$$\boxed{\frac{\partial u}{\partial t} + \frac{\partial \phi}{\partial x} = 0.} \tag{1.2.12}$$

In solids, chemicals spread out from regions of high concentration to regions of low concentration. According to **Fick's law of diffusion**, the flux is proportional to $\frac{\partial u}{\partial x}$ the spatial derivative of the chemical concentration:

$$\boxed{\phi = -k\frac{\partial u}{\partial x}.} \tag{1.2.13}$$

If the concentration $u(x,t)$ is constant in space, there is no flow of the chemical. If the chemical concentration is increasing to the right ($\frac{\partial u}{\partial x} > 0$), then atoms of chemicals migrate to the left, and vice versa. The proportionality constant k is called the chemical **diffusivity**, and it can be measured experimentally. When Fick's law (1.2.13) is used in the basic conservation law (1.2.12), we see that the chemical concentration satisfies the **diffusion equation**:

$$\boxed{\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},} \tag{1.2.14}$$

since we are assuming as an approximation that the diffusivity is constant. Fick's law of diffusion for chemical concentration is analogous to Fourier's law for heat diffusion. Our derivations are quite similar.

EXERCISES 1.2

1.2.1. Briefly explain the minus sign:

(a) in conservation law (1.2.3) or (1.2.5) if $Q = 0$

(b) in Fourier's law (1.2.8)

(c) in conservation law (1.2.12), $\frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x}$

(d) in Fick's law (1.2.13)

1.2.2. Derive the heat equation for a rod assuming constant thermal properties and no sources.

(a) Consider the total thermal energy between x and $x + \Delta x$.

(b) Consider the total thermal energy between $x = a$ and $x = b$.

1.2.3. Derive the heat equation for a rod assuming constant thermal properties with variable cross-sectional area $A(x)$ assuming no sources by considering the total thermal energy between $x = a$ and $x = b$.

1.2.4. Derive the diffusion equation for a chemical pollutant.

(a) Consider the total amount of the chemical in a thin region between x and $x + \Delta x$.

(b) Consider the total amount of the chemical between $x = a$ and $x = b$.

1.2.5. Derive an equation for the concentration $u(x, t)$ of a chemical pollutant if the chemical is produced due to chemical reaction at the rate of $\alpha u(\beta - u)$ per unit volume.

1.2.6. Suppose that the specific heat is a function of position and temperature, $c(x, u)$.

(a) Show that the heat energy per unit mass necessary to raise the temperature of a thin slice of thickness Δx from 0° to $u(x, t)$ is not $c(x)u(x, t)$, but instead $\int_0^u c(x, \bar{u})\, d\bar{u}$.

(b) Rederive the heat equation in this case. Show that (1.2.3) remains unchanged.

1.2.7. Consider conservation of thermal energy (1.2.4) for any segment of a one-dimensional rod $a \leq x \leq b$. By using the fundamental theorem of calculus,

$$\frac{\partial}{\partial b} \int_a^b f(x)\, dx = f(b),$$

derive the heat equation (1.2.9).

*1.2.8. If $u(x, t)$ is known, give an expression for the total thermal energy contained in a rod $(0 < x < L)$.

1.2.9. Consider a thin one-dimensional rod without sources of thermal energy whose lateral surface area is not insulated.

(a) Assume that the heat energy flowing out of the lateral sides per unit surface area per unit time is $w(x, t)$. Derive the partial differential equation for the temperature $u(x, t)$.

(b) Assume that $w(x, t)$ is proportional to the temperature difference between the rod $u(x, t)$ and a known outside temperature $\gamma(x, t)$. Derive that

$$c\rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0 \frac{\partial u}{\partial x}\right) - \frac{P}{A}[u(x, t) - \gamma(x, t)]h(x), \qquad (1.2.15)$$

where $h(x)$ is a positive x-dependent proportionality, P is the lateral perimeter, and A is the cross-sectional area.

(c) Compare (1.2.15) to the equation for a one-dimensional rod whose lateral surfaces are insulated, but with heat sources.

(d) Specialize (1.2.15) to a rod of circular cross section with constant thermal properties and 0° outside temperature.

*(e) Consider the assumptions in part (d). Suppose that the temperature in the rod is uniform [i.e., $u(x,t) = u(t)$]. Determine $u(t)$ if initially $u(0) = u_0$.

1.3 Boundary Conditions

In solving the heat equation, either (1.2.9) or (1.2.10), one **boundary condition** (BC) is needed at each end of the rod. The appropriate condition depends on the physical mechanism in effect at each end. Often the condition at the boundary depends on both the material inside and outside the rod. To avoid a more difficult mathematical problem, we will assume that the outside environment is known, not significantly altered by the rod.

Prescribed temperature. In certain situations, the temperature of the end of the rod, for example, $x = 0$, may be approximated by a **prescribed temperature**,

$$u(0,t) = u_B(t), \tag{1.3.1}$$

where $u_B(t)$ is the temperature of a fluid bath (or reservoir) with which the rod is in contact.

Insulated boundary. In other situations it is possible to prescribe the heat flow rather than the temperature,

$$-K_0(0)\frac{\partial u}{\partial x}(0,t) = \phi(t), \tag{1.3.2}$$

where $\phi(t)$ is given. This is equivalent to giving one condition for the first derivative, $\partial u/\partial x$, at $x = 0$. The slope is given at $x = 0$. Equation (1.3.2) *cannot* be integrated in x because the slope is known only at one value of x. The simplest example of the prescribed heat flow boundary condition is when an end is **perfectly insulated** (sometimes we omit the "perfectly"). In this case there is no heat flow at the boundary. If $x = 0$ is insulated, then

$$\frac{\partial u}{\partial x}(0,t) = 0. \tag{1.3.3}$$

Newton's law of cooling. When a one-dimensional rod is in contact at the boundary with a moving fluid (e.g., air), then neither the prescribed temperature nor the prescribed heat flow may be appropriate. For example, let us imagine a very warm rod in contact with cooler moving air. Heat will leave the rod, heating up the air. The air will then carry the heat away. This process of heat transfer is called **convection**. However, the air will be hotter near the rod. Again, this is a complicated problem; the air temperature will actually vary with distance from the rod (ranging between the bath and rod temperatures). Experiments show that, as a good approximation, the heat flow leaving the rod is proportional to the temperature