Math 221 Practice Test - December 16

1. a.) Show that $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$
Solution: $y = \tan^{-1}x$, so $\tan y = x$.
Taking the derivative of both sides, we see $\sec^2 y\frac{dy}{dx} = 1$
$\frac{dy}{dx} = \frac{1}{\sec^2 y}$
Using the Pythagorean Identity $\sec^2 a = 1+\tan^2 a$, we get $\frac{dy}{dx} = \frac{1}{1+\tan^2 y}$
Finally, using $\tan y = x$ from above, we get $\frac{dy}{dx} = \frac{1}{1+x^2}$
This can also be done using facts about the derivatives of inverse functions.

b.) Evaluate $\int \frac{x^2+2x-1}{x^2+9}dx$
Solution: $\int \frac{x^2+2x-1}{x^2+9}dx = \int \frac{x^2+9-9+2x-1}{x^2+9}dx$ Note: This step could also be done using polynomial long division, but this is a bit neater on my page.
$\int \frac{x^2+2x-1}{x^2+9}dx = \int(\frac{x^2+9}{x^2+9} + \frac{2x}{x^2+9} + \frac{-10}{x^2+9})dx$
So using the fact that we can split up integrals, we get $\int \frac{x^2+2x-1}{x^2+9}dx = \int \frac{x^2+9}{x^2+9}dx + \int \frac{2x}{x^2+9}dx + \int \frac{-10}{x^2+9}dx$
Taking each of these integrals in turn, we get $\int \frac{x^2+9}{x^2+9}dx = \int 1dx = x+C$
For $\int \frac{2x}{x^2+9}dx$, we use a substitution of $u = x^2+9$. In this case, $du = 2xdx$.
This means $\int \frac{2x}{x^2+9}dx = \int \frac{1}{x^2+9}2xdx = \int \frac{1}{u}du$
$\int \frac{2x}{x^2+9}dx = \ln|u| + C = \ln|x^2+9| + C$
Finally, the last part is the trickiest. $\int \frac{-10}{x^2+9}dx$ has a sum of squares in the denominator, so it looks like the integral will be of the same form as inverse tangent. Hence, we want something that looks like x^2+1 in the denominator. We will factor out a 9.
$\int \frac{-10}{x^2+9}dx = \frac{-10}{9}\int \frac{1}{\frac{x^2}{9}+1}dx$.
Using the substitution $3u = x$, (the same as $u = \frac{x}{3}$) we get $3du = dx$.
$\int \frac{-10}{x^2+9}dx = \frac{-10}{9}\int \frac{1}{u^2+1}3du = \frac{-10}{3}\int \frac{1}{u^2+1}du$
This is now a recognizable integral. $\int \frac{-10}{x^2+9}dx = \frac{-10}{3}\int \frac{1}{u^2+1}du = \frac{-10}{3}\tan^{-1}u + C = \frac{-10}{3}\tan^{-1}(\frac{x}{3}) + C$
Bringing all of these parts together, we get $\int \frac{x^2+2x-1}{x^2+9}dx = x + \ln(x^2+9) - \frac{10}{3}\tan^{-1}(\frac{x}{3}) + C$ (note that all of the C's combine into a single arbitrary constant).

2. Compute the derivatives $\frac{dy}{dx}$

a.) $y = \ln(\ln(x^2))$

Solution: $y' = (\ln(\ln(x^2)))'$. Using the chain rule, the outermost function is $\ln u$ and its derivative is $\frac{1}{u}$

$y' = \frac{1}{\ln(x^2)}(\ln(x^2))'$

Using a property of logarithms, $\ln(x^2) = 2\ln x$, so $y' = \frac{1}{2\ln x}(2\ln x)' = \frac{1}{2\ln x}\frac{2}{x}$. This simplifies to $\frac{1}{x\ln x}$.

b.) $\ln y = e^y \sin x$

Solution: $(\ln y)' = (e^y \sin x)'$

$\frac{y'}{y} = e^y y' \sin x + e^y \cos x$ using the product rule and implicit differentiation.

$\frac{y'}{y} - y'e^y \sin x = e^y \cos x$

$y'(\frac{1}{y} - e^y \sin x = e^y \cos x$

Dividing by the coefficient of y', we get $y' = \frac{e^y \cos x}{\frac{1}{y} - e^y \sin x}$, which is our derivative.

c.) $y = 2^{\sin(3x)}$

Solution: The only exponential we're comfortable differentiating is an exponential of e. Hence, we will remember that $2 = e^{\ln 2}$.

$y = (e^{\ln 2})^{\sin(3x)} = e^{(\ln 2)\sin(3x)}$

Differentiating, we remember the chain rule $y' = e^{(\ln 2)\sin(3x)}((\ln 2)\sin(3x))' = e^{(\ln 2)\sin(3x)}((\ln 2)3\cos(3x))$.

Simplifying this, we get $y' = 3(\ln 2)2^{\sin(3x)}\cos(3x)$.

3. $\frac{dV}{dt} = -\frac{1}{40}V$, $V(0) = V_0$

a.) Solve for $V(t)$

Solution: We first need to get all the V's and t's on their respective sides. To accomplish this, we "multiply" by dt.

$\frac{dV}{V} = -\frac{1}{40}dt$

Integrating both sides, we get $\ln|V| = -\frac{1}{40}t + c$

Exponentiating both sides, we get $e^{\ln|V|} = e^{-\frac{1}{40}t+c}$

$|V| = e^{-\frac{1}{40}t+c} = e^c e^{-\frac{1}{40}t}$

We notice that e^c can be any positive number. Together with eliminating the absolute value on V, this means e^c can be any (non-zero) number. We will call this number C.

$V = Ce^{-\frac{1}{40}t}$

$V(0) = V_0$, so $V_0 = Ce^{-\frac{1}{40}*0} = Ce^0 = C$.

$V(t) = V_0 e^{-\frac{1}{40}t}$

b.) Show how long it will take for V to reach ten percent of its initial value.
Solution: Ten percent of the initial value is $.1V_0$. We will plug this in and solve for t.
$.1V_0 = V_0 e^{-\frac{1}{40}t}$ At this point, we divide both sides by V_0, cancelling it off.
$.1 = e^{-\frac{1}{40}t}$ We need to get t out of the exponent, so we will take the natural logarithm of both sides.
$\ln .1 = \ln e^{-\frac{1}{40}t} = -\frac{1}{40}t$
$t = -40 \ln .1$. Using the reciprocal property of logarithms, we can simplify this to $t = 40 \ln 10$.

4. Find the volume of the figure obtained by rotating the region between $y = 2 - x^2$, $y = x^2$, and $x = 0$ about the y-axis. (Hint: the shell method will be easier here, but use whatever method you wish)

   Solution: Using the shell method, $V = \int\limits_a^b 2\pi r h dr$.
   The radius is the distance from the axis of rotation, hence the distance from the y-axis, so $r = x$. The radius ranges from 0 to where these functions meet.
   $2 - x^2 = x^2$, so $2 = 2x^2$, so $1 = x^2$. $x = 1$ or $x = -1$. As it turns out, these will rotate over each other (the function is even, so it matches up). We will, therefore, integrate from 0 to 1.
   The height is the distance from the top of the shell to the bottom of the shell. In this region, x^2 is smaller than $2 - x^2$. Therefore, the height is $(2 - x^2) - x^2 = 2 - 2x^2$.
   $V = \int\limits_0^1 2\pi x(2 - 2x^2)dx$
   $V = 4\pi \int\limits_0^1 (x - x^3)dx$
   $V = 4\pi(\frac{x^2}{2} - \frac{x^4}{4})|_0^1 = 4\pi(\frac{1^2}{2} - \frac{1^4}{4} - (\frac{0^2}{2} - \frac{0^4}{4}))$
   This works out to $V = \pi$.

5. Find the total area between $y = x\sqrt{a^2 - x^2}$ and $y = 0$ for $-a \leq x \leq a$

$(a > 0)$
Solution: We notice that when $x < 0$, the x is negative. Similarly, when $x > 0$, the x is positive. $\sqrt{a^2 - x^2}$ is always positive. Therefore, this function is positive when $x > 0$ and negative when $x < 0$.
$A = \int_a^b |f(x)|dx$
In this case, $A = \int_{-a}^{a} |x\sqrt{a^2 - x^2}|dx$.
As stated above, the sign flips at 0. In fact, with the absolute value, this is an even function, so the area simplifies to $A = 2\int_0^a x\sqrt{a^2 - x^2}dx$.
We use a substitution $u = a^2 - x^2$, which gives us $du = -2xdx$. Hence, when $x = 0$, $u = a^2$. When $x = a$, $u = 0$.
Our integral becomes $A = -\int_{a^2}^{0} \sqrt{u}du = \int_0^{a^2} \sqrt{u}du$ since negative signs flip the direction of integration.
$A = \frac{2}{3}u^{\frac{3}{2}}|_0^{a^2} = \frac{2}{3}((a^2)^{\frac{2}{3}} - 0^{\frac{3}{2}}) = \frac{2}{3}a^3$.

6. Sketch (including minima, maxima, intercepts, asymptotes and any other relevant data you know how to compute) $f(x) = x^{\frac{2}{3}}(\frac{5}{2} - x)$
Solution: See other solution set.

7. The sum of two non-negative numbers is 20. Find the numbers if:
a.) the product of one number with the square root of the other is to be as large as possible.
Solution: We have two numbers a and b. We have the relationship that $a + b = 20$. We are trying to maximize $M = a\sqrt{b}$.
Solving for a above, we find $a = 20 - b$.
Substituting, we find $M = (20 - b)\sqrt{b}$. We also notice that $0 \leq b \leq 20$. This is a continuous function on a closed interval, so it will have a maximum and a minimum (by the extreme value theorem).
$M' = \frac{10}{\sqrt{b}} - \frac{3}{2}\sqrt{b}$. We find where this is zero or non-existent to find critical points (and hence places we need to check for maxima). $b = 0$ is the only place of interest where the derivative does not exist.
$0 = \frac{10}{\sqrt{b}} - \frac{3}{2}\sqrt{b}$
Multiplying by $\sqrt{b}$, we get $0 = 10 - \frac{3}{2}b$. Solving, we get $b = \frac{20}{3}$.
The maximum exists. The First Derivative Test tells us it happens

either at an endpoint or at a critical point. We only have three points to check. $M(0) = 0$, $M(20) = 0$, and $M(\frac{20}{3}) = \frac{40}{3} * \sqrt{\frac{20}{3}}$. The largest of these is clearly at $\frac{20}{3}$. Hence, the numbers are $\frac{20}{3}$ and $\frac{40}{3}$.

b.) one number plus the square root of the other number is to be as large as possible.
Solution: We set this up in a similar matter to above. The only difference is $N = a + \sqrt{b}$.
Using the same solving for a, we get $N = 20 - b + \sqrt{b}$. We actually have several ways of finding the maximum of this (it's a parabola in $\sqrt{b}$), but let's use our old friend calculus.
$N' = -1 + \frac{1}{2\sqrt{b}}$.
As above, we find zeroes and discontinuities of N'. These occur at $\frac{1}{4}$ and 0 respectively.
Again, we notice this function is continuous on a closed, bounded domain, so it has a minimum and a maximum. $N(0) = 20$. $N(20) = \sqrt{20}$. $N(\frac{1}{4}) = 20 - \frac{1}{4} + \sqrt{\frac{1}{4}} = 20 - \frac{1}{4} + \frac{1}{2} = 20.25$. Clearly, 20.25 is the biggest. This means our numbers are 19.75 and .25.

8. Evaluate
a.) $\int \frac{\sin(2t+1)}{\cos^2(2t+1)} dt$
Solution: Set $u = \cos(2t + 1)$. This makes $du = -2\sin(2t + 1)dt$.
$\int \frac{\sin(2t+1)}{\cos^2(2t+1)} dt = -\frac{1}{2} \int \frac{1}{u^2} du$
$\int \frac{\sin(2t+1)}{\cos^2(2t+1)} dt = \frac{1}{2}\frac{1}{u} + C$
$\int \frac{\sin(2t+1)}{\cos^2(2t+1)} dt = \frac{1}{2\cos(2t+1)} + C$

b.) $\int \frac{1}{x^2} \sqrt{2 - \frac{1}{x}} dx$
Solution: In this case, we want $u = 2 - \frac{1}{x}$, so $du = \frac{1}{x^2}$
$\int \frac{1}{x^2} \sqrt{2 - \frac{1}{x}} dx = \int \sqrt{u} du$
$= \int u^{\frac{1}{2}} du$
$= \frac{2}{3} u^{\frac{3}{2}} + C = \frac{2}{3}(2 - \frac{1}{x})^{\frac{3}{2}} + C$

c.) $\int_0^{\frac{\pi}{2}} \frac{\sin x}{(3+2\cos x)^2} dx$
Solution: Here we choose $u = 3 + 2\cos x$, so $du = -2\sin x dx$. Also, when $x = 0$, $u = 3 + 2\cos 0 = 5$. When $x = \frac{\pi}{2}$, $u = 3 + \cos\frac{\pi}{2} = 3$.

$\int_{0}^{\frac{\pi}{2}} \frac{\sin x}{(3+2\cos x)^2} dx = -\frac{1}{2} \int_{5}^{3} \frac{1}{u^2} du$

$= \frac{1}{2} \int_{3}^{5} \frac{1}{u^2} du$ since a negative sign flips the direction of integration.

$= \frac{1}{2}(-\frac{1}{u})|_3^5 = \frac{1}{2}(-\frac{1}{5} - (-\frac{1}{3})) = \frac{1}{15}$.

Therefore, $\int_{0}^{\frac{\pi}{2}} \frac{\sin x}{(3+2\cos x)^2} dx = \frac{1}{15}$.