QUESTION 1

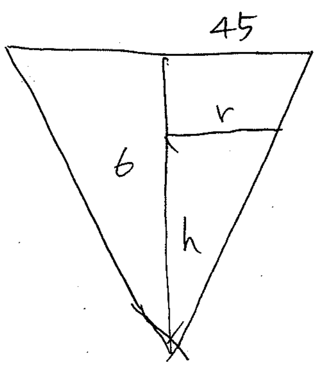
(____ / 20)

A draining conical reservoir Water is flowing at the rate of $50\,\mathrm{m}^3/\mathrm{min}$ from a shallow concrete conical reservoir (the cone points downwards) of base radius $45\,\mathrm{m}$ and height $6\,\mathrm{m}$. Given that the volume of a cone is $V = \frac{1}{3}\pi r^2 h$,

(a) At what rate (cm/min) is the water level changing when the water is $5\,\mathrm{m}$ deep? *[10 points]*

(b) How fast (in cm/min) is the radius of the water's surface changing then? *[10 points]*

ANSWER



$$\frac{r}{45} = \frac{h}{6}$$

$$\frac{dV}{dt} = -50.$$

$$V = \frac{1}{3}\pi r^2 h.$$

(a). $$r = \frac{45}{6}h = \frac{15}{2}h$$

$$V = \frac{1}{3}\pi\left(\frac{15}{2}h\right)^2 h$$

$$= \frac{75}{4}\pi h^3$$

$$\frac{dV}{dt} = \frac{75\pi}{4}(3h^2)\frac{dh}{dt} = \frac{225\pi}{4}\cdot 5^2\frac{dh}{dt}$$

$$-50 = \frac{225\pi}{4}\cdot 25\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = -\cancel{50}\,\frac{4}{\cancel{225}\cdot\cancel{25}\cdot\pi} = -\frac{8}{225\pi}\left(\frac{m}{min}\right)$$

$$= -\frac{8}{\cancel{225\pi}_{\,9}}\cdot\cancel{100}^{\,4}\left(\frac{cm}{min}\right) = -\frac{32}{9}\left(\frac{cm}{min}\right)$$

(b) $$\frac{dr}{dt} = \frac{d}{dt}\left(\frac{45}{6}h\right) = \frac{45}{6}\frac{dh}{dt} = \frac{15}{2}\frac{dh}{dt} = \frac{15}{2}\left(-\frac{32}{9}\right)\left(\frac{cm}{min}\right)$$

$$= -\frac{80}{3}\left(\frac{cm}{min}\right)$$

QUESTION 2 (____ / 20)

It costs you c dollars each to manufacture and distribute backpacks. If the backpacks sell at x dollars each, the number sold is given by

$$n = \frac{a}{x-c} + b\,(100 - x)$$

where a and b are positive constants. Use calculus to find the selling price that will bring a maximum profit (justify that it is a maximum).

[Hint: profit per backpack is the difference between selling price and cost!]

ANSWER

$$P_{(x)} = (x-c)\,n$$

$$= (x-c)\left(\frac{a}{x-c} + b(100-x)\right)$$

$$= a + b(100-x)(x-c)$$

maximize $P(x)$

$$p'(x) = 0 \Rightarrow b(-1)(x-c) + b(100-x) = 0$$

$$\Rightarrow x = 50 + \frac{c}{2}$$

$$\max P(x) = a + b\left(50 - \frac{c}{2}\right)^2$$

QUESTION 3 (____ / 20)

Evaluate the following indefinite integrals.

(a) $\displaystyle\int \tan(2t)\,\sec^2(2t)\,dt$ *[10 points]*

(b) $\displaystyle\int r^2\left(\frac{r^3}{18}-1\right)^5\,dr$ *[10 points]*

ANSWER

a) $\displaystyle\int \tan(2t)\cdot\sec^2(2t)\,dt.$ Let $u = \tan 2t.$

$du = 2\sec^2 2t\,dt$

$\Rightarrow dt = \dfrac{1}{2\sec^2 2t}\cdot du.$

$= \displaystyle\int u\cdot\cancel{\sec^2 2t}\cdot\frac{1}{2\cancel{\sec^2 2t}}\,du$

$= \displaystyle\int \frac{u}{2}\,du$

$= \frac{1}{4}u^2 + C = \frac{1}{4}\tan^2 2t + C$

b) $\displaystyle\int r^2\left(\frac{r^3}{18}-1\right)^5 dr.$ Let $u = \dfrac{r^3}{18} - 1.$

$du = \frac{1}{6}r^2\cdot dr.$

$\Rightarrow dr = \dfrac{6}{r^2}\cdot du.$

$= \displaystyle\int r^2\cdot u^5\cdot\frac{6}{r^2}\cdot du$

$= \displaystyle\int 6u^5\,du = u^6 + C$

$= \left(\frac{r^3}{18}-1\right)^6 + C.$

QUESTION 4 (____ / 20)

Find the total area between

$$y = x^{1/3} - x, \qquad -1 \le x \le 8$$

and the x axis.

ANSWER

First, solve $y=0$: (y is continuous)

$0 = x^{1/3} - x$

$x = x^{1/3}$

$x^3 = x$

$x^3 - x = 0$

$x(x^2-1) = 0$ so $x = -1, 0, 1$

$$A = \int_{-1}^{8} |y|\,dx = \left|\int_{-1}^{0} y\,dx\right| + \left|\int_{0}^{1} y\,dx\right| + \left|\int_{1}^{8} y\,dx\right|$$

Note that $\int y\,dx = \frac{3}{4}x^{4/3} - \frac{1}{2}x^2$. Call this $h(x)$.

$$A = \left|h(0) - h(-1)\right| + \left|h(1) - h(0)\right| + \left|h(8) - h(1)\right|$$

$h(-1) = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$ $\quad h(0) = 0 \quad$ $h(1) = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$

$h(8) = \frac{3}{4}2^4 - \frac{1}{2}8^2 = 3 \cdot 2^2 - \frac{1}{2}\cdot 64 = 12 - 32 = -20$

So

$$A = \left|0 - \tfrac{1}{4}\right| + \left|\tfrac{1}{4} - 0\right| + \left|-20 - \tfrac{1}{4}\right| = \tfrac{1}{4} + \tfrac{1}{4} + 20\tfrac{1}{4} = 20\tfrac{3}{4}$$

$\cancel{= \tfrac{1}{4} + \tfrac{1}{4} + 20\tfrac{1}{4} + 20\tfrac{3}{4}}$

QUESTION 5

(____ / 40)

For the function

$$f(x) = \frac{x^2}{x^2 - 1}$$

(a) What is the domain of $f(x)$? Is $f(x)$ even, odd, or neither? *[5 points]*

(b) Find and simplify the first derivative of $f(x)$. *[5 points]*

(c) Find the critical points of $f(x)$. Where is the function increasing/decreasing? *[5 points]*

(d) Determine if the critical points are local maxima, minima, or neither. Which ones are also global extrema, if any? *[5 points]*

(e) Where is $f(x)$ concave up/down? Are there any inflection points? If so, find them. *[5 points]*

(f) Are there any asymptotes? Find their position. *[5 points]*

(g) Use all the above information to sketch the graph of $y = f(x)$. *[10 points]*

ANSWER

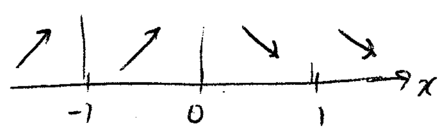
(a) domain is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

$f(x) = f(-x)$, so f is even

(b) $f'(x) = \dfrac{2x(x^2-1) - x^2(2x)}{(x^2-1)^2} = -\dfrac{2x}{(x^2-1)^2}$

(c) $f'(x) = 0$ at $x = 0$, undefined at $x = \pm 1$

Since $(x^2-1)^2 \geq 0$, the sign of $f'(x)$ is the same as sign of $-2x$, so



↗ = increasing
↘ = decreasing

(d) 0 is a local max since f' changes sign at $x = 0$ and f is continuous at $x = 0$.

NOT a global max, since $\lim_{x \to 1^+} \frac{x^2}{x^2-1} = \infty$

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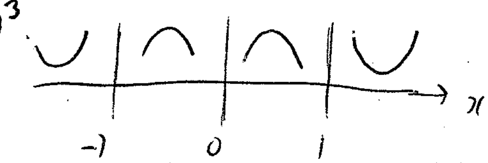
(e) $f''(x) = \dfrac{-2(x^2-1)^2 - (-2x)\,2(x^2-1)(2x)}{(x^2-1)^4}$

$= \dfrac{-2(x^2-1) + 8x^2}{(x^2-1)^3} = \dfrac{6x^2+2}{(x^2-1)^3}$

Numerator is > 0 so sign given by $(x^2-1)^3$.

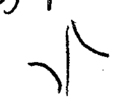


∪ = concave up, ∩ = down.

At $x = \pm 1$ the concavity changes but these are not in domain so not inflection points.

(f) $\lim_{x \to \pm\infty} \frac{x^2}{x^2-1} = 1$ so horizontal asymptote at $y = 1$.

$\left.\begin{array}{l} \lim_{x \to 1^+} \frac{x^2}{x^2-1} = \infty \\ \lim_{x \to 1^-} \frac{x^2}{x^2-1} = -\infty \end{array}\right\}$ vertical asymptote at $x = 1$



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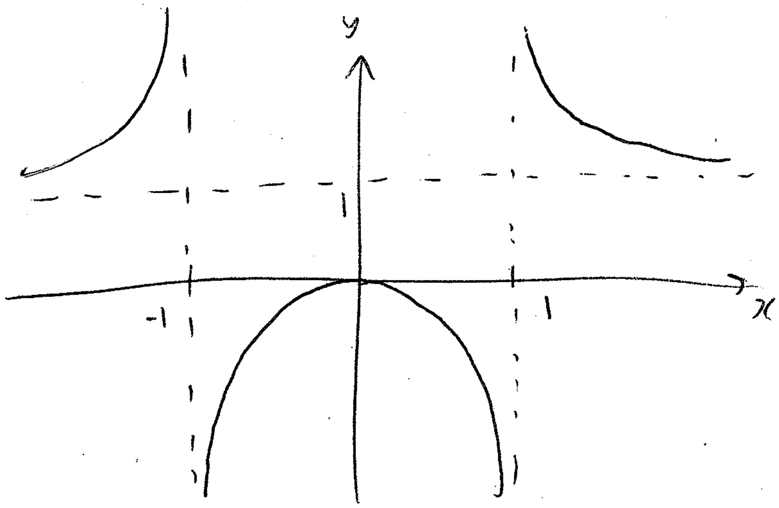
ANSWER

(f) (continued) $\left.\begin{array}{l} \lim\limits_{x\to -1^-} \dfrac{x^2}{x^2-1} = +\infty \\ \lim\limits_{x\to -1^+} \dfrac{x^2}{x^2-1} = -\infty \end{array}\right\}$ Vertical asymptote at $x = -1$

(g)



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