QUESTION 1 (____ / 25)

Use l'Hôpital's rule to evaluate

$$\lim_{x\to 0} \frac{1+x-\sqrt{1+2x}}{x^2}$$

ANSWER

$$\lim_{x\to 0} \frac{1+x-\sqrt{1+2x}}{x^2} = \text{"}\frac{1-1}{0}\text{"} = \text{"}\frac{0}{0}\text{"}$$

$$\stackrel{L.H.}{=} \lim_{x\to 0} \frac{1-\frac{1}{\sqrt{1+2x}}}{2x} = \text{"}\frac{0}{0}\text{"}$$

$$\stackrel{L.H.}{=} \lim_{x\to 0} \frac{\frac{1}{2}(1+2x)^{-3/2}\cdot 2}{2}$$

$$= \lim_{x\to 0} \frac{(1+2x)^{-3/2}}{2}$$

$$= \frac{(1+2\cdot 0)^{-3/2}}{2}$$

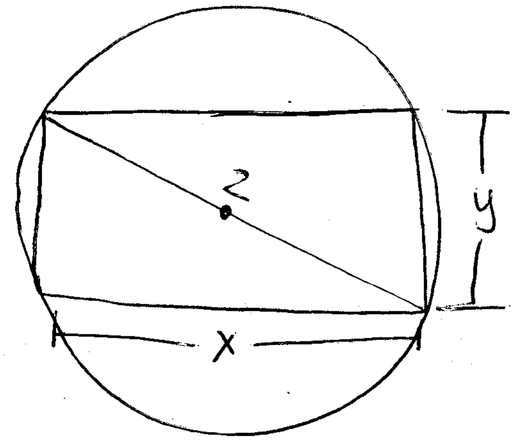
$$= \boxed{\frac{1}{2}}$$

QUESTION 2 (____/ 25)

Use calculus and optimization to find the maximum area of a rectangle that can be contained inside a circle of radius 1. Justify that the value you find is a maximum.

ANSWER

Let x be the length of the rectangle, and y be the height.



The diagonal is a diameter of the circle so

$$x^2 + y^2 = 2^2$$

$$x = \pm\sqrt{4-y^2}$$

$$x = +\sqrt{4-y^2}$$

(because $x \geq 0$)

$0 \leq x \leq 2$

$0 \leq y \leq 2$

$$A = xy = y\sqrt{4-y^2}$$

$$A' = \sqrt{4-y^2} + \frac{-y^2}{\sqrt{4-y^2}}$$

$$= \frac{4-2y^2}{\sqrt{4-y^2}}$$

Critical Points:

$A' = 0$ when $4 - 2y^2 = 0$

$y = \pm\sqrt{2}$

$y = +\sqrt{2}$

A' undefined at $y = 0$, $y = 2$

(over)

(2) cont)

$$A(0) = 0$$
$$A(2) = 0$$
$$A(\sqrt{2}) = 2$$

$A(y)$ is continuous, so by the Extreme Value Theorem, $A(\sqrt{2}) = 2$ is the global maximum.

QUESTION 3 (____ / 50)

For the function

$$y = f(x) = \sqrt{4 - x^2}$$

(a) What is the domain of $f(x)$? Is $f(x)$ even, odd, or neither? Find the intercepts of $y = f(x)$. *[10 points]*

(b) Find and simplify the first and second derivatives of $f(x)$. *[10 points]*

(c) Find the critical points of $f(x)$. Where is the function increasing/decreasing? *[8 points]*

(d) If it applies, use the second derivative test to determine if the critical points are local max/mins. Which ones are also global extrema, if any? *[4 points]*

(e) Where is $f(x)$ concave up/down? Are there any inflection points? If so, find them. *[8 points]*

(f) Use all the above information to sketch the graph of $y = f(x)$. *[10 points]*

ANSWER

(a) Domain: $4-x^2 \geq 0$

$(2-x)(2+x) \geq 0$

$[-2, 2]$

$f(-x) = \sqrt{4-(-x)^2}$

$= \sqrt{4-x^2}$

$= f(x)$

Even

y-int.: Set $x=0$

$y = \sqrt{4} = 2$

x-int: Set $y=0$

$0 = \sqrt{4-x^2}$

$0 = 4-x^2$

$0 = (2-x)(2+x)$

$x = 2 \quad x = -2$

(continue on the next page if you need more space)

ANSWER

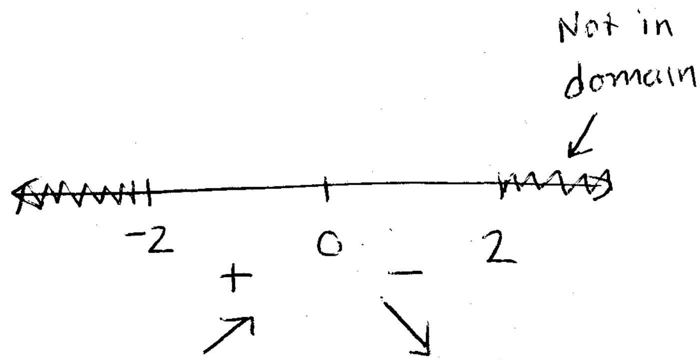
(b) $y' = \dfrac{-x}{\sqrt{4-x^2}}$

$$y'' = \frac{-\sqrt{4-x^2} + x\frac{-x}{\sqrt{4-x^2}}}{4-x^2} = \frac{\frac{-(4-x^2)-x^2}{\sqrt{4-x^2}}}{4-x^2}$$

$$= \frac{-4}{(4-x^2)^{3/2}}$$

(c) $y' = 0$ when $x = 0$

y' undefined at $x = \pm 2$



Critical Points:

$x = -2, 0, 2$

Increasing $[-2, 0]$

Decreasing $[0, 2]$

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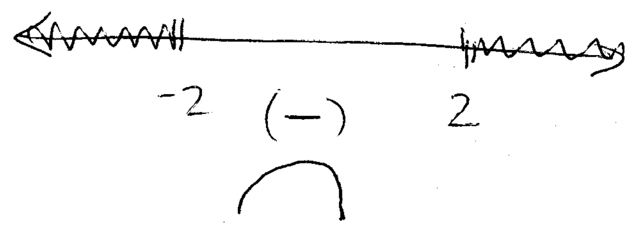
(d) $f''(-2) = \text{DNE}$
$f''(2) = \text{DNE}$ } 2^{nd} derivative test does not apply

$f''(0) = -\frac{1}{2} \leftarrow$ local max

Global max at $x=0$ because of increasing/
Global min at $x = \pm 2$ decreasing information

(e) $y'' = 0$ never

y'' undefined at $x = \pm 2$



Concave Down on $[-2, 2]$

No points of inflection.

(f)