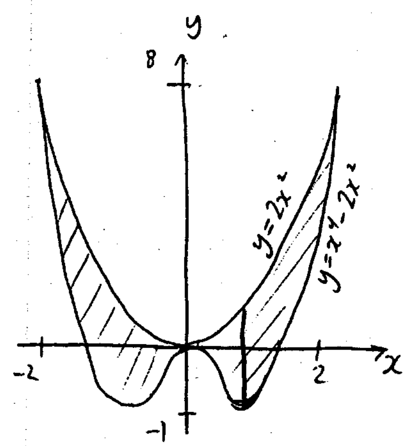
QUESTION 1

(____ / 25)

Find the area between the two curves $y = 2x^2$ and $y = x^4 - 2x^2$ for $-2 \leq x \leq 2$.



ANSWER

$$\text{Area} = \int_{-2}^{2} 2x^2 - (x^4 - 2x^2)\, dx$$

$$= 2 \int_0^2 2x^2 - (x^4 - 2x^2)\, dx$$

$$= 2 \int_0^2 (4x^2 - x^4)\, dx$$

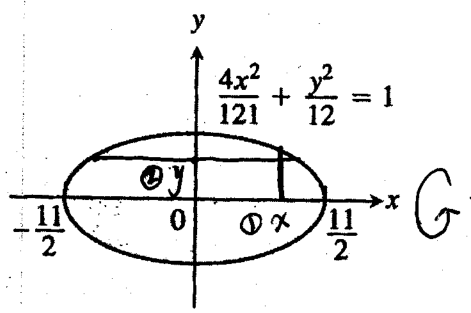
$$= 2\left(4 \cdot \frac{x^3}{3} - \frac{x^5}{5}\right)\Big|_{x=0}^{2}$$

$$= 2\left(4 \cdot \frac{2^3}{3} - \frac{2^5}{5}\right)$$

$$= 2\left(\frac{32}{3} - \frac{32}{5}\right) = \boxed{\frac{128}{15}}$$

QUESTION 2 (____/ 25)

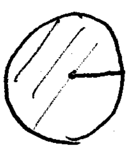
Using any calculus-based method, find the volume of a football with the profile given below. The cross-sections in the plane perpendicular to the x axis are circles.



To help simplify your answer, note that $(11/2)^2 = 121/4$. If you have any doubts about what a football looks like, raise your hand and ask an examiner.

ANSWER

Method ①, circle method



$r = y$

$\frac{y^2}{12} = 1 - \frac{4}{121}x^2$

$r = y = \pm\sqrt{12(1 - \frac{4}{121}x^2)}$

$A(x) = \pi r^2 = \pi y^2 = 12\pi(1 - \frac{4}{121}x^2)$

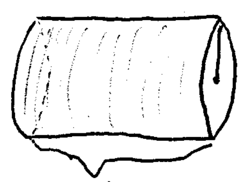
$$\text{Volume} = \int_{-\frac{11}{2}}^{\frac{11}{2}} A(x)\, dx$$

$$= \int_{-\frac{11}{2}}^{\frac{11}{2}} 12\pi\left(1 - \frac{4}{121}x^2\right) dx$$

$$= 12\pi\left(x - \frac{4}{121}\cdot\frac{x^3}{3}\right)\Big|_{x=-\frac{11}{2}}^{\frac{11}{2}}$$

$$= \boxed{88\pi}$$

Method ②, Shell method



radius y

length $2|x|$

$\frac{4}{121}x^2 = 1 - \frac{y^2}{12}$, $\quad x^2 = \frac{121}{4}\cdot\left(1 - \frac{y^2}{12}\right)$

$x = \pm\frac{11}{2}\sqrt{1 - \frac{y^2}{12}}$

$A(y) = 2\pi\cdot(\text{radius})\cdot(\text{length}/\text{height})$

$= 2\pi\, y\cdot 11\cdot\sqrt{1 - \frac{y^2}{12}} = 22\pi\cdot y\cdot\sqrt{1 - \frac{y^2}{12}}$

$$\text{Volume} = \int_0^{\sqrt{12}} A(y)\, dy = \int_0^{\sqrt{12}} 22\pi\cdot y\cdot\sqrt{1 - \frac{y^2}{12}}\, dy$$

let $u = 1 - \frac{y^2}{12}$

$du = -\frac{1}{6}y\, dy$

$y\, dy = -6\, du$

$$= 22\pi\int_1^0 (-6)\sqrt{u}\, du$$

$$= -22\cdot 6\pi\cdot\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\Big|_{u=1}^{0}$$

$$= \boxed{88\pi}$$

QUESTION 3

(____ / 25)

A thin rod lies along the x axis between $x = 0$ and $x = 3$ and has density function

$$\delta(x) = \sqrt{1+x}$$

(a) Find the total mass of the rod.

(b) Find position of the center of mass of the rod.

ANSWER

(a)

$$\text{Mass} = \int_0^3 \delta(x)\,dx = \int_0^3 \sqrt{1+x}\,dx$$

$$= \left.\frac{(1+x)^{\frac{3}{2}}}{\frac{3}{2}}\right|_{x=0}^{3}$$

$$= \frac{2}{3}(1+3)^{\frac{3}{2}} - \frac{2}{3}(1+0)^{\frac{3}{2}}$$

$$= \boxed{\frac{14}{3}}$$

(b)

$$\text{Moment} = \int_0^3 x\,\delta(x)\,dx = \int_0^3 x\sqrt{1+x}\,dx$$

let $u = 1+x$, $du = dx$

$$= \int_1^4 (u-1)\sqrt{u}\,du = \int_1^4 u^{\frac{3}{2}} - u^{\frac{1}{2}}\,du$$

$$= \left.\frac{u^{\frac{5}{2}}}{\frac{5}{2}} - \frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right|_{u=1}^{4}$$

$$= \left(\frac{2}{5}\cdot 4^{\frac{5}{2}} - \frac{2}{3}\cdot 4^{\frac{3}{2}}\right) - \left(\frac{2}{5} - \frac{2}{3}\right)$$

$$= \frac{116}{15}$$

Center: $$\bar{x} = \frac{\text{Moment}}{\text{Mass}} = \frac{\frac{116}{15}}{\frac{14}{3}} = \frac{116}{15}\cdot\frac{3}{14} = \boxed{\frac{58}{35}}$$

QUESTION 4 (____/ 25)

Use logarithmic differentiation to find the derivative of

$$y = \sqrt[3]{\frac{x\,(x-2)^{1/4}}{(x+1)(x^2+1)}}$$

ANSWER

$$\ln y = \ln\left(\frac{x\,(x-2)^{\frac{1}{4}}}{(x+1)(x^2+1)}\right)^{\frac{1}{3}}$$

$$= \frac{1}{3}\left(\ln x + \frac{1}{4}\ln(x-2) - \ln(x+1) - \ln(x^2+1)\right)$$

$\frac{d}{dx}$:

$$\frac{1}{y}\cdot y' = \frac{1}{3}\left(\frac{1}{x} + \frac{1}{4}\cdot\frac{1}{x-2} - \frac{1}{x+1} - \frac{1}{x^2+1}\cdot 2x\right)$$

$$y' = \frac{1}{3}\cdot y\left(\frac{1}{x} + \frac{1}{4(x-2)} - \frac{1}{x+1} - \frac{2x}{x^2+1}\right)$$

$$\underline{y' = \frac{1}{3}\cdot\sqrt[3]{\frac{x\,(x-2)^{1/4}}{(x+1)(x^2+1)}}\cdot\left(\frac{1}{x} + \frac{1}{4(x-2)} - \frac{1}{x+1} - \frac{2x}{x^2+1}\right)}$$