Only (2.2.12) is satisfied by $u \equiv 0$ (of the linear conditions) and hence is homogeneous. It is not necessary that a boundary condition be $u(0,t) = 0$ for $u \equiv 0$ to satisfy it.

EXERCISES 2.2

2.2.1. Show that any linear combination of linear operators is a linear operator.

2.2.2. (a) Show that $L(u) = \frac{\partial}{\partial x}\left[K_0(x)\frac{\partial u}{\partial x}\right]$ is a linear operator.

(b) Show that usually $L(u) = \frac{\partial}{\partial x}\left[K_0(x,u)\frac{\partial u}{\partial x}\right]$ is not a linear operator.

2.2.3. Show that $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(u,x,t)$ is linear if $Q = \alpha(x,t)u + \beta(x,t)$ and, in addition, homogeneous if $\beta(x,t) = 0$.

2.2.4. In this exercise we derive superposition principles for nonhomogeneous problems.

(a) Consider $L(u) = f$. If u_p is a particular solution, $L(u_p) = f$, and if u_1 and u_2 are homogeneous solutions, $L(u_i) = 0$, show that $u = u_p + c_1u_1 + c_2u_2$ is another particular solution.

(b) If $L(u) = f_1 + f_2$, where u_{pi} is a particular solution corresponding to f_i, what is a particular solution for $f_1 + f_2$?

2.2.5 If L is a linear operator, show that $L(\sum_{n=1}^{M} c_n u_n) = \sum_{n=1}^{M} c_n L(u_n)$. Use this result to show that the principle of superposition may be extended to any finite number of homogeneous solutions.

2.3 Heat Equation with Zero Temperatures at Finite Ends

2.3.1 Introduction

Partial differential equation (2.1.1) is linear but it is homogeneous only if there are no sources, $Q(x,t) = 0$. The boundary conditions (2.1.3) are also linear, and they too are homogeneous only if $T_1(t) = 0$ and $T_2(t) = 0$. We thus first propose to study

$$\text{PDE:} \quad \boxed{\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}} \quad \begin{array}{l} 0 < x < L \\ t > 0 \end{array} \tag{2.3.1}$$

$$\text{BC:} \quad \boxed{\begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0 \end{aligned}} \tag{2.3.2}$$