for any constant C_2. Unlike the first example (with fixed temperatures at both ends), here there is not a unique equilibrium temperature. Any constant temperature is an equilibrium temperature distribution for insulated boundary conditions. Thus, for the time-dependent initial value problem, we expect

$$\lim_{t\to\infty} u(x,t) = C_2;$$

if we wait long enough, a rod with insulated ends should approach a constant temperature. This seems physically quite reasonable. However, it does not make sense that the solution should approach an arbitrary constant; we ought to know what constant it approaches. In this case, the lack of uniqueness was caused by the complete neglect of the initial condition. In general, the equilibrium solution will not satisfy the initial condition. However, the particular constant equilibrium solution is determined by considering the initial condition for the time-dependent problem (1.4.11). Since both ends are insulated, the total thermal energy is constant. This follows from the integral conservation of thermal energy of the entire rod [see (1.2.4)]:

$$\frac{d}{dt}\int_0^L c\rho u\,dx = -K_0\frac{\partial u}{\partial x}(0,t) + K_0\frac{\partial u}{\partial x}(L,t). \tag{1.4.19}$$

Since both ends are insulated,

$$\int_0^L c\rho u\,dx = \text{constant}. \tag{1.4.20}$$

One implication of (1.4.20) is that the initial thermal energy must equal the final ($\lim_{t\to\infty}$) thermal energy. The initial thermal energy is $c\rho\int_0^L f(x)\,dx$ since $u(x,0) = f(x)$, while the equilibrium thermal energy is $c\rho\int_0^L C_2\,dx = c\rho C_2 L$ since the equilibrium temperature distribution is a constant $u(x,t) = C_2$. The constant C_2 is determined by equating these two expressions for the constant total thermal energy, $c\rho\int_0^L f(x)\,dx = c\rho C_2 L$. Solving for C_2 shows that the desired unique steady-state solution should be

$$\boxed{u(x) = C_2 = \frac{1}{L}\int_0^L f(x)\,dx,} \tag{1.4.21}$$

the **average** of the initial temperature distribution. It is as though the initial condition is not entirely forgotten. Later we will find a $u(x,t)$ that satisfies (1.4.10–1.4.13) and show that $\lim_{t\to\infty} u(x,t)$ is given by (1.4.21).

EXERCISES 1.4

1.4.1. Determine the equilibrium temperature distribution for a one-dimensional rod with constant thermal properties with the following sources and boundary conditions:

* (a)	$Q = 0,$	$u(0) = 0,$	$u(L) = T$
(b)	$Q = 0,$	$u(0) = T,$	$u(L) = 0$
(c)	$Q = 0,$	$\frac{\partial u}{\partial x}(0) = 0,$	$u(L) = T$
* (d)	$Q = 0,$	$u(0) = T,$	$\frac{\partial u}{\partial x}(L) = \alpha$
(e)	$\frac{Q}{K_0} = 1,$	$u(0) = T_1,$	$u(L) = T_2$
* (f)	$\frac{Q}{K_0} = x^2,$	$u(0) = T,$	$\frac{\partial u}{\partial x}(L) = 0$
(g)	$Q = 0,$	$u(0) = T,$	$\frac{\partial u}{\partial x}(L) + u(L) = 0$
* (h)	$Q = 0,$	$\frac{\partial u}{\partial x}(0) - [u(0) - T] = 0,$	$\frac{\partial u}{\partial x}(L) = \alpha$

In these you may assume that $u(x, 0) = f(x)$.

1.4.2. Consider the equilibrium temperature distribution for a uniform one-dimensional rod with sources $Q/K_0 = x$ of thermal energy, subject to the boundary conditions $u(0) = 0$ and $u(L) = 0$.

*(a) Determine the heat energy generated per unit time inside the entire rod.

(b) Determine the heat energy flowing out of the rod per unit time at $x = 0$ and at $x = L$.

(c) What relationships should exist between the answers in parts (a) and (b)?

1.4.3. Determine the equilibrium temperature distribution for a one-dimensional rod composed of two different materials in perfect thermal contact at $x = 1$. For $0 < x < 1$, there is one material ($c\rho = 1, K_0 = 1$) with a constant source ($Q = 1$), whereas for the other $1 < x < 2$ there are no sources ($Q = 0, c\rho = 2, K_0 = 2$) (see Exercise 1.3.2) with $u(0) = 0$ and $u(2) = 0$.

1.4.4. If both ends of a rod are insulated, derive *from the partial differential equation* that the total thermal energy in the rod is constant.

1.4.5. Consider a one-dimensional rod $0 \le x \le L$ of known length and known constant thermal properties without sources. Suppose that the temperature is an *unknown* constant T at $x = L$. Determine T if we know (in the steady state) both the temperature and the heat flow at $x = 0$.

1.4.6. The two ends of a uniform rod of length L are insulated. There is a constant source of thermal energy $Q_0 \neq 0$, and the temperature is initially $u(x, 0) = f(x)$.

(a) Show mathematically that there does not exist any equilibrium temperature distribution. Briefly explain physically.

(b) Calculate the total thermal energy in the entire rod.

1.4.7. For the following problems, determine an equilibrium temperature distribution (if one exists). For what values of β are there solutions? Explain physically.

* (a) $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + 1, \quad u(x,0) = f(x), \quad \dfrac{\partial u}{\partial x}(0,t) = 1, \quad \dfrac{\partial u}{\partial x}(L,t) = \beta$

(b) $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, \quad u(x,0) = f(x), \quad \dfrac{\partial u}{\partial x}(0,t) = 1, \quad \dfrac{\partial u}{\partial x}(L,t) = \beta$

(c) $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + x - \beta, \quad u(x,0) = f(x), \quad \dfrac{\partial u}{\partial x}(0,t) = 0, \quad \dfrac{\partial u}{\partial x}(L,t) = 0$

1.4.8. Express the integral conservation law for the entire rod with constant thermal properties. Assume the heat flow is known to be different constants at both ends By integrating with respect to time, determine the total thermal energy in the rod. (Hint: use the initial condition.)

(a) Assume there are no sources.

(b) Assume the sources of thermal energy are constant.

1.4.9. Derive the integral conservation law for the entire rod with constant thermal properties by integrating the heat equation (1.2.10) (assuming no sources). Show the result is equivalent to (1.2.4).

1.4.10. Suppose $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 4$, $u(x,0) = f(x)$, $\frac{\partial u}{\partial x}(0,t) = 5$, $\frac{\partial u}{\partial x}(L,t) = 6$. Calculate the total thermal energy in the one-dimensional rod (as a function of time).

1.4.11. Suppose $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x$, $u(x,0) = f(x)$, $\frac{\partial u}{\partial x}(0,t) = \beta$, $\frac{\partial u}{\partial x}(L,t) = 7$.

(a) Calculate the total thermal energy in the one-dimensional rod (as a function of time).

(b) From part (a), determine a value of β for which an equilibrium exists. For this value of β, determine $\lim_{t \to \infty} u(x,t)$.

1.4.12. Suppose the concentration $u(x,t)$ of a chemical satisfies Fick's law (1.2.13), and the initial concentration is given $u(x,0) = f(x)$. Consider a region $0 < x < L$ in which the flow is specified at both ends $-k\frac{\partial u}{\partial x}(0,t) = \alpha$ and $-k\frac{\partial u}{\partial x}(L,t) = \beta$. Assume α and β are constants.

(a) Express the conservation law for the entire region.

(b) Determine the total amount of chemical in the region as a function of time (using the initial condition).

(c) Under what conditions is there an equilibrium chemical concentration and what is it?

1.4.13. Do Exercise 1.4.12 if α and β are functions of time.

1.5 Derivation of the Heat Equation in Two or Three Dimensions

Introduction. In Sec. 1.2 we showed that for the conduction of heat in a one-dimensional rod the temperature $u(x,t)$ satisfies

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + Q.$$

In cases in which there are no sources ($Q = 0$) and the thermal properties are constant, the partial differential equation becomes

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

where $k = K_0/c\rho$. Before we solve problems involving these partial differential equations, we will formulate partial differential equations corresponding to heat flow problems in two or three spatial dimensions. We will find the derivation to be similar to the one used for one-dimensional problems, although important differences will emerge. We propose to derive new and more complex equations (before solving the simpler ones) so that, when we do discuss *techniques* for the solutions of PDEs, we will have more than one example to work with.

Heat energy. We begin our derivation by considering any *arbitrary subregion* R, as illustrated in Fig. 1.5.1. As in the one-dimensional case, conservation of heat energy is summarized by the following word equation:

rate of change of heat energy	=	heat energy flowing across the boundaries per unit time	+	heat energy generated inside per unit time,

where the heat energy within an arbitrary subregion R is

$$\text{heat energy} = \iiint_R c\rho u \, dV,$$

instead of the one-dimensional integral used in Sec. 1.2.