Table 2.4.1: Boundary Value Problems for $\frac{d^2\phi}{dx^2} = -\lambda\phi$

Boundary conditions	$\phi(0) = 0$ $\phi(L) = 0$	$\frac{d\phi}{dx}(0) = 0$ $\frac{d\phi}{dx}(L) = 0$	$\phi(-L) = \phi(L)$ $\frac{d\phi}{dx}(-L) = \frac{d\phi}{dx}(L)$
Eigenvalues λ_n	$\left(\frac{n\pi}{L}\right)^2$ $n = 1, 2, 3, \ldots$	$\left(\frac{n\pi}{L}\right)^2$ $n = 0, 1, 2, 3, \ldots$	$\left(\frac{n\pi}{L}\right)^2$ $n = 0, 1, 2, 3, \ldots$
Eigenfunctions	$\sin\frac{n\pi x}{L}$	$\cos\frac{n\pi x}{L}$	$\sin\frac{n\pi x}{L}$ and $\cos\frac{n\pi x}{L}$
Series	$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$	$f(x) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L}$	$f(x) = \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{L}$ $+\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$
Coefficients	$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$	$A_0 = \frac{1}{L}\int_0^L f(x)\,dx$ $A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$	$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$ $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$ $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$

EXERCISES 2.4

*2.4.1. Solve the heat equation $\partial u/\partial t = k\partial^2 u/\partial x^2$, $0 < x < L$, $t > 0$, subject to

$$\frac{\partial u}{\partial x}(0,t) = 0 \qquad t > 0$$
$$\frac{\partial u}{\partial x}(L,t) = 0 \qquad t > 0.$$

(a) $u(x,0) = \begin{cases} 0 & x < L/2 \\ 1 & x > L/2 \end{cases}$

(b) $u(x,0) = 6 + 4\cos\frac{3\pi x}{L}$

(c) $u(x,0) = -2\sin\frac{\pi x}{L}$

(d) $u(x,0) = -3\cos\frac{8\pi x}{L}$

*2.4.2. Solve

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \text{with} \quad \begin{aligned} &\frac{\partial u}{\partial x}(0,t) = 0 \\ &u(L,t) = 0 \\ &u(x,0) = f(x). \end{aligned}$$

For this problem you may assume that no solutions of the heat equation exponentially grow in time. You may also guess appropriate orthogonality conditions for the eigenfunctions.

*2.4.3. Solve the eigenvalue problem

$$\frac{d^2\phi}{dx^2} = -\lambda\phi$$

subject to

$$\phi(0) = \phi(2\pi) \quad \text{and} \quad \frac{d\phi}{dx}(0) = \frac{d\phi}{dx}(2\pi).$$

2.4.4. Explicitly show that there are no negative eigenvalues for

$$\frac{d^2\phi}{dx^2} = -\lambda\phi \quad \text{subject to} \quad \frac{d\phi}{dx}(0) = 0 \quad \text{and} \quad \frac{d\phi}{dx}(L) = 0.$$

2.4.5. This problem presents an alternative derivation of the heat equation for a thin wire. The equation for a circular wire of finite thickness is the two-dimensional heat equation (in polar coordinates). Show that this reduces to (2.4.25) if the temperature does not depend on r and if the wire is very thin.

2.4.6. Determine the equilibrium temperature distribution for the thin circular ring of Section 2.4.2:

(a) Directly from the equilibrium problem (see Sec. 1.4)

(b) By computing the limit as $t \to \infty$ of the time-dependent problem

2.4.7. Solve Laplace's equation inside a circle of radius a,

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

subject to the boundary condition

$$u(a,\theta) = f(\theta).$$

(*Hint:* If necessary, see Sec. 2.5.2.)