

QUESTION 1

(___ / 20)

Use the precise definition of a limit to show that

$$\lim_{x \rightarrow -3} \sqrt{1-5x} = 4$$

ANSWER

Given $\epsilon > 0$

We want

$$|\sqrt{1-5x} - 4| < \epsilon$$

$$-\epsilon < \sqrt{1-5x} - 4 < \epsilon$$

$4 - \epsilon < \sqrt{1-5x} < 4 + \epsilon$ Note: we may assume $\epsilon < 4$, so all of these are positive. Hence, squaring keeps the inequality direction

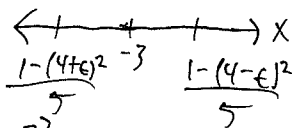
$$(4 - \epsilon)^2 < 1 - 5x < (4 + \epsilon)^2$$

$$(4 - \epsilon)^2 - 1 < -5x < -1 + (4 + \epsilon)^2$$

dividing by -5 switches the direction

$$\frac{1 - (4 - \epsilon)^2}{5} > x > \frac{1 - (4 + \epsilon)^2}{5}$$

so set $\delta = \min \left\{ -3 - \left(\frac{1 - (4 + \epsilon)^2}{5} \right), \frac{1 - (4 - \epsilon)^2}{5} + 3 \right\}$ (This is as far as was required)



Then if

$$0 < |x - (-3)| < \delta$$

$$|x - (-3)| < \delta$$

$$|x + 3| < \delta$$

$$-\delta < x + 3 < \delta$$

$$-3 - \delta < x < -3 + \delta$$

$$-3 - \left(-3 - \left(\frac{1 - (4 + \epsilon)^2}{5} \right) \right) < x < -3 + \left(\frac{1 - (4 - \epsilon)^2}{5} + 3 \right)$$

Page 2 of 6 $\frac{1 - (4 + \epsilon)^2}{5} < x < \frac{1 - (4 - \epsilon)^2}{5}$

$$1 - (4 + \epsilon)^2 < 5x < 1 - (4 - \epsilon)^2$$

$$-1 + (4 + \epsilon)^2 > -5x > -1 + (4 - \epsilon)^2$$

$$(4 + \epsilon)^2 > 1 - 5x > (4 - \epsilon)^2$$

As before, all of these are positive, so the inequalities stay the same.

$$4 + \epsilon > \sqrt{1 - 5x} > 4 - \epsilon$$

$$\epsilon > \sqrt{1 - 5x} - 4 > -\epsilon$$

$$|\sqrt{1 - 5x} - 4| < \epsilon$$

QUESTION 2

(___ / 20)

Use the precise definition of a derivative to show that

$$\frac{d}{dx} \cos x = -\sin x$$

[Hint: $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$]

ANSWER

$$\begin{aligned} \frac{d}{dx} \cos x &= \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cos x (\cos h - 1) - \sin x \sin h}{h} \\ &= \lim_{h \rightarrow 0} \cos x \cdot \frac{\cos h - 1}{h} - \sin x \cdot \frac{\sin h}{h} \\ &= \cos x \cdot 0 - \sin x \cdot 1 \\ &= -\sin x \end{aligned}$$

QUESTION 3

(___ / 20)

Evaluate the following limits (not necessarily using the rigorous definition). Justify each step.

(a) $\lim_{x \rightarrow -\infty} \frac{4 - 3x^3}{\sqrt{x^6 + 9}}$ [10 points]

(b) $\lim_{x \rightarrow 0^+} x \cos\left(\frac{1}{\sqrt{x}}\right)$ [10 points]

ANSWER

a) $\lim_{x \rightarrow -\infty} \frac{4 - 3x^3}{\sqrt{x^6 + 9}}$

$= \lim_{x \rightarrow -\infty} \frac{\frac{4}{x^3} - 3}{\frac{1}{x^3} \sqrt{x^6 + 9}}$

-note that for $x < 0$, $\frac{1}{x^3} = -\sqrt{x^6} -$

$= \lim_{x \rightarrow -\infty} \frac{\frac{4}{x^3} - 3}{-\sqrt{\frac{1}{x^6}} \sqrt{x^6 + 9}}$

$= \lim_{x \rightarrow -\infty} \frac{\frac{4}{x^3} - 3}{-\sqrt{1 + 9/x^6}}$

$= \frac{-3}{-1} = 3$

b)

$-1 \leq \cos\left(\frac{1}{\sqrt{x}}\right) \leq 1$

So for $x \geq 0$,

$-x \leq x \cos\left(\frac{1}{\sqrt{x}}\right) \leq x$

$\lim_{x \rightarrow 0^+} -x = \lim_{x \rightarrow 0^+} x = 0$

So by the sandwich theorem,

$\lim_{x \rightarrow 0^+} x \cos\left(\frac{1}{\sqrt{x}}\right) = 0$

QUESTION 4

(___ / 20)

Find the derivatives of the following functions (not necessarily starting from the definition):

(a) $f(x) = \frac{\cos(x^2)}{1 + \sin x}$ [10 points]

(b) $g(t) = \left(\frac{3t-4}{5t+2}\right)^{-5}$ [10 points]

ANSWER

$$\begin{aligned} \text{a) } \frac{df}{dx}(x) &= \frac{(1+\sin x) \frac{d \cos x^2}{dx} - (1+\sin x)' \cos x^2}{(1+\sin x)^2} \\ &= \frac{(1+\sin x)(-\sin x^2) 2x - \cos x \cos x^2}{(1+\sin x)^2} \end{aligned}$$

$$\begin{aligned} \text{b) } \frac{dg}{dt}(t) &= -5 \left(\frac{3t-4}{5t+2}\right)^{-6} \frac{d}{dt} \left(\frac{3t-4}{5t+2}\right) \\ &= -5 \left(\frac{3t-4}{5t+2}\right)^{-6} \frac{3(5t+2) - (3t-4)5}{(5t+2)^2} \\ &= -5 \left(\frac{3t-4}{5t+2}\right)^{-6} \frac{26}{(5t+2)^2} \\ &= -130 \frac{(5t+2)^4}{(3t-4)^6} \end{aligned}$$

QUESTION 5

(___ / 20)

Suppose that the distance an aircraft travels along a runway before takeoff is given by

$$D = (10/9)t^2 \text{ meters,}$$

measured from the starting point and t is measured in seconds from the time the brakes are released. The aircraft will become airborne when its speed reaches 60 meters per second.

How long will it take to become airborne, and what distance will it travel in that time?

ANSWER

The speed at time t ,

$$\begin{aligned} V(t) &= \left| \frac{dD}{dt} \right| = \left| \frac{d}{dt} \left(\frac{10}{9}t^2 \right) \right| \\ &= \frac{20}{9}t. \quad (t \geq 0) \end{aligned}$$

Because aircraft takes off when its speed $V(t) = 60$ m/s.

$$\Rightarrow V(t) = 60 \text{ m/s} \quad \text{i.e.} \quad \frac{20}{9}t = 60.$$

$$\Rightarrow t = \frac{60 \cdot 9}{20} = 27 \text{ secs.}$$

The Distance before take off,

$$D(27) = \frac{10}{9} \cdot (27)^2 = \frac{10}{9} \cdot 27 \cdot 27 = 810 \text{ meters.}$$

So, the aircraft will become airborne when $t = 27$ secs.

and it'll travel 810 meters before airborne.