Vanishing discount problems for Hamilton–Jacobi equations on changing domains

Graduate School of Mathematical Sciences
The University of Tokyo

Son N.T. Tu

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University of Wisconsin - Madison

The discount problem

Given $\lambda > 0$ as the discount factor, the equation of interest is

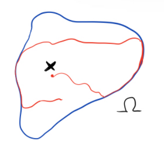
$$\begin{cases} \lambda u(x) + H(x, Du(x)) \leq 0 & \text{in } \Omega, \\ \lambda u(x) + H(x, Du(x)) \geq 0 & \text{on } \overline{\Omega} \end{cases} \tag{S_λ}$$

posed in an open, bounded subset $\Omega \subset \mathbb{R}^n$. Here it means that u solves the PDE in viscosity sense inside U,

$$\begin{cases} \lambda u(x) + H(x, Du(x)) = 0 & \text{in } \Omega, \\ u \text{ is the supersolution on the boundary } \partial\Omega. \end{cases}$$

State-constraint: If H is convex in p, let $L(x, v)$ be the Legendre transform of H, then

$$u(x) = \inf_{\eta(0)=x} \left\{ \int_0^\infty e^{-\lambda s} L(\eta(s), -\dot{\eta}(s))ds : \eta \in \mathrm{AC}, \eta([0,\infty)) \subset \overline{\Omega} \right\}.$$



Constraint: η stays in $\overline{\Omega}$.

Vanishing discount with changing domain

Let $\phi(\lambda) : (0,\infty) \to (0,\infty)$ (nondecreasing) and $r(\lambda) : (0,\infty) \to \mathbb{R}$ be continuous with

$$\lim_{\lambda \to 0^+} \phi(\lambda) = \lim_{\lambda \to 0^+} r(\lambda) = 0.$$

We look at

$$\begin{cases} \phi(\lambda)u_\lambda(x) + H(x, Du_\lambda(x)) \leq 0 & \text{in } (1 + r(\lambda))\Omega, \\ \phi(\lambda)u_\lambda(x) + H(x, Du_\lambda(x)) \geq 0 & \text{on } (1 + r(\lambda))\overline{\Omega}. \end{cases} \tag{S_λ}$$

Roughly speaking, along $\lambda_j \to 0^+$, $\phi(\lambda)u_\lambda \to c_0$ and $u_\lambda - u_\lambda(x_0) \to u$ and we have the ergodic problem

$$\begin{cases} H(x, Du(x)) \leq c_0 & \text{in } \Omega, \\ H(x, Du(x)) \geq c_0 & \text{on } \overline{\Omega}. \end{cases} \tag{S_0}$$

Difficulty: Solutions of (S_0) is not unique, even though c_0 is unique. Here c_0 is the so-called additive eigenvalue, can also be defined as

$$c_0 = \inf \left\{ c \in \mathbb{R} : H(x, Du(x)) \leq c \text{ in } \Omega \text{ has a solution} \right\}.$$

Main questions of interest: Assume $\frac{\phi(\lambda)}{r(\lambda)} \to \gamma$ as $\lambda \to 0^+$.

1. Behavior of u_λ as $\lambda \to 0^+$?
2. Behavior of c_λ, the additive eigenvalue of H in $(1 + r(\lambda))\Omega$ as $\lambda \to 0^+$?

Literature and previous results

State-constraint nested domains

- Qualitative results for nested domain: [Capuzzo-Dolcetta – Lions, 1990], [Armstrong-Tran, 2015], etc.
- (Quantitative results) [Kim-Tran-Tu] (2020).

Vanishing discount

- 1st-order on the torus, [Davini-Fathi-Iturriaga-Zavidovique, 2016].
- 2nd-order on the torus were studied, [Ishii-Mitake-Tran, 2017], [Mitake-Tran, 2017].
- Bounded domains with Neuumann boundary conditions [Al-Aidarous-Alzahrani-Ishii-Younas, 2016], [Ishii-Mitake-Tran, 2017].
- Problem in $\mathbb{R}^n$ [Ishii-Siconolfi, 2020].
- [Chen-Cheng-Ishii-Zhao, 2019] (vanishing discount with respect to changing Hamiltonians).

Summarizing of the main results

Two normalizations for solutions

$$\underbrace{\left\{u_\lambda + \frac{c_0}{\phi(\lambda)}\right\}_{\lambda>0}}_{I}, \qquad \text{and} \qquad \underbrace{\left\{u_\lambda + \frac{c_\lambda}{\phi(\lambda)}\right\}_{\lambda>0}}_{II}.$$

where c_λ is the additive eigenvalues of H in $(1 + r(\lambda))\Omega$.

Main results:

1. If $\gamma = 0$ then both I, II converge to the maximal solution in Ω.
2. If γ is finite then I converges to u^γ with description in terms of Mather measures. If $\gamma = \infty$ then I diverges (example).
3. The difference between I and II is $\frac{c_\lambda - c_0}{\phi(\lambda)}$. We show that

   $$\lim_{r(\lambda)\to 0^+} \frac{c_\lambda - c_0}{r(\lambda)}, \qquad \text{and} \qquad \lim_{r(\lambda)\to 0^-} \frac{c_\lambda - c_0}{r(\lambda)}$$

   exist (descriptions in terms if Mather measures). Thus II converges as well if γ is finite.
4. II is bounded even if $\gamma = \infty$, but we have example showing divergence for II when $\gamma = \infty$ and $r(\lambda) \leq 0$.

Main results 1 - Convergence of vanishing discount

Theorem 1. (Tu, 2020)

Assume H is locally Lipschitz, coercive in p uniformly in x and is convex in p. If $\gamma = 0$ then both families I and II converge to u^0 locally uniformly as $\lambda \to 0^+$.

Her u^0 is the maximal solution to the ergodic problem

$$\begin{cases} H(x, Du(x)) \leq c_0 & \text{in } \Omega, \\ H(x, Du(x)) \geq c_0 & \text{on } \overline{\Omega}. \end{cases} \tag{S_0}$$

It is well-known that u^0 is the limit of vanishing discount with fixed bounded domain. It can also be characterized in term of Mather measures $\mathcal{M}_0$.

Duality representation and Mather measures - I

By priori estimate: $|Du_\lambda| \leq h$, hence we can modify $H(x,p)$ with $|p| > h$. For a measure μ defined on $\overline{\Omega} \times \overline{B}_h$, we define

$$\langle \mu, f \rangle := \int_{\overline{\Omega} \times \overline{B}_h} f(x,v)\, d\mu(x,v), \qquad \text{for } f \in \mathrm{C}(\overline{\Omega} \times \overline{B}_h).$$

For each $f \in \mathrm{C}(\overline{\Omega} \times \overline{B}_h)$, define

$$H_f(x,p) = \max_{|v| \leq h} \Big(p \cdot v - f(x,v) \Big), \qquad (x,p) \in \overline{\Omega} \times \overline{B}_h.$$

Let $\mathcal{R}(\overline{\Omega} \times \overline{B}_h)$ be the space of Radon measures on $\overline{\Omega} \times \overline{B}_h$. For $\lambda > 0$, $z \in \overline{\Omega}$ we define

$$\mathcal{F}_{\lambda,\Omega} = \Big\{ (f,u) \in \mathrm{C}(\overline{\Omega} \times \overline{B}_h) \times \mathrm{C}(\overline{\Omega}) : \lambda u + H_f(x, Du) \leq 0 \text{ in } \Omega \Big\}$$

$$\mathcal{G}_{z,\lambda,\Omega} = \Big\{ f - \lambda u(z) : (f,u) \in \mathcal{F}_{\lambda,\Omega} \Big\}$$

$$\mathcal{G}'_{z,\lambda,\Omega} = \Big\{ \mu \in \mathcal{R}(\overline{\Omega} \times \overline{B}_h) : \langle \mu, f \rangle \geq 0 \text{ for all } f \in \mathcal{G}_{z,\lambda,\Omega} \Big\}.$$

[Mitake-Tran-Ishii, 2017]:

$$\lambda u(z) = \min_{\mu \in \mathcal{P} \cap \mathcal{G}'_{z,\lambda,\Omega}} \langle \mu, L \rangle = \min_{\mu \in \mathcal{P} \cap \mathcal{G}'_{z,\lambda,\Omega}} \int_{\overline{\Omega} \times \overline{B}_h} L(x,v)\, d\mu(x,v).$$

Duality representation and Mather measures - II

As $\lambda \to 0^+$, let us define

$$\mathcal{F}_{0,\Omega} = \Big\{(f,u) \in \mathrm{C}(\overline{\Omega} \times \overline{B}_h) \times \mathrm{C}(\overline{\Omega}) : H_f(x, Du(x)) \leq 0 \text{ in } \Omega\Big\}$$
$$\mathcal{G}_{0,\Omega} = \Big\{f : (f,u) \in \mathcal{F}_{0,\Omega} \text{ for some } u \in \mathrm{C}(\overline{\Omega})\Big\}$$
$$\mathcal{G}'_{0,\Omega} = \Big\{\mu \in \mathcal{R}(\overline{\Omega} \times \overline{B}_h) : \langle \mu, f\rangle \geq 0 \text{ for all } f \in \mathcal{G}_{0,\Omega}\Big\}.$$

Measures in $\mathcal{G}'_{0,\Omega}$ is called holonomic, s.t. $\langle \mu, v \cdot D\psi(x)\rangle = 0$ for all $\psi \in \mathrm{C}^1(\overline{\Omega})$. [Mitake-Tran-Ishii, 2017]:

$$-c_0 = \min_{\mu \in \mathcal{P} \cap \mathcal{G}'_{0,\Omega}} \langle \mu, L\rangle = \min_{\mu \in \mathcal{P} \cap \mathcal{G}'_{0,\Omega}} \int_{\overline{\Omega} \times \overline{B}_h} L(x,v)\, d\mu(x,v). \tag{1}$$

The set of all measures in $\mathcal{P} \cap \mathcal{G}'_{0,\Omega}$ that minimizing (1) is denoted $\mathcal{M}_0$.

Main results 1 - Convergence of vanishing discount

c_λ is the additive eigenvalues of H in $(1 - r(\lambda))\Omega$.

$$I = \left\{ u_\lambda + \frac{c_0}{\phi(\lambda)} \right\}_{\lambda>0}, \qquad \text{and} \qquad II = \left\{ u_\lambda + \frac{c_\lambda}{\phi(\lambda)} \right\}_{\lambda>0}.$$

Theorem 1. (Tu, 2020)

Assume H is locally Lipschitz, coercive in p uniformly in x and is convex in p. If $\gamma = 0$ then both families I and II converge to u^0 locally uniformly as $\lambda \to 0^+$.

Her u^0 is the maximal solution to the ergodic problem

$$\begin{cases} H(x, Du(x)) \leq c_0 & \text{in } \Omega, \\ H(x, Du(x)) \geq c_0 & \text{on } \overline{\Omega}. \end{cases} \qquad \text{and} \qquad u^0 = \sup_{w \in \mathcal{E}} w$$

where

$$\mathcal{E} = \Big\{ w : H(x, Dw) \leq c_0 \text{ in } \Omega \ : \ \langle \mu, w \rangle \leq 0 \text{ for all } \mu \in \mathcal{M}_0 \Big\}.$$

$\mathcal{M}_0$ is the set of all $\mu \in \mathcal{P} \cap \mathcal{G}_0'$ such that $-c_0 = \langle \mu, L \rangle$.

Main results 2 - Convergence with the first normalization

Assume $\lim_{\lambda\to 0^+} \frac{\phi(\lambda)}{r(\lambda)} = \gamma$, our normalization $I = \left\{u_\lambda + \frac{c_0}{\phi(\lambda)}\right\}_{\lambda>0}$.

$$\begin{cases} \phi(\lambda)u_\lambda(x) + H(x, Du_\lambda(x)) \leq 0 & \text{in } (1+r(\lambda))\Omega, \\ \phi(\lambda)u_\lambda(x) + H(x, Du_\lambda(x)) \geq 0 & \text{on } (1+r(\lambda))\overline{\Omega}. \end{cases}$$

Theorem 2. (Tu, 2020)

Assume H is locally Lipschitz, coercive in p uniformly in x and is convex in p and $\lambda \mapsto L((1+\lambda)x, v)$ is C^1. If $\gamma \in \mathbb{R}$ then the family I converge to u^γ (solve the ergodic problem) locally uniformly in Ω as $\lambda \to 0^+$. Furthermore

$$u^\gamma = \sup_{w\in\mathcal{E}^\gamma} w,$$

where $\mathcal{E}^\gamma$ denotes the family of subsolutions v to the ergodic problem (S_0) such that

$$\gamma\big\langle\mu, (-x)\cdot D_xL(x,v)\big\rangle + \langle\mu, w\rangle \leq 0 \qquad \text{for all } \mu \in \mathcal{M}_0.$$

- If $\gamma = \infty$ then I is unbounded (counter example).
- The mapping $\gamma \mapsto u^\gamma(\cdot)$ from $\mathbb{R}$ to $\mathrm{C}(\overline{\Omega})$ is concave and decreasing.

Main results 2 - Second normalization - eigenvalue behavior

Assume $\lim_{\lambda\to 0^+} \frac{\phi(\lambda)}{r(\lambda)} = \gamma$, $I = \left\{u_\lambda + \frac{c_0}{\phi(\lambda)}\right\}_{\lambda>0}$ and $II = \left\{u_\lambda + \frac{c_\lambda}{\phi(\lambda)}\right\}_{\lambda>0}$.

$$\begin{cases} \phi(\lambda)u_\lambda(x) + H(x, Du_\lambda(x)) \leq 0 & \text{in } (1+r(\lambda))\Omega, \\ \phi(\lambda)u_\lambda(x) + H(x, Du_\lambda(x)) \geq 0 & \text{on } (1+r(\lambda))\overline{\Omega}. \end{cases}$$

The difference between II and I is $\lim_{\lambda\to 0^+}\left(\frac{c_\lambda - c_0}{\phi(\lambda)}\right) = \gamma \lim_{\lambda\to 0^+}\left(\frac{c_\lambda - c_0}{r(\lambda) - r(0)}\right)$.

The latter only depends on $r(\lambda)$, can we write $c_\lambda = c_0 + c_{(1)}\lambda + o(\lambda)$ as $\lambda \to 0^+$?

Theorem 3. (Tu, 2020)

Assume H is locally Lipschitz, coercive in p uniformly in x and is convex in p and $\lambda \mapsto L((1+\lambda)x, v)$ is C^1.

$$\lim_{\substack{\lambda\to 0^+ \\ r(\lambda)>0}}\left(\frac{c_\lambda - c_0}{r(\lambda)}\right) = \max_{\mu\in\mathcal{M}_0} \langle \mu, (-x)\cdot D_xL(x,v)\rangle,$$

$$\lim_{\substack{\lambda\to 0^+ \\ r(\lambda)<0}}\left(\frac{c_\lambda - c_0}{r(\lambda)}\right) = \min_{\mu\in\mathcal{M}_0} \langle \mu, (-x)\cdot D_xL(x,v)\rangle.$$

Thus $\frac{c_\lambda - c_0}{r(\lambda)}$ converges if and only if $\langle \mu, (-x)\cdot D_xL(x,v)\rangle = c_{(1)}$ for all $\mu \in \mathcal{M}_0$. To study $c_{(1)}$ we can assume $r(\lambda) = \lambda$.

Main results 3 - Second normalization, divergence result

Assume $\lim_{\lambda\to 0^+} \frac{\phi(\lambda)}{r(\lambda)} = \gamma$, $I = \left\{u_\lambda + \frac{c_0}{\phi(\lambda)}\right\}_{\lambda>0}$ and $II = \left\{u_\lambda + \frac{c_\lambda}{\phi(\lambda)}\right\}_{\lambda>0}$.

$$\begin{cases} \phi(\lambda)u_\lambda(x) + H(x, Du_\lambda(x)) \le 0 & \text{in } (1+r(\lambda))\Omega, \\ \phi(\lambda)u_\lambda(x) + H(x, Du_\lambda(x)) \ge 0 & \text{on } (1+r(\lambda))\overline{\Omega}. \end{cases}$$

Corollary.

Assume H is locally Lipschitz, coercive in p uniformly in x and is convex in p and $\lambda \mapsto L((1+\lambda)x, v)$ is C^1, and $\gamma \in \mathbb{R}\backslash\{0\}$, then

$$\lim_{\lambda\to 0^+} \left(u_\lambda(x) + \frac{c_\lambda}{\phi(\lambda)}\right) = u^\gamma(x) + \gamma \lim_{\lambda\to 0^+} \left(\frac{c_\lambda - c_0}{r(\lambda)}\right)$$

locally uniformly in Ω.

II is bounded even if $\gamma = \infty$, but we have divergence example.

Theorem 4. (S. Tu, 2020)

There exists a Hamiltonian where given any $r(\lambda) \le 0$ we can construct $\phi(\lambda)$ such that along a subsequence $\lambda_j \to 0^+$ we have $r(\lambda_j)/\phi(\lambda_j) \to -\infty$ and II diverges.

Sketch of proof for the convergence of family I

Key ingredient: Scaling of domains which preserves gradient.

$$\begin{cases} \phi(\lambda)u_\lambda(x) + H(x, Du_\lambda(x)) \leq 0 & \text{in } \ (1+r(\lambda))\Omega, \\ \phi(\lambda)u_\lambda(x) + H(x, Du_\lambda(x)) \geq 0 & \text{on } (1+r(\lambda))\overline{\Omega}. \end{cases}$$

Denote $\Omega_\lambda = (1+r(\lambda))\Omega$, scale $u_\lambda \in \mathrm{C}(\overline{\Omega}_\lambda)$ to $\tilde{u}_\lambda \in \mathrm{C}(\overline{\Omega})$. Assume

$$\begin{cases} \lambda_j \to 0 \\ \delta_j \to 0 \end{cases} \quad \text{such that} \quad \begin{cases} \tilde{u}_{\lambda_j} + \phi(\lambda_j)^{-1}c_0 \to u \\ \tilde{u}_{\delta_j} + \phi(\delta_j)^{-1}c_0 \to w \end{cases} \quad \text{locally uniformly.}$$

Fix $z \in \Omega$, we show $u(z) = w(z)$. Let $\mu_\lambda \in \mathcal{P} \cap \mathcal{G}'_{z,\phi(\lambda),\Omega_\lambda}$ such that

$$\phi(\lambda)u_\lambda(z) = \langle \mu_\lambda, L \rangle_{\overline{\Omega}_\lambda \times \overline{B}_h}.$$

Rescale μ_λ to $\tilde{\mu}_\lambda$ on $\overline{\Omega} \times \overline{B}_h$ by

$$\langle \tilde{\mu}_\lambda, f \rangle_{\overline{\Omega} \times \overline{B}_h} = \left\langle \mu_\lambda, f\left((1+r(\lambda))^{-1}x, v\right)\right\rangle_{\overline{\Omega}_\lambda \times \overline{B}_h}, \qquad f \in \mathrm{C}(\overline{\Omega} \times \overline{B}_h).$$

Assume $\tilde{\mu}_\lambda \rightharpoonup \mu_0$ in measure, then $\mu_0 \in \mathcal{M}_0$.

Sketch of proof for the convergence of family I

(Scaling up Ω to Ω_λ) $H(x, Dw(x)) \le c_0$ in Ω, thus $H((1+r(\lambda)x, Dw_\lambda)) \le c_0$ in Ω_λ by an appropriate scaling w_λ, thus

$$\phi(\lambda)w_\lambda(x) + H_{L\left(\frac{x}{1+r(\lambda)}, v\right)+\phi(\lambda)w_\lambda(x)+c_0}(x, Dw_\lambda) \le 0 \qquad \text{in } (1+r(\lambda))\Omega,$$

As μ_λ is the minimizing measures, $\langle \mu_\lambda, L\rangle = \phi(\lambda)u_\lambda(z)$ and

$$0 \le \left\langle \mu_\lambda, L\left(\frac{x}{1+r(\lambda)}, v\right) + \phi(\lambda)w_\lambda(x) - \phi(\lambda)w_\lambda(z) + c_0 \right\rangle$$

Rescaling to $\tilde{\mu}_\lambda$, divide both sides by $r(\lambda)$ we have

$$\frac{r(\lambda)}{\phi(\lambda)}\left\langle \tilde{\mu}_\lambda, \frac{L(x,v) - L((1+r(\lambda))x, v)}{r(\lambda)}\right\rangle + (1+r(\lambda))\langle \tilde{\mu}_\lambda, w\rangle + \left(u_\lambda(z) + \frac{c_0}{\phi(\lambda)}\right) \ge w_\lambda(z).$$

Since $\tilde{\mu}_{\lambda_j} \rightharpoonup \mu_0$ we deduce that

$$\boxed{\gamma\langle \mu_0, (-x)\cdot D_xL(x,v)\rangle + \langle \mu_0, w\rangle + (u(z) - \gamma c_0) \ge w(z)} \tag{UP}$$

where we note that $(\tilde{u}_\lambda - u_\lambda) \to \gamma c_0$.

Sketch of proof for the convergence of family I

(Scaling down Ω_λ to Ω) As $\phi(\lambda)(1+r(\lambda))\tilde{u}_\lambda + H((1+r(\lambda))x, D\tilde{u}_\lambda) \leq 0$ in Ω

$$L\big((1+r(\lambda))x, v\big) - \phi(\lambda)(1+r(\lambda))\tilde{u}_\lambda(x) \in \mathcal{F}_{0,\Omega}.$$

Recall that $-c_0 = \langle \mu, L\rangle$ for all $\mu \in \mathcal{M}_0$, we have

$$\frac{r(\lambda)}{\phi(\lambda)}\left\langle \mu, \frac{L((1+r(\lambda))x, v) - L(x,v)}{r(\lambda)} \right\rangle \geq (1+r(\lambda))\left\langle \mu, \tilde{u}_\lambda(x) + \frac{c_0}{\phi(\lambda)} \right\rangle - \frac{r(\lambda)}{\phi(\lambda)} c_0$$

for all $\mu \in \mathcal{M}_0$. Let $\lambda = \delta_j \to 0$ then $\gamma\langle \mu, x\cdot D_x L(x,v)\rangle \geq \langle \mu, w\rangle - \gamma c_0$. In other words,

$$\boxed{\gamma\langle \mu, (-x)\cdot D_x L(x,v)\rangle + \langle \mu, w\rangle - \gamma c_0 \leq 0, \qquad \text{for all } \mu \in \mathcal{M}_0.} \tag{DOWN}$$

$$\boxed{\gamma\left\langle \mu_0, (-x)\cdot D_x L(x,v)\right\rangle + \langle \mu_0, w\rangle + \big(u(z) - \gamma c_0\big) \geq w(z)} \tag{UP}$$

We deduce that $u(z) \geq w(z)$, by symmetry we obtain $u(z) = w(z)$ and thus $u \equiv w$. Here $u = w = u^\gamma + \gamma c_0$, we deduce $u^\gamma = \sup \mathcal{E}^\gamma$.

(Scaling down Ω_λ to Ω) Ergodic problem on $\Omega_\lambda = (1 + r(\lambda))\Omega$

$$\begin{cases} H(x, Dw_\lambda(x)) \leq c_\lambda & \text{in } \Omega_\lambda, \\ H(x, Dw_\lambda(x)) \geq c_\lambda & \text{on } \overline{\Omega}_\lambda. \end{cases} \tag{S_0}$$

Let $\nu_\lambda \in \mathcal{P} \cap \mathcal{G}'_{0,\Omega_\lambda}$ such that $-c_\lambda = \langle \nu_\lambda, L \rangle_{\overline{\Omega}_\lambda \times \overline{B}_h} = \langle \tilde{\nu}_\lambda, L \rangle_{\overline{\Omega} \times \overline{B}_h}$ as the scaling measure. If $\tilde{\nu}_\lambda \rightharpoonup \nu_0$ then $\nu_0 \in \mathcal{M}_0$.

Let $r(\lambda) \geq 0$. Let $\tilde{w}_\lambda$ be its scaling of w_λ then

$$H_{L((1+r(\lambda))x,v)+c_\lambda}(x, D\tilde{w}_\lambda(x)) \leq 0 \qquad \text{in } \Omega \quad \Longrightarrow \quad L((1 + r(\lambda))x, v) + c_\lambda \in \mathcal{F}_{0,\Omega}.$$

For any $\mu \in \mathcal{M}_0$, $\langle \mu, L \rangle = -c_0$, thus

$$\langle \mu, L((1 + r(\lambda))x, v) - L(x, v) \rangle + (c_\lambda - c_0) \geq 0.$$

Thus if $r(\lambda) > 0$ then for all $\mu \in \mathcal{M}_0$ we have

$$\left\langle \mu, \frac{L((1 + r(\lambda))x, v) - L(x, v)}{r(\lambda)} \right\rangle + \left(\frac{c_\lambda - c_0}{r(\lambda)} \right) \geq 0.$$

As $r(\lambda)$ is not identically zero near 0

$$\boxed{\langle \mu, x \cdot D_x L(x, v) \rangle + \liminf_{\lambda \to 0^+} \left(\frac{c_\lambda - c_0}{r(\lambda)} \right) \geq 0 \qquad \text{for all } \mu \in \mathcal{M}_0.} \tag{INF}$$

Sketch of proof for the convergence eigenvalue

Let $\lambda_j \to 0^+$ be the subsequence such that

$$\limsup_{\lambda\to 0^+}\left(\frac{c_\lambda - c_0}{r(\lambda)}\right) = \lim_{j\to\infty}\left(\frac{c_{\lambda_j} - c_0}{r(\lambda)}\right).$$

(Scaling up Ω to Ω_λ) Assume that $\tilde{\nu}_{\lambda_j} \rightharpoonup \nu_0$ and $\nu_0 \in \mathcal{M}_0$. Let w such that $H(x, Dw) \leq c_0$ in Ω, then scaling up to Ω_λ we have

$$H_{L\left(\frac{x}{1+r(\lambda)}, v\right)+c_0}(x, D\tilde{w}(x)) \leq 0 \qquad \text{in } (1+r(\lambda))\Omega.$$

As $v_\lambda \in \mathcal{P} \cap \mathcal{G}'_{0,\Omega_\lambda}$ and $\langle \nu_\lambda, L\rangle = -c_\lambda$, we obtain that

$$\left\langle \nu_\lambda, L\left(\frac{x}{1+r(\lambda)}, v\right) - L(x, v)\right\rangle - c_\lambda + c_0 \geq 0.$$

By definition of $\tilde{\nu}_\lambda$, it is equivalent to

$$\langle \tilde{\nu}_\lambda, L(x, v) - L((1+r(\lambda))x, v)\rangle \geq c_\lambda - c_0.$$

As $r(\lambda_j) \geq 0$, let $\lambda_j \to 0^+$ we obtain

$$\boxed{\langle \nu_0, (-x)\cdot D_x L(x, v)\rangle \geq \limsup_{\lambda\to 0^+}\left(\frac{c_\lambda - c_0}{r(\lambda)}\right).} \tag{SUP}$$

Sketch of proof for the convergence eigenvalue

$$\boxed{\langle \mu, x \cdot D_x L(x,v) \rangle + \liminf_{\lambda \to 0^+} \left(\frac{c_\lambda - c_0}{r(\lambda)} \right) \geq 0 \qquad \text{for all } \mu \in \mathcal{M}_0.} \tag{INF}$$

$$\boxed{\langle \nu_0, (-x) \cdot D_x L(x,v) \rangle \geq \limsup_{\lambda \to 0^+} \left(\frac{c_\lambda - c_0}{r(\lambda)} \right).} \tag{SUP}$$

We deduce that

$$\lim_{\lambda \to 0^+} \left(\frac{c_\lambda - c_0}{r(\lambda)} \right) = \langle \nu_0, (-x) \cdot D_x L(x,v) \rangle = \sup_{\mu \in \mathcal{M}_0} \langle \mu, (-x) \cdot D_x L(x,v) \rangle .$$

Similarly, if $r(\lambda) \leq 0$ as $\lambda \to 0^+$ then

$$\lim_{\lambda \to 0^+} \left(\frac{c_\lambda - c_0}{r(\lambda)} \right) = \min_{\mu \in \mathcal{M}_0} \langle \mu, (-x) \cdot D_x L(x,v) \rangle .$$

Divergence of family II - a counter example

Let $r(\lambda) \geq 0$, c_λ be the eigenvalue of H in $\Omega_\lambda = (1 - r(\lambda)\Omega$. Let

$$H(x,p) = |p| - V(x), \qquad (x,p) \in \overline{\Omega} \times \mathbb{R}.$$

where $V : \overline{\Omega} \to \mathbb{R}$, $V \geq 0$ and $V \in \mathrm{BUC}(\overline{\Omega})$. Given $r(\lambda)$, construct $\phi(\lambda)$ so that $\{u_\lambda + \phi(\lambda)^{-1} c_\lambda\}_{\lambda>0}$ is divergent as $\lambda \to 0^+$.

Main tool: The instability of the Aubry set $\mathcal{A}_{\Omega_\lambda}$ of H on Ω_λ, when $\lambda \to 0^+$.

Aubry set: Let $S_\Omega(x,y) = \sup\left\{u(x) - u(y) : u \text{ s.t. } H(x,Du(x)) \leq c_0 \text{ in } \Omega\right\}$ then

$$\begin{cases} H(x,Du(x)) \leq c_0 & \text{in } \Omega, \\ H(x,Du(x)) \geq c_0 & \text{on } \overline{\Omega}\backslash\{y\}. \end{cases}$$

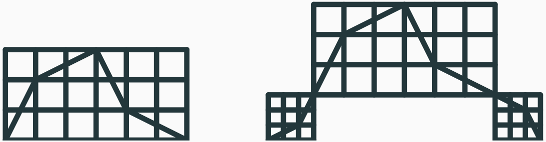
The Aury set $\mathcal{A}_\Omega = \left\{z \in \overline{\Omega} : x \mapsto S_\Omega(x,z) \text{ is a state-constraint solution}\right\}$.

One can show that, with $H(x,p) = |p| - V(x)$ and $V : \overline{\Omega} \to \mathbb{R}$, $V \geq 0$ and $V \in \mathrm{BUC}(\overline{\Omega})$ then

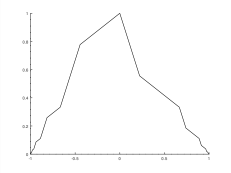
$$-c_0 = \min_{\overline{\Omega}} V \qquad \text{and} \qquad \mathcal{A}_\Omega = \left\{x \in \overline{\Omega} : V(x) = \min_{\overline{\Omega}} V\right\}.$$

Key point: If $\mathcal{A}_\Omega = \{z_0\}$ is a singleton then $u^0(x) \equiv S_\Omega(x,z_0)$ where u^0 is the maximal solution on Ω.

Divergence of family II - a counter example



Switching the small box with this construction, we obtain



Lemma.

Let $\Omega_\lambda = (-1 + r(\lambda), 1 - r(\lambda))$. Then the maximal solution on Ω_λ, denoted by $u^0_\lambda(x)$, does not converge as $\lambda \to 0^+$.

This is intuitive as we can choose two subsequences where the mimimum points of V over Ω_λ converge to the two edges.

Divergence of family II - a counter example

Consider

$$\begin{cases} \delta u_\delta(x) + H(x, Du_\delta(x)) \le 0 & \text{in } \Omega_\lambda, \\ \delta u_\delta(x) + H(x, Du_\delta(x)) \ge 0 & \text{on } \overline{\Omega}_\lambda. \end{cases} \tag{2}$$

Let c_λ be the eigenvalue of H over Ω_λ, we know that

$$\lim_{\delta \to 0^+} \left(u_\delta(x) + \frac{c_\lambda}{\delta} \right) \to u_\lambda^0(x)$$

uniformly on $\overline{\Omega}$, where $u_\lambda^0(x)$ is a maximal solution on Ω_λ. For each $\lambda > 0$, we can find $\tau(\lambda) > 0$ such that

$$\sup_{x \in \overline{\Omega}_\lambda} \left| \left(u_\delta(x) + \frac{c_\lambda}{\delta} \right) - u_\lambda^0(x) \right| \le r(\lambda) \qquad \text{for all } \delta \le \tau(\lambda). \tag{3}$$

Set $\phi(\lambda) = \tau(\lambda) r(\lambda)^2$, then $\phi(\lambda) \to 0$ as $\lambda \to 0^+$ and $\gamma = \infty$ (u_λ means $u_{\phi(\lambda)}$).

Now by (3) and above Lemma, along two subsequences λ_j and δ_j we have

$$\lim_{\lambda_j \to 0^+} \left(u_{\lambda_j}(x) + \frac{c_{\lambda_j}}{\phi(\lambda_j)} \right) = S_\Omega(x, -1) \ne S_\Omega(x, 1) = \lim_{\delta_j \to 0^+} \left(u_{\delta_j}(x) + \frac{c_{\delta_j}}{\phi(\delta_j)} \right).$$

Thus we have the divergence of $\{u_\lambda + \phi(\lambda)^{-1} c_\lambda\}_{\lambda > 0}$ in this case.

Concluding remarks

Open questions

1. The function $\lambda \mapsto c(\lambda)$ is Lipschitz, increasing, left and right differentiable, and the set of λ where $c'_+(\lambda) \neq c'_-(\lambda)$ (one-sided derivatives) is at most countable. Can we go further to higher order derivative of $c(\lambda)$ (if they exist, for example, if $(x,p) \mapsto H(x,p)$ is convex then $c \mapsto c(\lambda)$ is convex)? This requires a better understanding about the struture of Mather measures on slightly different domains.
2. Instead of scaling, say $\Omega(\lambda)$ changes with repsect to some law (mean curvatuve flows, curve shortening flow, etc) and if we can find a transformation that "scale" solutions between $\Omega(\lambda_1)$ and $\Omega(\lambda_2)$, can we derive the same results?
3. Unbounded domains?
4. Similar results for the viscous case?

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THANK YOU FOR YOUR ATTENTION!