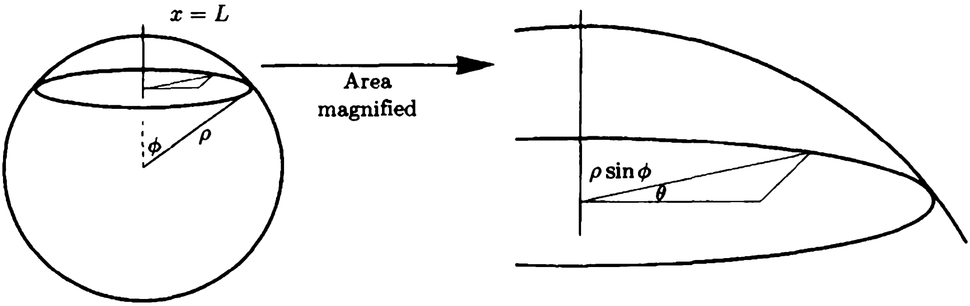

Figure 1.5.3 Spherical coordinates.

EXERCISES 1.5

1.5.1. Let $c(x, y, z, t)$ denote the concentration of a pollutant (the amount per unit volume).

(a) What is an expression for the total amount of pollutant in the region R?

(b) Suppose that the flow $\boldsymbol{J}$ of the pollutant is proportional to the gradient of the concentration. (Is this reasonable?) Express conservation of the pollutant.

(c) Derive the partial differential equation governing the diffusion of the pollutant.

*1.5.2. For conduction of thermal energy, the heat flux vector is $\boldsymbol{\phi} = -K_0\nabla u$. If in addition the molecules move at an average velocity $\boldsymbol{V}$, a process called **convection**, then briefly explain why $\boldsymbol{\phi} = -K_0\nabla u + c\rho u\boldsymbol{V}$. Derive the corresponding equation for heat flow, including both conduction and convection of thermal energy (assuming constant thermal properties with no sources).

1.5.3. Consider the polar coordinates

$$x = r\cos\theta$$
$$y = r\sin\theta.$$

(a) Since $r^2 = x^2 + y^2$, show that $\frac{\partial r}{\partial x} = \cos\theta$, $\frac{\partial r}{\partial y} = \sin\theta$, $\frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$, and $\frac{\partial \theta}{\partial x} = \frac{-\sin\theta}{r}$.

(b) Show that $\hat{\boldsymbol{r}} = \cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{\theta}} = -\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}$.

(c) Using the chain rule, show that $\nabla = \hat{\boldsymbol{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta}$ and hence $\nabla u = \frac{\partial u}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial u}{\partial \theta}\hat{\boldsymbol{\theta}}$.

(d) If $\boldsymbol{A} = A_r\hat{\boldsymbol{r}} + A_\theta\hat{\boldsymbol{\theta}}$, show that $\nabla\cdot\boldsymbol{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(A_\theta)$, since $\partial\hat{\boldsymbol{r}}/\partial\theta = \hat{\boldsymbol{\theta}}$ and $\partial\hat{\boldsymbol{\theta}}/\partial\theta = -\hat{\boldsymbol{r}}$ follows from part (b).

(e) Show that $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$.

1.5.4. Using Exercise 1.5.3(a) and the chain rule for partial derivatives, derive the special case of Exercise 1.5.3(e) if $u(r)$ only.

1.5.5. Assume that the temperature is circularly symmetric: $u = u(r,t)$, where $r^2 = x^2 + y^2$. We will derive the heat equation for this problem. Consider any circular annulus $a \le r \le b$.

(a) Show that the total heat energy is $2\pi \int_a^b c\rho u r \, dr$.

(b) Show that the flow of heat energy per unit time out of the annulus at $r = b$ is $-2\pi b K_0 \partial u/\partial r \,|_{r=b}$. A similar result holds at $r = a$.

(c) Use parts (a) and (b) to derive the circularly symmetric heat equation without sources:

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right).$$

1.5.6. Modify Exercise 1.5.5 if the thermal properties depend on r.

1.5.7. Derive the heat equation in two dimensions by using Green's theorem, (1.5.16), the two-dimensional form of the divergence theorem.

1.5.8. If Laplace's equation is satisfied in three dimensions, show that

$$\oiint \nabla u \cdot \hat{n} \, dS = 0$$

for any closed surface. (*Hint*: Use the divergence theorem.) Give a physical interpretation of this result (in the context of heat flow).

1.5.9. Determine the equilibrium temperature distribution inside a circular annulus ($r_1 \le r \le r_2$):

*(a) if the outer radius is at temperature T_2 and the inner at T_1

(b) if the outer radius is insulated and the inner radius is at temperature T_1

1.5.10. Determine the equilibrium temperature distribution inside a circle ($r \le r_0$) if the boundary is fixed at temperature T_0.

*1.5.11. Consider

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \qquad a < r < b$$

subject to

$$u(r,0) = f(r), \quad \frac{\partial u}{\partial r}(a,t) = \beta, \text{ and } \frac{\partial u}{\partial r}(b,t) = 1.$$

Using physical reasoning, for what value(s) of β does an equilibrium temperature distribution exist?

1.5.12. Assume that the temperature is spherically symmetric, $u = u(r,t)$, where r is the distance from a fixed point ($r^2 = x^2 + y^2 + z^2$). Consider the heat flow (without sources) between any two concentric spheres of radii a and b.

(a) Show that the total heat energy is $4\pi \int_a^b c\rho u r^2 \, dr$.

(b) Show that the flow of heat energy per unit time out of the spherical shell at $r = b$ is $-4\pi b^2 K_0 \, \partial u/\partial r \,|_{r=b}$. A similar result holds at $r = a$.

(c) Use parts (a) and (b) to derive the spherically symmetric heat equation

$$\frac{\partial u}{\partial t} = \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right).$$

*1.5.13. Determine the *steady-state* temperature distribution between two concentric *spheres* with radii 1 and 4, respectively, if the temperature of the outer sphere is maintained at 80° and the inner sphere at 0° (see Exercise 1.5.12).

1.5.14. Isobars are lines of constant temperature. Show that isobars are perpendicular to any part of the boundary that is insulated.

1.5.15. Derive the heat equation in three dimensions assuming constant thermal properties and no sources.

1.5.16. Express the integral conservation law for any three-dimensional object. Assume there are no sources. Also assume the heat flow is specified, $\nabla u \cdot \hat{\boldsymbol{n}} = g(x, y, z)$, on the entire boundary and does not depend on time. By integrating with respect to time, determine the total thermal energy. (Hint: Use the initial condition.)

1.5.17. Derive the integral conservation law for any three dimensional object (with constant thermal properties) by integrating the heat equation (1.5.11) (assuming no sources). Show that the result is equivalent to (1.5.1).

Orthogonal curvilinear coordinates. A coordinate system (u, v, w) may be introduced and defined by $x = x(u,v,w)$, $y = y(u,v,w)$ and $z = z(u,v,w)$. The radial vector $\boldsymbol{r} \equiv x\hat{\imath} + y\hat{\jmath} + z\hat{\boldsymbol{k}}$. Partial derivatives of $\boldsymbol{r}$ with respect to a coordinate are in the direction of the coordinate. Thus, for example, a vector in the u-direction $\partial \boldsymbol{r}/\partial u$ can be made a unit vector $\hat{\boldsymbol{e}}_u$ in the u-direction by dividing by its length $h_u = |\partial \boldsymbol{r}/\partial u|$ called the **scale factor**: $\hat{\boldsymbol{e}}_u = \frac{1}{h_u}\partial \boldsymbol{r}/\partial u$.

1.5.18. Determine the scale factors for cylindrical coordinates.

1.5.19. Determine the scale factors for spherical coordinates.

1.5.20. The gradient of a scalar can be expressed in terms of the new coordinate system $\nabla g = a\,\partial \boldsymbol{r}/\partial u + b\,\partial \boldsymbol{r}/\partial v + c\,\partial \boldsymbol{r}/\partial w$, where you will determine the scalars a, b, c. Using $dg = \nabla g \cdot d\boldsymbol{r}$, derive that the **gradient** in an orthogonal curvilinear coordinate system is given by

$$\nabla g = \frac{1}{h_u}\frac{\partial g}{\partial u}\hat{\boldsymbol{e}}_u + \frac{1}{h_v}\frac{\partial g}{\partial v}\hat{\boldsymbol{e}}_v + \frac{1}{h_w}\frac{\partial g}{\partial w}\hat{\boldsymbol{e}}_w. \qquad (1.5.23)$$

An expression for the divergence is more difficult to derive, and we will just state that if a vector $\boldsymbol{p}$ is expressed in terms of this new coordinate system $\boldsymbol{p} = p_u \hat{\boldsymbol{e}}_u + p_v \hat{\boldsymbol{e}}_v + p_w \hat{\boldsymbol{e}}_w$, then the **divergence** satisfies

$$\nabla \cdot \boldsymbol{p} = \frac{1}{h_u h_v h_w} \left[\frac{\partial}{\partial u} (h_v h_w p_u) + \frac{\partial}{\partial v} (h_u h_w p_v) + \frac{\partial}{\partial w} (h_u h_v p_w) \right]. \quad (1.5.24)$$

1.5.21. Using (1.5.23) and (1.5.24), derive the **Laplacian** in an orthogonal curvilinear coordinate system:

$$\nabla^2 T = \frac{1}{h_u h_v h_w} \left[\frac{\partial}{\partial u} \left(\frac{h_v h_w}{h_u} \frac{\partial T}{\partial u} \right) + \frac{\partial}{\partial v} \left(\frac{h_u h_w}{h_v} \frac{\partial T}{\partial v} \right) + \frac{\partial}{\partial w} \left(\frac{h_u h_v}{h_w} \frac{\partial T}{\partial w} \right) \right]. \quad (1.5.25)$$

1.5.22. Using (1.5.25), derive the Laplacian for cylindrical coordinates.

1.5.23. Using (1.5.25), derive the Laplacian for spherical coordinates.

Appendix to 1.5: Review of Gradient and a Derivation of Fourier's Law of Heat Conduction

Experimentally, for isotropic[6] materials (i.e., without preferential directions) **heat flows from hot to cold in the direction in which temperature differences are greatest**. The heat flow is proportional (with proportionality constant K_0, the thermal conductivity) to the rate of change of temperature in this direction.

The change in the temperature Δu is

$$\Delta u = u(\boldsymbol{x} + \Delta \boldsymbol{x}, t) - u(\boldsymbol{x}, t) \approx \frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y + \frac{\partial u}{\partial z} \Delta z.$$

In the direction $\hat{\boldsymbol{\alpha}} = \alpha_1 \hat{\boldsymbol{i}} + \alpha_2 \hat{\boldsymbol{j}} + \alpha_3 \hat{\boldsymbol{k}}$, $\Delta \boldsymbol{x} = \Delta s \hat{\boldsymbol{\alpha}}$, where Δs is the distance between $\boldsymbol{x}$ and $\boldsymbol{x} + \Delta \boldsymbol{x}$. Thus, the rate of change of the temperature in the direction $\hat{\boldsymbol{\alpha}}$ is the **directional derivative**:

$$\lim_{\Delta s \to 0} \frac{\Delta u}{\Delta s} = \alpha_1 \frac{\partial u}{\partial x} + \alpha_2 \frac{\partial u}{\partial y} + \alpha_3 \frac{\partial u}{\partial z} = \boldsymbol{\alpha} \cdot \nabla u,$$

where it has been convenient to define the following *vector*:

$$\boxed{\nabla u \equiv \frac{\partial u}{\partial x} \hat{\boldsymbol{i}} + \frac{\partial u}{\partial y} \hat{\boldsymbol{j}} + \frac{\partial u}{\partial z} \hat{\boldsymbol{k}},} \quad (1.5.26)$$

called the **gradient** of the temperature. From the property of dot products, if θ is the angle between $\hat{\boldsymbol{\alpha}}$ and ∇u, then the directional derivative is $|\nabla u| \cos \theta$ since

[6] Examples of nonisotropic materials are certain crystal and grainy woods.