of standing waves, it can be shown that this solution is a combination of just two waves (each rather complicated)—one traveling to the left at velocity $-c$ with fixed shape and the other to the right at velocity c with a different fixed shape. We are claiming that the solution to the one-dimensional wave equation can be written as

$$u(x,t) = R(x - ct) + S(x + ct),$$

even if the boundary conditions are not fixed at $x = 0$ and $x = L$. We will show and discuss this further in the Exercises and in Chapter 12.

EXERCISES 4.4

4.4.1. Consider vibrating strings of uniform density ρ_0 and tension T_0.

*(a) What are the natural frequencies of a vibrating string of length L fixed at both ends?

*(b) What are the natural frequencies of a vibrating string of length H, which is fixed at $x = 0$ and "free" at the other end [i.e., $\partial u/\partial x(H,t) = 0$]? Sketch a few modes of vibration as in Fig. 4.4.1.

(c) Show that the modes of vibration for the *odd* harmonics (i.e., $n = 1, 3, 5, \ldots$) of part (a) are identical to modes of part (b) if $H = L/2$. Verify that their natural frequencies are the same. Briefly explain using symmetry arguments.

4.4.2. In Sec. 4.2 it was shown that the displacement u of a nonuniform string satisfies

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} + Q,$$

where Q represents the vertical component of the body force per unit length. If $Q = 0$, the partial differential equation is homogeneous. A slightly different homogeneous equation occurs if $Q = \alpha u$.

(a) Show that if $\alpha < 0$, the body force is restoring (toward $u = 0$). Show that if $\alpha > 0$, the body force tends to push the string further away from its unperturbed position $u = 0$.

(b) Separate variables if $\rho_0(x)$ and $\alpha(x)$ but T_0 is constant for physical reasons. Analyze the time-dependent ordinary differential equation.

*(c) Specialize part (b) to the constant coefficient case. Solve the initial value problem if $\alpha < 0$:

$$u(0,t) = 0 \qquad u(x,0) = 0$$

$$u(L,t) = 0 \qquad \frac{\partial u}{\partial t}(x,0) = f(x).$$

What are the frequencies of vibration?

4.4.3. Consider a slightly damped vibrating string that satisfies

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t}.$$

(a) Briefly explain why $\beta > 0$.

*(b) Determine the solution (by separation of variables) that satisfies the boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0$$

and the initial conditions

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x).$$

You can assume that this frictional coefficient β is relatively small ($\beta^2 < 4\pi^2 \rho_0 T_0 / L^2$).

4.4.4. Redo Exercise 4.4.3(b) by the eigenfunction expansion method.

4.4.5. Redo Exercise 4.4.3(b) if $4\pi^2 \rho_0 T_0/L^2 < \beta^2 < 16\pi^2 \rho_0 T_0 / L^2$.

4.4.6. For (4.4.1)–(4.4.3), from (4.4.11) show that

$$u(x,t) = R(x - ct) + S(x + ct),$$

where R and S are some functions.

4.4.7. If a vibrating string satisfying (4.4.1)–(4.4.3) is initially at rest, $g(x) = 0$, show that

$$u(x,t) = \frac{1}{2}[F(x - ct) + F(x + ct)],$$

where $F(x)$ is the odd periodic extension of $f(x)$. *Hints*:

1. For all x, $F(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}$.
2. $\sin a \cos b = \frac{1}{2}[\sin(a+b) + \sin(a-b)]$.

Comment: This result shows that *the practical difficulty of summing an infinite number of terms of a Fourier series may be avoided for the one-dimensional wave equation.*

4.4.8. If a vibrating string satisfying (4.4.1)–(4.4.3) is initially unperturbed, $f(x) = 0$, with the initial velocity given, show that

$$u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} G(\bar{x})\, d\bar{x},$$

where $G(x)$ is the odd periodic extension of $g(x)$. *Hints*:

1. For all x, $G(x) = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L} \sin \frac{n\pi x}{L}$.

2. $\sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$.

See the comment after Exercise 4.4.7.

4.4.9 From (4.4.1), derive conservation of energy for a vibrating string,

$$\frac{dE}{dt} = c^2 \frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\bigg|_0^L, \tag{4.4.15}$$

where the total energy E is the sum of the kinetic energy, defined by $\int_0^L \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 dx$, and the potential energy, defined by $\int_0^L \frac{c^2}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx$.

4.4.10. What happens to the total energy E of a vibrating string (see Exercise 4.4.9)

(a) If $u(0,T) = 0$ and $u(L,t) = 0$

(b) If $\frac{\partial u}{\partial x}(0,t) = 0$ and $u(L,t) = 0$

(c) If $u(0,t) = 0$ and $\frac{\partial u}{\partial x}(L,t) = -\gamma u(L,t)$ with $\gamma > 0$

(d) If $\gamma < 0$ in part (c)

4.4.11. Show that the potential and kinetic energies (defined in Exercise 4.4.9) are equal for a traveling wave, $u = R(x - ct)$.

4.4.12. Using (4.4.15), prove that the solution of (4.4.1)–(4.4.3) is unique.

4.4.13. (a) Using (4.4.15), calculate the energy of one normal mode.

(b) Show that the total energy, when $u(x,t)$ satisfies (4.4.11), is the sum of the energies contained in each mode.

4.5 Vibrating Membrane

The heat equation in one spatial dimension is $\partial u/\partial t = k\partial^2 u/\partial x^2$. In two or three dimensions, the temperature satisfies $\partial u/\partial t = k\nabla^2 u$. In a similar way, the vibration of a string (one dimension) can be extended to the vibration of a membrane (two dimensions).

The vertical displacement of a vibrating string satisfies the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

There are important physical problems that solve

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u, \tag{4.5.1}$$

known as the two- or three-dimensional wave equation. An example of a physical problem that satisfies a two-dimensional wave equation is the vibration of a highly stretched membrane. This can be thought of as a two-dimensional vibrating string. We will give a *brief* derivation in the manner described by Kaplan [1981], omitting