This approximation is not very good if $a_1 = 0$, in which case (5.4.14) should begin with the first nonzero term. However, often the initial temperature $f(x)$ is non-negative (and not identically zero). In this case, we will show from (5.4.13) that $a_1 \neq 0$:

$$a_1 = \frac{\int_0^L f(x)\phi_1(x)c(x)\rho(x)\, dx}{\int_0^L \phi_1^2(x)c(x)\rho(x)\, dx}. \tag{5.4.15}$$

It follows that $a_1 \neq 0$, because $\phi_1(x)$ is the eigenfunction corresponding to the lowest eigenvalue and has no zeros; $\phi_1(x)$ is of one sign. Thus, if $f(x) > 0$ it follows that $a_1 \neq 0$, since $c(x)$ and $\rho(x)$ are positive physical functions. In order to šketch the solution for large fixed t, (5.4.14) shows that all that is needed is the first eigenfunction. At the very least, a numerical calculation of the first eigenfunction is easier than the computation of the first hundred.

For large time, the "shape" of the temperature distribution in space stays approximately the same in time. Its amplitude grows or decays in time depending on whether $\lambda_1 > 0$ or $\lambda_1 < 0$ (it would be constant in time if $\lambda_1 = 0$). Since this is a heat flow problem with no sources and with zero temperature at $x = 0$, we certainly expect the temperature to be exponentially decaying toward $0°$ (i.e., we expect that $\lambda_1 > 0$). Although the right end is insulated, heat energy should flow out the left end since there $u = 0$. We now prove mathematically that all $\lambda > 0$. Since $p(x) = K_0(x), q(x) = 0$, and $\sigma(x) = c(x)\rho(x)$, it follows from the Rayleigh quotient that

$$\lambda = \frac{\int_0^L K_0(x)(d\phi/dx)^2\, dx}{\int_0^L \phi^2 c(x)\rho(x)\, dx}, \tag{5.4.16}$$

where the boundary contribution to (5.4.16) vanished due to the specific homogeneous boundary conditions, (5.4.7) and (5.4.8). It immediately follows from (5.4.16) that all $\lambda \geq 0$, since the thermal coefficients are positive. Furthermore, $\lambda > 0$, since $\phi = $ constant is not an allowable eigenfunction [because $\phi(0) = 0$]. Thus, we have shown that $\lim_{t\to\infty} u(x,t) = 0$ for this example.

EXERCISES 5.4

5.4.1.   Consider

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + \alpha u,$$

where c, ρ, K_0, α are functions of x, subject to

$$\begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0 \\ u(x,0) &= f(x). \end{aligned}$$

Assume that the appropriate eigenfunctions are known.

(a) Show that the eigenvalues are positive if $\alpha < 0$ (see Sec. 5.2.1).

(b) Solve the initial value problem.

(c) Briefly discuss $\lim_{t\to\infty} u(x,t)$.

*5.4.2. **Consider**

$$c\rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0 \frac{\partial u}{\partial x}\right),$$

where c, ρ, K_0 are functions of x, subject to

$$\frac{\partial u}{\partial x}(0,t) = 0$$

$$\frac{\partial u}{\partial x}(L,t) = 0$$

$$u(x,0) = f(x).$$

Assume that the appropriate eigenfunctions are known. Solve the initial value problem, briefly discussing $\lim_{t\to\infty} u(x,t)$.

*5.4.3. **Solve**

$$\frac{\partial u}{\partial t} = k\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$$

with $u(r,0) = f(r), u(0,t)$ bounded, and $u(a,t) = 0$. You may assume that the corresponding eigenfunctions, denoted $\phi_n(r)$, are known and are complete. (*Hint:* See Sec. 5.2.2.)

5.4.4. Consider the following boundary value problem:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \text{with} \quad \frac{\partial u}{\partial x}(0,t) = 0 \quad \text{and} \quad u(L,t) = 0.$$

Solve such that $u(x,0) = \sin \pi x/L$ (initial condition). (*Hint:* If necessary, use a table of integrals.)

5.4.5. **Consider**

$$\rho\frac{\partial u^2}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} + \alpha u,$$

where $\rho(x) > 0, \alpha(x) < 0$, and T_0 is constant, subject to

$$u(0,t) = 0 \qquad u(x,0) = f(x)$$
$$u(L,t) = 0 \qquad \frac{\partial u}{\partial t}(x,0) = g(x).$$

Assume that the appropriate eigenfunctions are known. Solve the initial value problem.

*5.4.6. Consider the vibrations of a *nonuniform* string of mass density $\rho_0(x)$. Suppose that the left end at $x = 0$ is fixed and the right end obeys the elastic boundary condition: $\partial u/\partial x = -(k/T_0)u$ at $x = L$. Suppose that the string is *initially at rest* with a known initial position $f(x)$. Solve this initial value problem. (*Hints: Assume* that the appropriate eigenvalues and corresponding eigenfunctions are known. What differential equations with what boundary conditions do they satisfy? The eigenfunctions are orthogonal with what weighting function?)