues.

(c) Determine all negative eigenvalues.

(d) Is $\lambda = 0$ an eigenvalue?

(e) Is it possible that there are other eigenvalues besides those determined in parts (b) through (d)? *Briefly* explain.

5.8.8. Consider the boundary value problem

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \quad \text{with} \quad \begin{aligned} \phi(0) - \frac{d\phi}{dx}(0) &= 0 \\ \phi(1) + \frac{d\phi}{dx}(1) &= 0. \end{aligned}$$

(a) Using the Rayleigh quotient, show that $\lambda \geq 0$. Why is $\lambda > 0$?

(b) Prove that eigenfunctions corresponding to different eigenvalues are orthogonal.

*(c) Show that

$$\tan\sqrt{\lambda} = \frac{2\sqrt{\lambda}}{\lambda - 1}.$$

Determine the eigenvalues graphically. Estimate the large eigenvalues.

(d) Solve

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

with

$$\begin{aligned} u(0,t) - \frac{\partial u}{\partial x}(0,t) &= 0 \\ u(1,t) + \frac{\partial u}{\partial x}(1,t) &= 0 \\ u(x,0) &= f(x). \end{aligned}$$

You may call the relevant eigenfunctions $\phi_n(x)$ and assume that they are known.

5.8.9. Consider the eigenvalue problem

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \text{ with } \phi(0) = \frac{d\phi}{dx}(0) \text{ and } \phi(1) = \beta\frac{d\phi}{dx}(1).$$

For what values (if any) of β is $\lambda = 0$ an eigenvalue?

5.8.10. Consider the special case of the eigenvalue problem of Sec. 5.8:

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \text{ with } \phi(0) = 0 \text{ and } \frac{d\phi}{dx}(1) + \phi(1) = 0.$$

*(a) Determine the lowest eigenvalue to at least two or three significant figures using tables or a calculator.

*(b) Determine the lowest eigenvalue using a root finding algorithm (e.g.. Newton's method) on a computer.

(c) Compare either part (a) or (b) to the bound obtained using the Rayleigh quotient [see Exercise 5.6.1(c)].

5.8.11. Determine all negative eigenvalues for

$$\frac{d^2\phi}{dx^2} + 5\phi = -\lambda\phi \text{ with } \phi(0) = 0 \text{ and } \phi(\pi) = 0.$$

5.8.12. Consider $\partial^2 u/\partial t^2 = c^2\partial^2 u/\partial x^2$ with the boundary conditions

$$\begin{aligned} u &= 0 && \text{at } x = 0 \\ m\frac{\partial^2 u}{\partial t^2} &= -T_0\frac{\partial u}{\partial x} - ku && \text{at } x = L. \end{aligned}$$

(a) Give a brief physical interpretation of the boundary conditions.

(b) Show how to determine the frequencies of oscillation. Estimate the large frequencies of oscillation.

(c) *Without* attempting to use the Rayleigh quotient, explicitly determine if there are any separated solutions that do not oscillate in time. (*Hint*: There are none.)

(d) Show that the boundary condition is *not* self-adjoint: that is, show

$$\int_0^L \left(u_n \frac{d^2 u_n}{dx^2} - u_m \frac{d^2 u_n}{dx^2} \right) dx \neq 0$$

even when u_n and u_m are eigenfunctions corresponding to different eigenvalues.

*5.8.13. Simplify $\int_0^L \sin^2 \sqrt{\lambda} x \, dx$ when λ is given by (5.8.15).

5.9 Large Eigenvalues (Asymptotic Behavior)

For the variable coefficient case, the eigenvalues for the Sturm-Liouville differential equation,

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + [\lambda\sigma(x) + q(x)]\phi = 0, \tag{5.9.1}$$

usually must be calculated numerically. We know that there will be an infinite number of eigenvalues with no largest one. Thus, there will be an infinite sequence of large eigenvalues. In this section we state and explain reasonably good approximations to these large eigenvalues and corresponding eigenfunctions. Thus, numerical solutions will be needed only for the first few eigenvalues and eigenfunctions.

A careful derivation with adequate explanations of the asymptotic method would be lengthy. Nonetheless, some motivation for our result will be presented. We begin by attempting to approximate solutions of the differential equation (5.9.1) if the unknown eigenvalue λ is large ($\lambda \gg 1$). Interpreting (5.9.1) as a spring-mass system (x is time, ϕ is position) with time-varying parameters is helpful. Then (5.9.1) has a large restoring force $[-\lambda\sigma(x)\phi]$ such that we expect the solution to have rapid oscillation in x. Alternatively, we know that eigenfunctions corresponding to large eigenvalues have many zeros. Since the solution oscillates rapidly, over a few periods (each small) the variable coefficients are approximately constant. Thus, near any point x_0, the differential equation may be approximated crudely by one with constant coefficients:

$$p(x_0)\frac{d^2\phi}{dx^2} + \lambda\sigma(x_0)\phi \approx 0, \tag{5.9.2}$$

since in addition $\lambda\sigma(x) \gg q(x)$. According to (5.9.2), the solution is expected to oscillate with "local" spatial (circular) frequency

$$\text{frequency} = \sqrt{\frac{\lambda\sigma(x_0)}{p(x_0)}}. \tag{5.9.3}$$