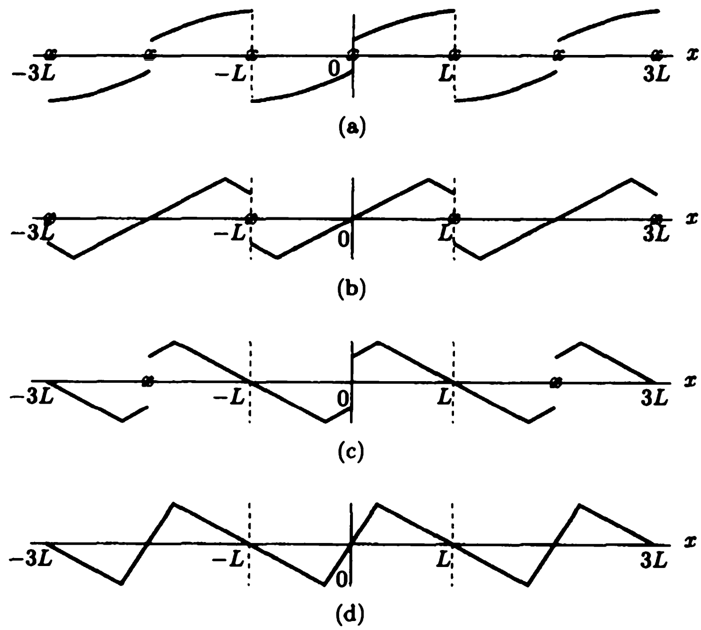

Figure 3.3.19 Fourier sine series of a continuous function with (a) $f(0) \neq 0$ and $f(L) \neq 0$; (b) $f(0) = 0$ but $f(L) \neq 0$; (c) $f(L) = 0$ but $f(0) \neq 0$; and (d) $f(0) = 0$ and $f(L) = 0$.

$-L \leq x \leq L$. Also note that the even extension is the same at $\pm L$. Thus, the periodic extension will automatically be continuous at the endpoints.

Compare this result to what happens for a Fourier sine series. Four examples are considered in Fig. 3.3.19, all continuous functions for $0 \leq x \leq L$. From the first three figures, we see that it is possible for the Fourier sine series of a continuous function to be discontinuous. It is seen that

> For piecewise smooth functions $f(x)$, the Fourier sine series of $f(x)$ is continuous and converges to $f(x)$ for $0 \leq x \leq L$ if and only if $f(x)$ is continuous and both $f(0) = 0$ *and* $f(L) = 0$.

If $f(0) \neq 0$, then the odd extension of $f(x)$ will have a jump discontinuity at $x = 0$, as illustrated in Figs. 3.3.19a and c. If $f(L) \neq 0$, then the odd extension at $x = -L$ will be of opposite sign from $f(L)$. Thus, the periodic extension will not be continuous at the endpoints if $f(L) \neq 0$ as in Figs. 3.3.19a and b.

EXERCISES 3.3

3.3.1. For the following functions, sketch $f(x)$, the Fourier series of $f(x)$, the Fourier sine series of $f(x)$, and the Fourier cosine series of $f(x)$.

(a) $f(x) = 1$

(b) $f(x) = 1 + x$

(c) $f(x) = \begin{cases} x & x < 0 \\ 1 + x & x > 0 \end{cases}$

*(d) $f(x) = e^x$

(e) $f(x) = \begin{cases} 2 & x < 0 \\ e^{-x} & x > 0 \end{cases}$

3.3.2. For the following functions, sketch the Fourier sine series of $f(x)$ and determine its Fourier coefficients.

(a) $f(x) = \cos \pi x/L$ [Verify formula (3.3.13).]

(b) $f(x) = \begin{cases} 1 & x < L/6 \\ 3 & L/6 < x < L/2 \\ 0 & x > L/2 \end{cases}$

(c) $f(x) = \begin{cases} 0 & x < L/2 \\ x & x > L/2 \end{cases}$

*(d) $f(x) = \begin{cases} 1 & x < L/2 \\ 0 & x > L/2 \end{cases}$

3.3.3. For the following functions, sketch the Fourier sine series of $f(x)$. Also, roughly sketch the sum of a *finite* number of nonzero terms (at least the first two) of the Fourier sine series:

(a) $f(x) = \cos \pi x/L$ [Use formula (3.3.13).]

(b) $f(x) = \begin{cases} 1 & x < L/2 \\ 0 & x > L/2 \end{cases}$

(c) $f(x) = x$ [Use formula (3.3.12).]

3.3.4. Sketch the Fourier cosine series of $f(x) = \sin \pi x/L$. Briefly discuss.

3.3.5. For the following functions, sketch the Fourier cosine series of $f(x)$ and determine its Fourier coefficients:

(a) $f(x) = x^2$

(b) $f(x) = \begin{cases} 1 & x < L/6 \\ 3 & L/6 < x < L/2 \\ 0 & x > L/2 \end{cases}$

(c) $f(x) = \begin{cases} 0 & x < L/2 \\ x & x > L/2 \end{cases}$

3.3.6. For the following functions, sketch the Fourier cosine series of $f(x)$. Also, roughly sketch the sum of a finite number of nonzero terms (at least the first two) of the Fourier cosine series:

(a) $f(x) = x$ [Use formulas (3.3.22) and (3.3.23).]

(b) $f(x) = \begin{cases} 0 & x < L/2 \\ 1 & x > L/2 \end{cases}$ [Use carefully formulas (3.2.6) and (3.2.7).]

(c) $f(x) = \begin{cases} 0 & x < L/2 \\ 1 & x > L/2 \end{cases}$ [*Hint*: Add the functions in parts (b) and (c).]

3.3.7. Show that e^x is the sum of an even and an odd function.

3.3.8. (a) Determine formulas for the even extension of any $f(x)$. Compare to the formula for the even part of $f(x)$.

(b) Do the same for the odd extension of $f(x)$ and the odd part of $f(x)$.

(c) Calculate and sketch the four functions of parts (a) and (b) if

$$f(x) = \begin{cases} x & x > 0 \\ x^2 & x < 0. \end{cases}$$

Graphically add the even and odd parts of $f(x)$. What occurs? Similarly, add the even and odd extensions. What occurs then?

3.3.9. What is the sum of the Fourier sine series of $f(x)$ and the Fourier cosine series of $f(x)$? [What is the sum of the even and odd extensions of $f(x)$?]

*3.3.10. If $f(x) = \begin{cases} x^2 & x < 0 \\ e^{-x} & x > 0 \end{cases}$, what are the even and odd parts of $f(x)$?

3.3.11. Given a sketch of $f(x)$, describe a procedure to sketch the even and odd parts of $f(x)$.

3.3.12. (a) Graphically show that the even terms (n even) of the Fourier sine series of any function on $0 \le x \le L$ are odd (antisymmetric) around $x = L/2$.

(b) Consider a function $f(x)$ that is odd around $x = L/2$. Show that the odd coefficients (n odd) of the Fourier sine series of $f(x)$ on $0 \le x \le L$ are zero.

*3.3.13. Consider a function $f(x)$ that is even around $x = L/2$. Show that the even coefficients (n even) of the Fourier sine series of $f(x)$ on $0 \le x \le L$ are zero.

3.3.14. (a) Consider a function $f(x)$ that is even around $x = L/2$. Show that the odd coefficients (n odd) of the Fourier cosine series of $f(x)$ on $0 \le x \le L$ are zero.

(b) Explain the result of part (a) by considering a Fourier cosine series of $f(x)$ on the interval $0 \le x \le L/2$.

3.3.15. Consider a function $f(x)$ that is odd around $x = L/2$. Show that the even coefficients (n even) of the Fourier cosine series of $f(x)$ on $0 \le x \le L$ are zero.

3.3.16. Fourier series can be defined on other intervals besides $-L \le x \le L$. Suppose that $g(y)$ is defined for $a \le y \le b$. Represent $g(y)$ using periodic trigonometric functions with period $b - a$. Determine formulas for the coefficients. [*Hint*: Use the linear transformation

$$y = \frac{a+b}{2} + \frac{b-a}{2L}x.]$$

3.3.17. Consider

$$\int_0^1 \frac{dx}{1+x^2}.$$

(a) Evaluate explicitly.

(b) Use the Taylor series of $1/(1+x^2)$ (itself a geometric series) to obtain an infinite series for the integral.

(c) Equate part (a) to part (b) in order to derive a formula for π.

3.3.18. For continuous functions,

(a) Under what conditions does $f(x)$ equal its Fourier series for all x, $-L \le x \le L$?

(b) Under what conditions does $f(x)$ equal its Fourier sine series for all x, $0 \le x \le L$?

(c) Under what conditions does $f(x)$ equal its Fourier cosine series for all x, $0 \le x \le L$?

3.4 Term-by-Term Differentiation of Fourier Series

In solving partial differential equations by the method of separation of variables, the homogeneous boundary conditions sometimes suggest that the desired solution is either an infinite series of sines or cosines. For example, we consider one-dimensional heat conduction with zero boundary conditions. As before, we want to solve the initial boundary value problem

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \tag{3.4.1}$$

$$u(0,t) = 0, \quad u(L,t) = 0, \quad u(x,0) = f(x). \tag{3.4.2}$$

By the method of separation of variables combined with the principle of superposition (taking a *finite* linear combination of solutions), we know that

$$u(x,t) = \sum_{n=1}^{N} B_n \sin\frac{n\pi x}{L} e^{-(n\pi/L)^2 kt}$$

solves the partial differential equation and the two homogeneous boundary conditions. To satisfy the initial conditions, in general an infinite series is needed. Does the infinite series

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-(n\pi/L)^2 kt} \tag{3.4.3}$$

satisfy our problem? The theory of Fourier sine series shows that the Fourier coefficients B_n can be determined to satisfy any (piecewise smooth) initial condition