The Fourier series of a continuous function $u(x,t)$ (depending on a parameter $t$)
$$u(x,t) = a_0(t) + \sum_{n=1}^{\infty} \left[a_n(t)\cos\frac{n\pi x}{L} + b_n(t)\sin\frac{n\pi x}{L}\right]$$
can be differentiated term by term with respect to the parameter $t$, yielding
$$\frac{\partial}{\partial t}u(x,t) \sim a_0'(t) + \sum_{n=1}^{\infty} \left[a_n'(t)\cos\frac{n\pi x}{L} + b_n'(t)\sin\frac{n\pi x}{L}\right]$$
if $\partial u/\partial t$ is piecewise smooth.

We omit its proof (see Exercise 3.4.7), which depends on the fact that

$$\frac{\partial}{\partial t}\int_{-L}^{L} g(x,t)\,dx = \int_{-L}^{L}\frac{\partial g}{\partial t}dx$$

is valid if $g$ is continuous.

In summary, we have verified that the Fourier sine series is actually a solution of the heat equation satisfying the boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$. Now we have two reasons for choosing a Fourier sine series for this problem. First, the method of separation of variables implies that if $u(0,t) = 0$ and $u(L,t) = 0$, then the appropriate eigenfunctions are $\sin n\pi x/L$. Second, we now see that all the differentiations of the infinite sine series are justified, where we need to assume that $u(0,t) = 0$ and $u(L,t) = 0$, exactly the physical boundary conditions.

## EXERCISES 3.4

3.4.1. The integration-by-parts formula

$$\int_a^b u\frac{dv}{dx}\,dx = uv\Big|_a^b - \int_a^b v\frac{du}{dx}\,dx$$

is known to be valid for functions $u(x)$ and $v(x)$, which are continuous and have continuous first derivatives. However, we will assume that $u$, $v$, $du/dx$, and $dv/dx$ are continuous only for $a \le x \le c$ and $c \le x \le b$; we assume that all quantities may have a jump discontinuity at $x = c$.

*(a) Derive an expression for $\int_a^b u\, dv/dx\, dx$ in terms of $\int_a^b v\, du/dx\, dx$.

(b) Show that this reduces to the integration-by-parts formula if $u$ and $v$ are continuous across $x = c$. It is *not* necessary for $du/dx$ and $dv/dx$ to be continuous at $x = c$.

3.4.2. Suppose that $f(x)$ and $df/dx$ are piecewise smooth. Prove that the Fourier series of $f(x)$ can be differentiated term by term if the Fourier series of $f(x)$ is continuous.

3.4.3. Suppose that $f(x)$ is continuous [except for a jump discontinuity at $x = x_0$, $f(x_0^-) = \alpha$ and $f(x_0^+) = \beta$] and $df/dx$ is piecewise smooth.

*(a) Determine the Fourier sine series of $df/dx$ in terms of the Fourier cosine series coefficients of $f(x)$.

(b) Determine the Fourier cosine series of $df/dx$ in terms of the Fourier sine series coefficients of $f(x)$.

3.4.4. Suppose that $f(x)$ and $df/dx$ are piecewise smooth.

(a) Prove that the Fourier sine series of a continuous function $f(x)$ can only be differentiated term by term if $f(0) = 0$ and $f(L) = 0$.

(b) Prove that the Fourier cosine series of a continuous function $f(x)$ can be differentiated term by term.

3.4.5. Using (3.3.13) determine the Fourier cosine series of $\sin \pi x/L$.

3.4.6. There are some things wrong in the following demonstration. Find the mistakes and correct them.

In this exercise we attempt to obtain the Fourier cosine coefficients of $e^x$:

$$e^x = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}. \tag{3.4.22}$$

Differentiating yields

$$e^x = -\sum_{n=1}^{\infty} \frac{n\pi}{L} A_n \sin \frac{n\pi x}{L},$$

the Fourier sine series of $e^x$. Differentiating again yields

$$e^x = -\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 A_n \cos \frac{n\pi x}{L}. \tag{3.4.23}$$

Since equations (3.4.22) and (3.4.23) give the Fourier cosine series of $e^x$, they must be identical. Thus,

$$\left.\begin{aligned} A_0 &= 0 \\ A_n &= 0 \end{aligned}\right\} \text{ (obviously wrong!).}$$

By correcting the mistakes, you should be able to obtain $A_0$ and $A_n$ *without* using the typical technique, that is, $A_n = 2/L \int_0^L e^x \cos n\pi x/L \, dx$.

3.4.7. Prove that the Fourier series of a continuous function $u(x,t)$ can be differentiated term by term with respect to the parameter $t$ if $\partial u/\partial t$ is piecewise smooth.

3.4.8. Consider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

subject to

$$\partial u/\partial x(0,t) = 0, \quad \partial u/\partial x(L,t) = 0, \quad \text{and} \quad u(x,0) = f(x).$$

Solve in the following way. Look for the solution as a Fourier cosine series. Assume that $u$ and $\partial u/\partial x$ are continuous and $\partial^2 u/\partial x^2$ and $\partial u/\partial t$ are piecewise smooth. Justify all differentiations of infinite series.

*3.4.9 Consider the heat equation with a known source $q(x,t)$:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + q(x,t) \text{ with } u(0,t) = 0 \text{ and } u(L,t) = 0.$$

Assume that $q(x,t)$ (for each $t > 0$) is a piecewise smooth function of $x$. Also assume that $u$ and $\partial u/\partial x$ are continuous functions of $x$ (for $t > 0$) and $\partial^2 u/\partial x^2$ and $\partial u/\partial t$ are piecewise smooth. Thus,

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin\frac{n\pi x}{L}.$$

What ordinary differential equation does $b_n(t)$ satisfy? Do not solve this differential equation.

3.4.10. Modify Exercise 3.4.9 if instead $\partial u/\partial x(0,t) = 0$ and $\partial u/\partial x(L,t) = 0$.

3.4.11. Consider the *nonhomogeneous* heat equation (with a steady heat source):

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + g(x).$$

Solve this equation with the initial condition

$$u(x,0) = f(x)$$

and the boundary conditions

$$u(0,t) = 0 \text{ and } u(L,t) = 0.$$

Assume that a continuous solution exists (with continuous derivatives). [*Hints*: Expand the solution as a Fourier sine series (i.e., use the method of eigenfunction expansion). Expand $g(x)$ as a Fourier sine series. Solve for the Fourier sine series of the solution. Justify all differentiations with respect to $x$.]

*3.4.12. Solve the following *nonhomogeneous* problem:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + e^{-t} + e^{-2t}\cos\frac{3\pi x}{L} \quad [\text{assume that } 2 \neq k(3\pi/L)^2]$$

subject to

$$\frac{\partial u}{\partial x}(0,t) = 0, \frac{\partial u}{\partial x}(L,t) = 0, \quad \text{and} \quad u(x,0) = f(x).$$

Use the following method. Look for the solution as a Fourier cosine series. Justify all differentiations of infinite series (assume appropriate continuity).

3.4.13. Consider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

subject to

$$u(0,t) = A(t), \quad u(L,t) = 0, \quad \text{and} \quad u(x,0) = g(x).$$

Assume that $u(x,t)$ has a Fourier sine series. Determine a differential equation for the Fourier coefficients (assume appropriate continuity).

## 3.5 Term-By-Term Integration of Fourier Series

In doing mathematical manipulations with infinite series, we must remember that some properties of finite series do not hold for infinite series. In particular, Sec. 3.4 indicated that we must be especially careful differentiating term by term an infinite Fourier series. The following theorem however, enables us to integrate Fourier series without caution:

> A Fourier series of piecewise smooth $f(x)$ can always be integrated term by term and the result is a *convergent* infinite series that always *converges* to the integral of $f(x)$ for $-L \le x \le L$ (even if the original Fourier series has jump discontinuities).

Remarkably, the new series formed by term-by-term integration is continuous. However, the new series may not be a Fourier series.

To quantify this statement, let us suppose that $f(x)$ is piecewise smooth and hence has a Fourier series in the range $-L \le x \le L$ (not necessarily continuous):

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}. \tag{3.5.1}$$

We will prove our claim that we can just integrate this result term by term:

$$\int_{-L}^{x} f(t)\,dt \sim a_0(x+L) + \sum_{n=1}^{\infty}\left(a_n\int_{-L}^{x}\cos\frac{n\pi t}{L}\,dt + b_n\int_{-L}^{x}\sin\frac{n\pi t}{L}\,dt\right).$$