State-Constraint static Hamilton-Jacobi equations in nested domains

PDE and Geometric Analysis Seminar
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Outline

- Introduction.
- State-Constraint Boundary conditions.
- State-Constraint equation in nested domain.
- Rate of convergences for various settings.
- Sketch of the proof for convex Hamiltonians.
- Sketch of the proof for general Hamiltonians.
- Examples.
- Open questions.

Motivation - optimal control

Let Ω be an open subset of $\mathbb{R}^n$, U be a compact metric space, A control $\alpha : [0, \infty) \mapsto U$ is a Borel measurable map.

$$\begin{cases} b = b(x,a) : \overline{\Omega} \times U \to \mathbb{R}^n, \\ f = f(x,a) : \overline{\Omega} \times U \to \mathbb{R} \qquad \text{is the (running) cost.} \end{cases}$$

For $x \in \overline{\Omega}$ and a control $\alpha(\cdot)$, let $y^{x,\alpha}(t)$ solves

$$\begin{cases} \dot{y}(t) & = b(y(t), \alpha(t)), \qquad t > 0, \\ y(0) & = x \in \overline{\Omega}. \end{cases}$$

Let $\mathcal{A}_x$ be the set of all controls such that $y^{x,\alpha}(t) \in \overline{\Omega}$ for all $t \geq 0$.
<u>**QUESTION.**</u> Minimize the cost functional

$$u(x) = \inf_{\alpha \in \mathcal{A}_x} \int_0^\infty e^{-s} f\Big(y^{x,\alpha}(s), \alpha(s)\Big)\, ds.$$

Define $H(x,p) = \sup_{v \in U} \Big(- b(x,v) \cdot p - f(x,v) \Big)$ then u solves

$$\begin{cases} u(x) + H(x, Du(x)) \leq 0 & \text{in } \Omega, \\ u(x) + H(x, Du(x)) \geq 0 & \text{on } \overline{\Omega} \end{cases}$$

in the viscosity sense.

Literature

- Viscosity solution to Hamilton-Jacobi equations: L. Evans, M. Crandall, P-L. Lions, etc.
- State-constraint problem in convex setting: first by H. Soner (1986).
- Qualitative results on state-constraint Hamilton-Jacobi equations: I. Capuzzo-Dolcetta, P-L. Lions (1990), J. M. Lasry, and P.-L Lions (1989), H. Ishii and S. Koike (1996), F. Camilli, and M. Falcone (1996), etc.
- Qualitative results for nested domain: I. Capuzzo-Dolcetta, P-L. Lions (1990), S. Armstrong and H. Tran (2015), etc.
- Vanishing discount: H. Ishi, H. Mitake, H. Tran (2017), etc.

In this work, we provide **quantitative results** for state-constraint problem on nested domains.

Motivation Example

Let $n = 1$ and $U = [-1, 1]$. Let us consider the following Hamiltonian defined as

$$H(x,p) = \sup_{a \in [-1,1]} \left\{ -a \cdot p - e^{-|x|} \right\} = |p| - e^{-|x|}, \qquad (x,p) \in \mathbb{R} \times \mathbb{R}.$$

The domain is $[-k, k]$. For each $x_0 \in [-k, k]$, to find the value function $u_k(x_0)$ one needs to find a control $\alpha(t)$ that minimizes

$$\int_0^\infty e^{-s-|y(s)|}\, ds \qquad \text{subject to} \qquad \begin{cases} \dot{y}(t) & = \alpha(t) \in [-1,1], \\ y(0) & = x_0, \\ y(t) & \in [-k,k] \ \forall\, t \geq 0. \end{cases}$$

An optimal control for the constrained problem on $[-k, k]$ with $x_0 \geq 0$ ($x_0 < 0$) is $\alpha(t) \equiv 1$ on $[0, k - x_0]$ and 0 elsewhere ($\alpha(t) \equiv -1$ on $[0, k + x_0]$ and 0 elsewhere, respectively).

Viscosity solutions

Consider the equation

$$u(x) + H(x, Du(x)) = 0 \qquad \text{in } \Omega.$$

No classical solution in general, viscosity solution is a way to define an appropriate ”weak solution” as follows.

- u is a viscosity subsolution if for every $x \in \Omega$ and every C^1 function φ such that $u - \varphi$ has a local max over Ω at x then $u(x) + H(x, D\varphi(x)) \leq 0$.
- u is a viscosity supersolution if for every $x \in \Omega$ and every C^1 function φ such that $u - \varphi$ has a local min over Ω at x then $u(x) + H(x, D\varphi(x)) \geq 0$.

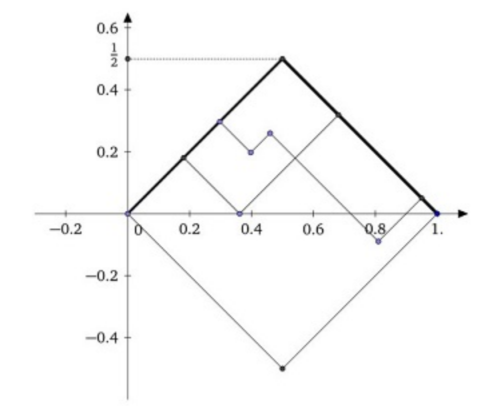
We call u a viscosity solution if it is both viscosity subsolution and viscosity supersolution.

Comparison principle: Under some appropriate conditions, a comparison principle for this kind of equation is a theorem saying that ”subsolution” $\leq$ ”supersolution”.

Example

Eikonal equation

$$\begin{cases} |u'(x)| = 1 & x \in (0,1), \\ u(0) = u(1) = 0. \end{cases}$$



State-Constraint boundary condition

$u \in \mathrm{BUC}(\overline{\Omega})$ is a **state-constraint viscosity solution** to $u(x) + H(x, Du(x)) = 0$ in Ω if it is a subsolution in Ω and is a supersolution on $\overline{\Omega}$, i.e.,

$$\begin{cases} u(x) + H(x, Du(x)) \leq 0 & \text{in } \Omega, \\ u(x) + H(x, Du(x)) \geq 0 & \text{on } \overline{\Omega}. \end{cases}$$

Heuristically, if u is C^1 then

$$\begin{cases} u(x) + H(x, Du(x)) = 0 & \text{in } \Omega, \\ H(x, Du(x)) \leq H(x, Du(x) + \beta\nu(x)) & \text{for all } \beta > 0, x \in \partial\Omega. \end{cases}$$

or if H is differentiable in p then

$$\begin{cases} u(x) + H(x, Du(x)) = 0 & \text{in } \Omega, \\ D_pH(x, Du(x)) \cdot \nu(x) \geq 0 & \text{on } \partial\Omega. \end{cases}$$

where $\nu(x)$ is the outward normal vector to $\partial\Omega$ at x.

Theorem (Perron's method - Existence).

Under some appropriate conditions, there exists a state-constraint viscosity solution $u \in \mathrm{C}(\overline{\Omega}) \cap \mathrm{W}^{1,\infty}(\Omega)$ and it is the maximal viscosity subsolution provided that comparison principle holds.

We consider a sequence of domains $\Omega_k \subset \Omega_{k+1}$ and $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$. Typically we will consider the following two prototypes

(P1) $\Omega_k = B(0, k)$.

(P2) $\Omega_k = B\left(0, 1 - 1/k\right)$.

Let u_k be the state-constraint viscosity solution to the problem on Ω_k, then under some appropriate conditions, we have that

- Under (P1), $u_k \to u$ locally uniformly and u is the viscosity solution to $u + H(x, Du) = 0$ in $\mathbb{R}^n$.
- Under (P2), $u_k \to u$ locally uniformly and u is the state-constraint viscosity solution to $u + H(x, Du) = 0$ on Ω.

Main question: What is the rate of convergence?

Why is this problem interesting?

- The problem on $\mathbb{R}^n$ is usually hard to investigate due to lack of compactness. It is interesting that we can recover the information of the global problem from the local problem on $\Omega_k \Longrightarrow$ to **quantify localization effects**.
- Possibly a way to approximate numerically the global problem from the local problem with rate of convergence.
- Understanding Dirichlet boundary condition.

$$\begin{cases} u + H(x, Du) = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega \end{cases}$$

- It is helpful for the analysis as well, since in most of the problems, people prefer to stay away from the boundary.

Rate of convergence on unbounded domain

Theorem.

(i) Assume $\Omega_k = B(0,k)$, in general (H may be nonconvex)

$$0 \le u_k(x) - u(x) \le \frac{C_R}{k^2} \qquad \text{for } k \in \mathbb{N} \text{ and } |x| \le R.$$

(ii) Assume $\Omega_k = B(0,k)$, if $p \mapsto H(x,p)$ is convex for every x then

$$0 \le u_k(x) - u(x) \le C_R e^{-C_R k} \qquad \text{for } k \in \mathbb{N} \text{ and } |x| \le R.$$

Moreover, this rate is optimal.

(iii) Assume $\Omega_k = B(0, 1 - 1/k)$ then (H maybe nonconvex)

$$0 \le u_k(x) - u(x) \le \frac{C}{k} \qquad \text{for } k \in \mathbb{N}.$$

Moreover, this rate is optimal.

Sketch of the proof for the convex case

Using a priori estimate $|u(x)| + |Du(x)| \leq C$, we can assume that H has quadratic growth, and so

$$A|v|^2 - K \leq L(x, v) \leq B|v|^2 + K \qquad \text{for all } (x, v).$$

The solution u_k and u can be written as

$$u_k(x) = \inf_{\eta \in \mathcal{A}_x^k} \int_0^\infty e^{-s} L(\eta(s), -\dot{\eta}(s))\, ds$$

where $\mathcal{A}_x^k = \{\eta \in \mathrm{AC}([0,\infty), \mathbb{R}^n) : \eta(0) = x, \eta(t) \in \overline{\Omega}_k \text{ for all } t \geq 0\}$ and

$$u(x) = \inf_{\eta \in \mathcal{A}_x} \int_0^\infty e^{-s} L(\eta(s), -\dot{\eta}(s))\, ds$$

where $\mathcal{A}_x = \{\eta \in \mathrm{AC}([0,\infty), \mathbb{R}^n) : \eta(0) = x \text{ for all } t \geq 0\}$.

Since $\mathcal{A}_x^k \subset \mathcal{A}_x$, one has $\boxed{u_k(x) \geq u(x)}$. For the converse, for $x \in \mathbb{R}^n$ we look for a minimizer η such that $\eta(0) = x$ and

$$u(x) = \int_0^\infty e^{-s} L(\eta(s), -\dot{\eta}(s))\, ds.$$

Sketch of the proof for the convex case

We show the **existence of a minimizer with bounded velocity**, i.e., $|\dot{\eta}(t)| \leq C$
$\Longrightarrow$ estimate on the first time η exits the domain:

$$k = |\eta(t)| \leq |\eta(0)| + \int_0^t |\dot{\eta}(s)|\, ds \leq |x| + Ct \qquad \Rightarrow \qquad t \geq \frac{k - |x|}{C}.$$

Dynamic Programming Principle

$$\begin{aligned} u(x) &= \int_0^t e^{-s} L\left(\eta(s), -\dot{\eta}(s)\right)\, ds + e^{-t} u(\eta(t)) \\ &\geq \int_0^t e^{-s} L\left(\gamma(s), -\dot{\gamma}(s)\right)\, ds - Ce^{-t} \\ &\geq \int_0^\infty e^{-s} L\left(\gamma(s), -\dot{\gamma}(s)\right)\, ds - C'e^{-t} - C_H e^{-t} \\ &\geq u_k(x) - \left((C + C')e^{\frac{|x|}{C}}\right) e^{-\frac{k}{C}} \qquad \Longrightarrow \qquad \boxed{u(x) \geq u_k(x) - Ce^{-Ck}.} \end{aligned}$$

where

$$\gamma(s) = \begin{cases} \eta(s) & \text{if } s \in [0, t], \\ \eta(t) & \text{if } s \in [t, \infty). \end{cases}$$

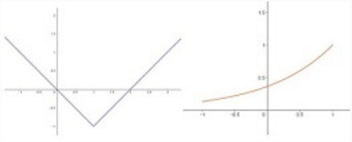
Example. Let $n = 1$ and $U = [-1, 1]$, we consider $b(x, a) = a$ and $f(x, a) = e^{-|x|}$. For a fix $k \in \mathbb{N}$, we consider $\Omega_k = (-k, k)$.

$$H(x, p) = \sup_{a \in [-1,1]} \left\{ -a \cdot p - e^{-|x|} \right\} = |p| - e^{-|x|}, \qquad (x, p) \in \mathbb{R} \times \mathbb{R}.$$

Then, $u_k(x) = \frac{e^{-|x|}}{2} + \frac{e^{|x|-2k}}{2}$ for $x \in [-k, k]$, while $u(x) = \frac{e^{-|x|}}{2}$. Hence, the exponential rate of convergence is obtained.

Example. Let $H(x, p) = H(p) = |p - 1| - 1$ in $\mathbb{R}$, then the unique state-constraint viscosity solution on $[-k, k]$ is

$$u_k(x) = e^{x-k}.$$



A special class of Hamiltonians

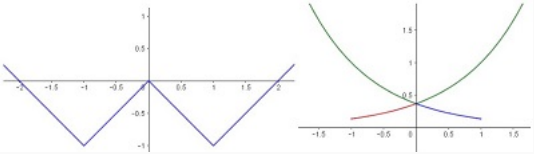
Theorem (A special class of Hamiltonians).
Assume that $H(x,p) = a(x)K(p)$ with a Lispchitz function K such that $K(0) = 0$ and $0 < \alpha \leq a(x) \leq \beta$, then

$$0 \leq u_k(x) \leq C_R e^{-C_R k}$$

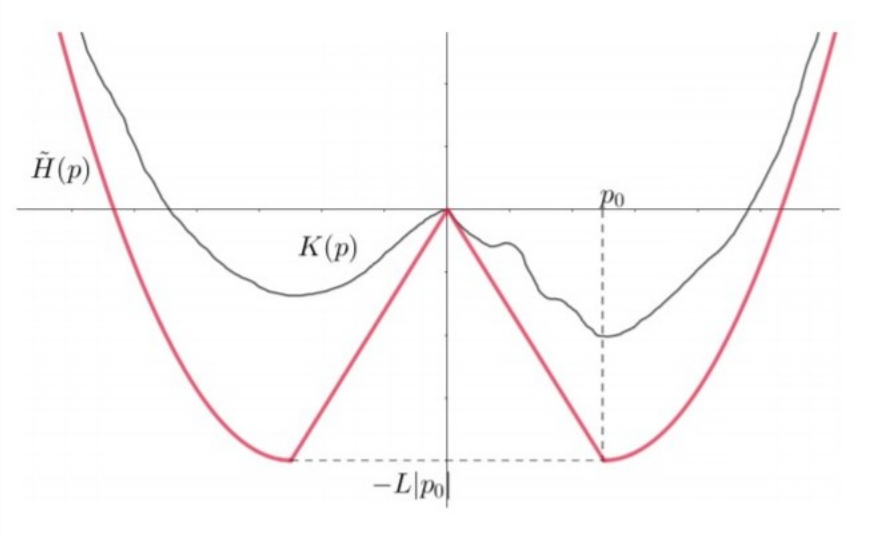
for $|x| \leq R$ and this rate is optimal.

Example. Let $H(x,p) = H(p) = -|p|$ if $|p| \leq 1$ and $H(p) = |p-2|$ if $|p| \geq 1$, then the unique state-constaint viscosity solution on $[-k, k]$ is

$$u_k(x) = \max\{e^{x-k}, x^{-x-k}\} = e^{|x|-k}.$$



Sketch of the proof - A picture



Example. Assume $H(x,p) = K(p) + V(x)$ where $V(x) = e^{-|x|}$ and $K : \mathbb{R} \to \mathbb{R}$ defined by

$$K(p) = \begin{cases} -|p| & \text{for } |p| \leq 1, \\ |p| - 2 & \text{for } |p| \geq 1. \end{cases}$$

The unique state-constraint viscosity solution is

$$u_k(x) = -\frac{1}{2}e^{-|x|} + \left(e^{-k} - \frac{1}{2}e^{-2k}\right)e^{|x|}, \qquad x \in [-k, k],$$

and

$$u(x) = -\frac{1}{2}e^{-|x|}, \qquad x \in \mathbb{R}.$$

We have $u_k \to u$ locally uniformly in $\mathbb{R}$ with rate $\mathcal{O}(e^{-k})$.

Sketch of the proof for rate $\mathcal{O}\left(\frac{1}{k^2}\right)$

Lower bound.

$$\begin{cases} u_k(x) + H(x, Du_k(x)) \geq 0 & \text{on } \overline{B(0,k)}, \\ u(x) + H(x, Du(x)) \leq 0 & \text{in } B(0,k). \end{cases} \implies u_k(x) \geq u(x) \quad \text{for } x \in \overline{B(0,k)}.$$

Upper bound. Define

$$\Phi(x,y) = u_k(x) - u(y) - 2C_H k^2 |x-y|^2 - \frac{8C_H}{k^2}|y|^2, \qquad (x,y) \in \overline{B(0,k)} \times \mathbb{R}^n.$$

$\Phi^k \leq 2C_H$, if $|y| > \frac{k}{2}$, then

$$\Phi^k(0,0) - \Phi^k(x,y) \geq \underbrace{-u_k(x) + u_k(0) - u(0) + u(y)}_{\geq -4C_H} + 2C_H k^2 |x-y|^2 + \frac{8C_H}{k^2}|y|^2 > 0,$$

Thus $\Phi^k(x,y)$ has a max over $\overline{B(0,k)} \times \mathbb{R}^n$ at $(x_k, y_k) \in \overline{B(0,k)} \times \overline{B\left(0, \frac{k}{2}\right)}$.

$\Phi^k(x_k, y_k) \geq \Phi^k(y_k, y_k)$

$$2C_H k^2 |x_k - y_k|^2 \leq u_k(x_k) - u_k(y_k) \leq C_H |x_k - y_k|.$$

Therefore, we deduce that

$$|x_k| \leq |y_k| + \frac{1}{2k^2} < k \tag{1}$$

for all $k \geq 1$ since $|y_k| \leq \frac{k}{2}$.

Sketch of the proof for rate $\mathcal{O}\left(\frac{1}{k^2}\right)$

$x \mapsto \Phi^k(x, y_k)$ has a max at x_k with $|x_k| < k$,

$$u_k(x_k) + H\left(x_k, p_k\right) \leq 0, \tag{2}$$

where $p_k = 4C_H k^2(x_k - y_k)$.

$y \mapsto \Phi^k(x_k, y)$ has a max at y_k, thus $u(y) - \left(-2C_H k^2 \left|x_k - y_k\right|^2 - \frac{8C_H}{k^2}\left|y\right|^2\right)$ has a min at y_k.

$$u(y_k) + H(y_k, p_k + q_k) \geq 0 \tag{3}$$

where $q_k = -\frac{16C_H}{k^2} y_k$.

Using (2), (3) and some Lipschitz assumption on H, there exists a constant $\tilde{C}_H$ such that

$$\begin{aligned} u_k(x_k) - u(y_k) &\leq H(y_k, p_k + q_k) - H(x_k, p_k) \\ &= H(y_k, p_k + q_k) - H(y_k, p_k) + H(y_k, p_k) - H(x_k, p_k) \\ &\leq \tilde{C}_H|q_k| + \tilde{C}_H|x_k - y_k| \\ &\leq \frac{16\tilde{C}_H C_H}{k^2}|y_k| + \frac{\tilde{C}_H}{k^2} \leq \frac{8\tilde{C}_H C_H}{k} + \frac{\tilde{C}_H}{k^2}. \end{aligned} \tag{4}$$

Sketch of the proof for general nonconvex Hamiltonians under (P1)

If we stop here, the fact that $\Phi^k(x_k, y_k) \geq \Phi^k(x, x)$ for $x \in B(0, k)$ gives

$$u_k(x) - u(x) \leq u_k(x_k) - u(y_k) + \frac{8C_H}{k^2}|x|^2 \leq \frac{C}{k} + \frac{C(1+|x|^2)}{k^2}$$

for all $k \geq 2 \implies$ a typical rate of convergence $\mathcal{O}\left(\frac{1}{k}\right)$.

Nevertheless, a key new point here is to bootstrap once more to improve this rate.
We use that $\Phi^k(x_k, y_k) \geq \Phi^k(0, 0)$ together with (4) to yield

$$\begin{aligned} 2C_H k^2 |x_k - y_k|^2 + \frac{8C_H}{k^2}|y_k|^2 &\leq u_k(x_k) - u_k(0) + u(0) - u(y_k) \\ &\leq u_k(x_k) - u(y_k) \\ &\leq \frac{16\tilde{C}_H C_H}{k^2}|y_k| + \frac{\tilde{C}_H}{k^2}. \end{aligned}$$

Therefore,

$$|y_k|^2 \leq 2\tilde{C}_H|y_k| + \frac{\tilde{C}_H}{8C_H} \leq \frac{1}{2}|y_k|^2 + 2\tilde{C}_H^2 + \frac{\tilde{C}_H}{8C_H} = \frac{1}{2}|y_k|^2 + C.$$

In particular, $|y_k| \leq C$.

Sketch of the proof for general nonconvex Hamiltonians under (P1)

Now for any $x \in \overline{B(0,k)}$, clearly we have that $\Phi^k(x_k, y_k) \geq \Phi^k(x,x)$. This, together with (4) and $|y_k| \leq C$, implies

$$u_k(x) - u(x) \leq u_k(x_k) - u(y_k) + \frac{8C_H}{k^2}|x|^2 \leq \frac{C(1+|x|^2)}{k^2}$$

for all $k \geq 2$. If $|x| \leq R$, then

$$0 \leq u_k(x) - u(x) \leq \frac{C(1+R^2)}{k^2},$$

which gives the desired result.

Rate of convergence on bounded domain

Theorem (Bounded domain).

Under (P2), H is coercive and some appropriate assumptions we have

$$0 \leq u_k(x) - u(x) \leq \frac{C}{k} \qquad \text{for } k \in \mathbb{N}.$$

Moreover, this rate is optimal.

Example. (Bounded domain) Let $H(x,p) = H(p) = -|p|$ if $|p| \leq 1$ and $H(p) = |p-2|$ if $|p| \geq 1$, , we see that $u_k(x) = e^{x-\left(1-\frac{1}{k}\right)}$ and $u(x) = e^{x-1}$, therefore

$$0 \leq u_k(x) - u(x) = e^{x-1}\left(e^{\frac{1}{k}} - 1\right) \leq \frac{2}{k}$$

for $x \in \left[-\left(1-\frac{1}{k}\right), 1-\frac{1}{k}\right]$. Besides, $e^{\frac{1}{k}} - 1 \geq \frac{1}{k}$, and so, $\mathcal{O}\left(\frac{1}{k}\right)$ is optimal.

Questions

Open questions

- In the first prototype (P1) case, what is the optimal rate of convergence of u_k to u in the general nonconvex setting?
- Assume (P1), and $H(x,p) = K(p) + V(x)$, where $K \in \mathrm{Lip}(\mathbb{R}^n)$ is coercive and nonconvex, and $V \in \mathrm{BUC}(\mathbb{R}^n)$. Is it true that we always have exponential rate of convergence of u_k to u?
- Other kinds of boundary conditions?
- Similar results for the viscous case?

THANK YOU!