EXERCISES 5.5

5.5.1. A Sturm-Liouville eigenvalue problem is called self-adjoint if

$$p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b = 0$$

since then $\int_a^b [uL(v) - vL(u)]\ dx = 0$ for any two functions u and v satisfying the boundary conditions. Show that the following yield self-adjoint problems.

(a) $\phi(0) = 0$ and $\phi(L) = 0$

(b) $\frac{d\phi}{dx}(0) = 0$ and $\phi(L) = 0$

(c) $\frac{d\phi}{dx}(0) - h\phi(0) = 0$ and $\frac{d\phi}{dx}(L) = 0$

(d) $\phi(a) = \phi(b)$ and $p(a)\frac{d\phi}{dx}(a) = p(b)\frac{d\phi}{dx}(b)$

(e) $\phi(a) = \phi(b)$ and $\frac{d\phi}{dx}(a) = \frac{d\phi}{dx}(b)$ [self-adjoint only if $p(a) = p(b)$]

(f) $\phi(L) = 0$ and [in the situation in which $p(0) = 0$] $\phi(0)$ bounded and $\lim_{x\to 0} p(x)\frac{d\phi}{dx} = 0$

*(g) Under what conditions is the following self-adjoint (if p is constant)?

$$\begin{aligned}\phi(L) + \alpha\phi(0) + \beta\frac{d\phi}{dx}(0) &= 0\\ \frac{d\phi}{dx}(L) + \gamma\phi(0) + \delta\frac{d\phi}{dx}(0) &= 0\end{aligned}$$

5.5.2. Prove that the eigenfunctions corresponding to different eigenvalues (of the following eigenvalue problem) are orthogonal:

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + q(x)\phi + \lambda\sigma(x)\phi = 0$$

with the boundary conditions

$$\begin{aligned}\phi(1) &= 0\\ \phi(2) - 2\frac{d\phi}{dx}(2) &= 0.\end{aligned}$$

What is the weighting function?

5.5.3. Consider the eigenvalue problem $L(\phi) = -\lambda\sigma(x)\phi$, subject to a given set of homogeneous boundary conditions. Suppose that

$$\int_a^b [uL(v) - vL(u)]\ dx = 0$$

for all functions u and v satisfying the same set of boundary conditions. Prove that eigenfunctions corresponding to different eigenvalues are orthogonal (with what weight?).

5.5.4. Give an example of an eigenvalue problem with more than one eigenfunction corresponding to an eigenvalue.

5.5.5. Consider

$$L = \frac{d^2}{dx^2} + 6\frac{d}{dx} + 9.$$

(a) Show that $L(e^{rx}) = (r+3)^2 e^{rx}$.

(b) Use part (a) to obtain solutions of $L(y) = 0$ (a second-order constant-coefficient differential equation).

(c) If z depends on x and a parameter r, show that

$$\frac{\partial}{\partial r} L(z) = L\left(\frac{\partial z}{\partial r}\right).$$

(d) Using part (c), evaluate $L(\partial z/\partial r)$ if $z = e^{rx}$.

(e) Obtain a second solution of $L(y) = 0$, using part (d).

5.5.6. Prove that if x is a root of a sixth-order polynomial *with real coefficients*, then $\overline{x}$ is also a root.

5.5.7. For

$$L = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q$$

with p and q real, carefully show that

$$\overline{L(\phi)} = L(\overline{\phi}).$$

5.5.8. Consider a fourth-order linear differential operator,

$$L = \frac{d^4}{dx^4}.$$

(a) Show that $uL(v) - vL(u)$ is an exact differential.

(b) Evaluate $\int_0^1 [uL(v) - vL(u)]\ dx$ in terms of the boundary data for any functions u and v.

(c) Show that $\int_0^1 [uL(v) - vL(u)]\ dx = 0$ if u and v are any two functions satisfying the boundary conditions

$$\begin{aligned} \phi(0) &= 0 \quad & \phi(1) &= 0 \\ \frac{d\phi}{dx}(0) &= 0 \quad & \frac{d^2\phi}{dx^2}(1) &= 0. \end{aligned}$$

(d) Give another example of boundary conditions such that

$$\int_0^1 [uL(v) - vL(u)]\ dx = 0.$$

(e) For the eigenvalue problem [using the boundary conditions in part (c)]

$$\frac{d^4\phi}{dx^4} + \lambda e^x \phi = 0,$$

show that the eigenfunctions corresponding to different eigenvalues are orthogonal. What is the weighting function?

*5.5.9. For the eigenvalue problem

$$\frac{d^4\phi}{dx^4} + \lambda e^x \phi = 0$$

subject to the boundary conditions

$$\begin{aligned} \phi(0) &= 0 & \phi(1) &= 0 \\ \tfrac{d\phi}{dx}(0) &= 0 & \tfrac{d^2\phi}{dx^2}(1) &= 0, \end{aligned}$$

show that the eigenvalues are less than or equal to zero ($\lambda \leq 0$). (Don't worry; in a physical context that is exactly what is expected.) Is $\lambda = 0$ an eigenvalue?

5.5.10. (a) Show that (5.5.22) yields (5.5.23) if at least one of the boundary conditions is of the regular Sturm-Liouville type.

(b) Do part (a) if one boundary condition is of the singular type.

5.5.11. *(a) Suppose that

$$L = p(x)\frac{d^2}{dx^2} + r(x)\frac{d}{dx} + q(x).$$

Consider

$$\int_a^b vL(u)\, dx.$$

By repeated integration by parts, determine the adjoint operator L^* such that

$$\int_a^b [uL^*(v) - vL(u)]\, dx = H(x)\Big|_a^b.$$

What is $H(x)$? Under what conditions does $L = L^*$, the **self-adjoint** case? [*Hint*: Show that

$$L^* = p\frac{d^2}{dx^2} + \left(2\frac{dp}{dx} - r\right)\frac{d}{dx} + \left(\frac{d^2p}{dx^2} - \frac{dr}{dx} + q\right)\Bigg].$$

(b) If

$$u(0) = 0 \quad \text{and} \quad \frac{du}{dx}(L) + u(L) = 0,$$

what boundary conditions should $v(x)$ satisfy for $H(x)|_0^L = 0$, called the adjoint boundary conditions?

5.5.12. Consider nonself-adjoint operators as in Exercise 5.5.11. The eigenvalues λ may be complex as well as their corresponding eigenfunctions ϕ.

(a) Show that if λ is a complex eigenvalue with corresponding eigenfunction ϕ, then the complex conjugate $\overline{\lambda}$ is also an eigenvalue with eigenfunction $\overline{\phi}$.

(b) The eigenvalues of the adjoint L^* may be different from the eigenvalues of L. Using the result of Exercise 5.5.11, show that the eigenfunctions of $L(\phi) + \lambda\sigma\phi = 0$ are orthogonal with weight σ(in a complex sense) to eigenfunctions of $L^*(\psi) + \nu\sigma\psi = 0$ if the eigenvalues are different. Assume that ψ satisfies adjoint boundary conditions. You should also use part (a).

5.5.13. Using the result of Exercise 5.5.11, prove the following part of the **Fredholm alternative** (for operators that are not necessarily self-adjoint): A solution of $L(u) = f(x)$ subject to homogeneous boundary conditions may exist only if $f(x)$ is orthogonal to all solutions of the homogeneous adjoint problem.

5.5.14. If L is the following first-order linear differential operator

$$L = p(x)\frac{d}{dx},$$

then determine the adjoint operator L^* such that

$$\int_a^b [uL^*(v) - vL(u)]\,dx = B(x)\Big|_a^b.$$

What is $B(x)$? [*Hint*: Consider $\int_a^b vL(u)\,dx$ and integrate by parts.]

Appendix to 5.5: Matrix Eigenvalue Problem and Orthogonality of Eigenvectors

The matrix eigenvalue problem

$$\boldsymbol{Ax} = \lambda \boldsymbol{x}, \tag{5.5.26}$$

where $\boldsymbol{A}$ is an $n \times n$ real matrix (with entries a_{ij}) and $\boldsymbol{x}$ is an n-dimensional column vector (with components x_i), has many properties similar to those of the Sturm-Liouville eigenvalue problem.

Eigenvalues and eigenvectors. For all values of λ, $\boldsymbol{x}= 0$ is a "trivial" solution of the homogeneous linear system (5.5.26). We ask, for what values of λ are there nontrivial solutions? In general, (5.5.26) can be rewritten as

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}, \tag{5.5.27}$$

where $\boldsymbol{I}$ is the identity matrix. According to the theory of linear equations (elementary linear algebra), a nontrivial solution exists only if

$$\det[\boldsymbol{A} - \lambda \boldsymbol{I}] = 0. \tag{5.5.28}$$