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Particle representations for a class of nonlinear SPDEs

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Abstract

An infinite system of stochastic differential equations for the locations and weights of a collection of particles is considered. The particles interact through their weighted empirical measure, V, and V is shown to be the unique solution of a nonlinear stochastic partial differential equation (SPDE). Conditions are given under which the weighted empirical measure has an L_2-density with respect to Lebesgue measure. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

We consider a class of nonlinear stochastic partial differential equations of the form

$$\mathrm{d}v(t,x) = \left(\frac{1}{2}\sum_{i,j=1}^{d} \partial_{x_i}\partial_{x_j}[a_{ij}(x,v(t,\cdot))v(t,x)] - \sum_{i=1}^{d} \partial_{x_i}[b_i(x,v(t,\cdot))v(t,x)] + d(x,v(t,\cdot))v(t,x) \right)\mathrm{d}t$$
$$- \int_U \left(\beta(x,v(t,\cdot),u)v(t,x) + \sum_{i=1}^{d} \partial_{x_i}[\alpha_i(x,v(t,\cdot),u)] \right) W(\mathrm{d}u\,\mathrm{d}t), \qquad (1.1)$$

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where W is a space–time Gaussian white noise on $U \times [0,\infty)$. We are interested in representations of the solution in terms of weighted empirical measures of the form

$$V(t) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} A_i(t)\delta_{X_i(t)}, \tag{1.2}$$

where δ_x is the Dirac measure at x and the limit exists in the weak* topology on $\mathscr{M}(\mathbb{R}^d)$, the collection of all finite signed Borel measures on $\mathbb{R}^d$. We think of $\{X_i(t)\colon t\geqslant 0, i\in\mathbb{N}\}$ as a system of particles with locations in $\mathbb{R}^d$ and time-varying weights $\{A_i(t)\colon t\geqslant 0, i\in\mathbb{N}\}$.

Suppose $\{X_i, A_i, V\}$ is governed by the following equations:

$$X_i(t) = X_i(0) + \int_0^t \sigma(X_i(s),V(s))\,\mathrm{d}B_i(s) + \int_0^t c(X_i(s),V(s))\,\mathrm{d}s + \int_{U\times[0,t]} \alpha(X_i(s),V(s),u)W(\mathrm{d}u\,\mathrm{d}s) \tag{1.3}$$

and

$$A_i(t) = A_i(0) + \int_0^t A_i(s)\gamma^{\mathrm{T}}(X_i(s),V(s))\,\mathrm{d}B_i(s) + \int_0^t A_i(s)\,d(X_i(s),V(s))\,\mathrm{d}s + \int_{U\times[0,t]} A_i(s)\beta(X_i(s),V(s),u)W(\mathrm{d}u\,\mathrm{d}s), \tag{1.4}$$

where the B_i are independent, standard $\mathbb{R}^d$-valued Brownian motions and W, independent of $\{B_i\}$, is Gaussian white noise with

$$\mathbb{E}[W(A,t)W(B,t)] = \mu(A\cap B)t.$$

For simplicity, assume that μ is a Borel measure on a complete, separable metric space U.

Assume that $\{(A_i(0),X_i(0))\}$ is exchangeable (for example, iid) and independent of $\{B_i\}$ and W. Applying Itô's formula to (1.3) and (1.4), for every $\phi\in C_b^2(\mathbb{R}^d)$, we have

$$\begin{aligned} A_i(t)\phi(X_i(t)) = A_i(0)\phi(X_i(0)) &+ \int_0^t A_i(s)\phi(X_i(s))\gamma^{\mathrm{T}}(X_i(s),V(s))\,\mathrm{d}B_i(s) \\ &+ \int_0^t A_i(s)\phi(X_i(s))\,d(X_i(s),V(s))\,\mathrm{d}s \\ &+ \int_{U\times[0,t]} A_i(s)\phi(X_i(s))\beta(X_i(s),V(s),u)W(\mathrm{d}u\,\mathrm{d}s) \\ &+ \int_0^t A_i(s)L(V(s))\phi(X_i(s))\,\mathrm{d}s \\ &+ \int_0^t A_i(s)\nabla^{\mathrm{T}}\phi(X_i(s))\sigma(X_i(s),V(s))\,\mathrm{d}B_i(s) \\ &+ \int_{U\times[0,t]} A_i(s)\nabla^{\mathrm{T}}\phi(X_i(s))\alpha(X_i(s),V(s),u)W(\mathrm{d}u\,\mathrm{d}s), \end{aligned} \tag{1.5}$$

where

$$L(v)\phi(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x,v)\partial_{x_i}\partial_{x_j}\phi(x) + \sum_{i=1}^{d} b_i(x,v)\partial_{x_i}\phi(x)$$

with

$$a(x,v) = \sigma(x,v)\sigma^{\mathrm{T}}(x,v) + \int_U \alpha(x,v,u)\alpha^{\mathrm{T}}(x,v,u)\mu(\mathrm{d}u)$$

and

$$b(x,v) = c(x,v) + \sigma(x,v)\gamma(x,v) + \int_U \beta(x,v,u)\alpha(x,v,u)\mu(\mathrm{d}u).$$

Averaging both sides of (1.5), we will show that V given by (1.2) satisfies

$$\langle\phi, V(t)\rangle = \langle\phi, V(0)\rangle + \int_0^t \langle d(\cdot, V(s))\phi + L(V(s))\phi, V(s)\rangle\,\mathrm{d}s$$
$$+ \int_{U\times[0,t]} \langle\beta(\cdot, V(s),u)\phi + \alpha^{\mathrm{T}}(\cdot, V(s),u)\nabla\phi, V(s)\rangle W(\mathrm{d}u\,\mathrm{d}s),$$
$$\forall\phi\in C_b^2(\mathbb{R}^d), \tag{1.6}$$

and, hence, is a weak solution of the stochastic partial differential equation (SPDE) (1.1) where, if it exists, v is the density

$$V(t,B) = \int_B v(t,x)\,\mathrm{d}x.$$

Our goal is to give conditions under which there exists a unique solution of the system (1.3), (1.4) and as a consequence obtain existence and uniqueness of the SPDE (1.6).

Limits of empirical measure processes for systems of interacting diffusions have been studied by various authors (see, for example, Chiang et al., 1991; Graham, 1992; Hitsuda and Mitoma, 1986; Kallianpur and Xiong, 1994, 1995; Méléard, 1996; Morien, 1996) since the pioneering work by McKean (1967). Typically, the driving processes in the models are assumed to be independent. The limit is then a deterministic, measure-valued function.

Florchinger and Le Gland (1992) consider particle approximations for stochastic partial differential equations in a setting that, in the notation above, corresponds to taking $\gamma=\sigma=0$ and the other coefficients independent of V. Florchinger and Le Gland were motivated by approximations to the Zakai equation of nonlinear filtering. Del Moral (1995) specifically studies this example. Kotelenez (1995) introduces a model of n-particles with the same driving process for each particle and studies the empirical process as the solution of a SPDE. His model corresponds to taking $\gamma=\sigma=d=\beta=0$, but the other coefficients are allowed to depend on V. In particular, the weights A_i are constants. Dawson and Vaillancourt (1995) consider a model given as a solution of a martingale problem that corresponds to taking $A_i(t)\equiv 1$ in the current model. Bernard et al. (1994) consider a system with time-varying weights and a deterministic limit.

The paper is organized as follows: In Section 2, we prove that the system (1.2)–(1.4) has a unique solution. Since the system does not satisfy a global Lipschitz condition, we cannot directly apply the results developed by Kurtz and Protter (1996) (cf. Theorem 9.1). Instead, a truncation technique is employed. In Section 3, we prove existence and uniqueness for (1.6). We achieve this goal by considering a corresponding linear equation first. As a by-product from this linear equation, the existence of the density $v(t,x)$ is obtained. Uniqueness for the system (1.2)–(1.4) and for the linear equation implies uniqueness for the SPDE (1.1). In Section 4, we briefly discuss the relationship of (1.6) to the equations of nonlinear filtering theory.

2. Existence and uniqueness of the solution for the system

In this section, we establish existence and uniqueness for the solution of the system (1.2)–(1.4). For $v_1, v_2 \in \mathscr{M}_+(\mathbb{R}^d)$, the Wasserstein metric is defined by

$$\rho(v_1, v_2) = \sup\{|\langle \phi, v_1\rangle - \langle \phi, v_2\rangle|\colon \phi \in \mathbb{B}_1\},$$

where

$$\mathbb{B}_1 = \{\phi\colon |\phi(x) - \phi(y)| \leqslant |x-y|, |\phi(x)| \leqslant 1, \forall x, y \in \mathbb{R}^d\}.$$

Note that the metric ρ determines the topology of weak convergence on $\mathscr{M}_+(\mathbb{R}^d)$.

We assume that $\sigma : \mathbb{R}^d \times \mathscr{M}(\mathbb{R}^d) \to \mathbb{R}^{d\times d}$, $c : \mathbb{R}^d \times \mathscr{M}(\mathbb{R}^d) \to \mathbb{R}^d$, $\alpha : \mathbb{R}^d \times \mathscr{M}(\mathbb{R}^d) \times U \to \mathbb{R}^d$, $\gamma : \mathbb{R}^d \times \mathscr{M}(\mathbb{R}^d) \to \mathbb{R}^d$, $d : \mathbb{R}^d \times \mathscr{M}(\mathbb{R}^d) \to \mathbb{R}$ and $\beta : \mathbb{R}^d \times \mathscr{M}(\mathbb{R}^d) \times U \to \mathbb{R}$ satisfy the following conditions (S1) and (S2):

(S1) There exists a constant K such that for each $x \in \mathbb{R}^d$, $v \in \mathscr{M}(\mathbb{R}^d)$

$$\begin{aligned}&|\sigma(x,v)|^2 + |c(x,v)|^2 + \int_U |\alpha(x,v,u)|^2 \mu(\mathrm{d}u)\\ &\quad + |\gamma(x,v)|^2 + |d(x,v)|^2 + \int_U \beta(x,v,u)^2 \mu(\mathrm{d}u) \leqslant K^2.\end{aligned}$$

(S2) For each $x_1, x_2 \in \mathbb{R}^d$ $v_1, v_2 \in \mathscr{M}(\mathbb{R}^d)$ and any representation $v_i = v_i^+ - v_i^-, v_i^+, v_i^- \in \mathscr{M}_+(\mathbb{R}^d)$

$$\begin{aligned}&|\sigma(x_1,v_1) - \sigma(x_2,v_2)|^2 + |c(x_1,v_1) - c(x_2,v_2)|^2\\ &\quad + |\gamma(x_1,v_1) - \gamma(x_1,v_1)|^2 + \int_U |\alpha(x_1,v_1,u) - \alpha(x_2,v_2,u)|^2 \mu(\mathrm{d}u)\\ &\quad + |d(x_1,v_1) - d(x_2,v_2)|^2 + \int_U |\beta(x_1,v_1,u) - \beta(x_2,v_2,u)|^2 \mu(\mathrm{d}u)\\ &\quad \leqslant K^2(|x_1 - x_2|^2 + \rho(v_1^+, v_2^+)^2 + \rho(v_1^-, v_2^-)^2).\end{aligned}$$

Let (X, A, V) be a solution of (1.2)–(1.4). In order to apply the Lipschitz condition, we identify a canonical decomposition $V(t) = V^+(t) - V^-(t)$. For simplicity of notation, define

$$M(t) = \int_0^t \gamma^{\mathrm{T}}(X_i(s), V(s))\,\mathrm{d}B_i(s) + \int_{U\times[0,t]} \beta(X_i(s), V(s), u) W(\mathrm{d}u\,\mathrm{d}s).$$

Then $M(t)$ is a martingale and

$$\langle M\rangle_t = \int_0^t |\gamma(X_i(s),V(s))|^2\,\mathrm{d}s + \int_{U\times[0,t]} \beta(X_i(s),V(s),u)^2\mu(\mathrm{d}u)\,\mathrm{d}s$$
$$\leqslant K^2 t.$$

An application of Itô's formula shows that the solution of (1.4) is given by

$$A_i(t) = A_i(0)\exp\left(M(t) - \frac{1}{2}\langle M\rangle_t + \int_0^t d(X_i(s),V(s))\,\mathrm{d}s\right). \tag{2.1}$$

Note that if $A_i(0) > 0$, then $A_i(t) > 0$ for all $t > 0$ and similarly if $A_i(0) < 0$. Let $A_i^+(t) = A_i(t)$ if $A_i(t) > 0$ and $A_i^+(t) = 0$ otherwise, and let $A_i^-(t) = -A_i(t)$ if $A_i(t) < 0$ and $A_i^-(t) = 0$ otherwise. Then we define

$$V^+(t) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n A_i^+(t)\delta_{X_i(t)}, \qquad V^-(t) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n A_i^-(t)\delta_{X_i(t)}. \tag{2.2}$$

A truncation argument will require the following estimate.

Proposition 2.1. *Suppose that Assumption* (S1) *holds and*

$$\mathbb{E}A_1(0)^2 + \mathbb{E}|X_1(0)|^2 < \infty. \tag{2.3}$$

If (X,A,V) *is a solution of* (1.2)–(1.4), *then for every* $t\geqslant 0$,

$$\mathbb{E}\sup_{0\leqslant s\leqslant t}(A_i(s)^2 + |X_i(s)|^2) < \infty. \tag{2.4}$$

Proof. By Doob's inequality, we have

$$\mathbb{E}\sup_{0\leqslant s\leqslant t}|X_i(s)|^2 \leqslant 4\mathbb{E}|X_i(0)|^2 + 16\mathbb{E}\int_0^t |\sigma(X_i(s),V(s))|^2\,\mathrm{d}s$$
$$+4t\mathbb{E}\int_0^t |c(X_i(s),V(s))|^2\,\mathrm{d}s$$
$$+16\mathbb{E}\int_{U\times[0,t]} |\alpha(X_i(s),V(s),u)|^2\mu(\mathrm{d}u)\,\mathrm{d}s$$
$$\leqslant 4\mathbb{E}|X_i(0)|^2 + 32K^2t + 4K^2t^2 < \infty.$$

By (2.1), we have

$$A_i^2(t) = A_i^2(0)\exp\left(2M(t) - \langle M\rangle_t + \int_0^t 2d(X_i(s),V(s))\,\mathrm{d}s\right).$$

Since $\exp(2M(t) - 2\langle M\rangle_t)$ is a martingale, the bounds on $\langle M\rangle$ and d and Doob's inequality imply

$$\mathbb{E}\sup_{0\leqslant s\leqslant t}A_i(s)^2 \leqslant 4\mathrm{e}^{2Kt+K^2t}\mathbb{E}A_i(0)^2. \qquad \square$$

Theorem 2.1. *Under Assumptions* (S1), (S2) *and* (2.3), *the system has at most one solution.*

Proof. Let (X,A,V) and $(\tilde{X},\tilde{A},\tilde{V})$ be two solutions of (1.2)–(1.4) with the same initial conditions, and define V^+, V^-, $\tilde{V}^+$, and $\tilde{V}^-$ as in (2.2).

Recall that by the exchangeability, we have the existence of $\lim_{n\to\infty}(1/n)\sum_{i=1}^n A_i(t)^2$. Let

$$\tau_m = \inf\left\{t\colon \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n A_i(t)^2 > m^2\right\}.$$

$\tilde{\tau}_m$ is defined similarly. Let $\eta_m = \tau_m \wedge \tilde{\tau}_m$. Then

$$\begin{aligned}
&\mathbb{E}|X_i(t\wedge\eta_m) - \tilde{X}_i(t\wedge\eta_m)|^2\\
&\quad\leqslant 12\mathbb{E}\int_0^t |\sigma(X_i(s),V(s)) - \sigma(\tilde{X}_i(s),\tilde{V}(s))|^2 1_{s\leqslant\eta_m}\,\mathrm{d}s\\
&\qquad +3t\mathbb{E}\int_0^t |c(X_i(s),V(s)) - c(\tilde{X}_i(s),\tilde{V}(s))|^2 1_{s\leqslant\eta_m}\,\mathrm{d}s\\
&\qquad +12\mathbb{E}\int_{U\times[0,t]} |\alpha(X_i(s),V(s),u) - \alpha(\tilde{X}_i(s),\tilde{V}(s),u)|^2 1_{s\leqslant\eta_m}\mu(\mathrm{d}u)\,\mathrm{d}s\\
&\quad\leqslant 3K^2(8+t)\mathbb{E}\int_0^t (|X_i(s) - \tilde{X}_i(s)|^2 + \rho(V^+(s),\tilde{V}^+(s))^2\\
&\qquad +\rho(V^-(s),\tilde{V}^-(s))^2)1_{s\leqslant\eta_m}\,\mathrm{d}s.
\end{aligned}\tag{2.5}$$

For $s\leqslant\eta_m$,

$$\begin{aligned}
\rho(V^+(s),\tilde{V}^+(s)) &= \sup_{\phi\in\mathbb{B}_1}\left|\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n (A_i^+(s)\phi(X_i(s)) - \tilde{A}_i^+(s)\phi(\tilde{X}_i(s)))\right|\\
&\leqslant \sup_{\phi\in\mathbb{B}_1}\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n A_i^+(s)|\phi(X_i(s)) - \phi(\tilde{X}_i(s))|\\
&\quad + \sup_{\phi\in\mathbb{B}_1}\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n |A_i^+(s) - \tilde{A}_i^+(s)||\phi(\tilde{X}_i(s))|\\
&\leqslant \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n A_i^+(s)|X_i(s) - \tilde{X}_i(s)|\\
&\quad + \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n |A_i^+(s) - \tilde{A}_i^+(s)|
\end{aligned}$$

and a similar estimate holds for $\rho(V^-(t),\tilde{V}^-(t))$. Consequently,

$$\begin{aligned}
&\rho(V^+(s),\tilde{V}^+(s)) + \rho(V^-(s),\tilde{V}^-(s))\\
&\quad\leqslant \left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n A_i(s)^2\right)^{1/2}\left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n |X_i(s) - \tilde{X}_i(s)|^2\right)^{1/2}
\end{aligned}$$

$$+\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}|A_i(s)-\tilde{A}_i(s)|$$

$$\leqslant m\left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}|X_i(s)-\tilde{X}_i(s)|^2\right)^{1/2}+\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}|A_i(s)-\tilde{A}_i(s)|.$$

Let

$$f_m(t)=\mathbb{E}|X_i(t\wedge\eta_m)-\tilde{X}_i(t\wedge\eta_m)|^2$$

and

$$g_m(t)=\mathbb{E}\left[\left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}|A_i(t\wedge\eta_m)-\tilde{A}_i(t\wedge\eta_m)|\right)^2\right].$$

By (2.5) and Fatou's lemma, we have

$$f_m(t)\leqslant 3K^2(8+t)\int_0^t (f_m(s)+2m^2 f_m(s)+2g_m(s))\,\mathrm{d}s. \tag{2.6}$$

By (2.1), making use of the fact that $|\mathrm{e}^x-\mathrm{e}^y|\leqslant(\mathrm{e}^x\vee\mathrm{e}^y)|x-y|$, we have

$$\begin{aligned}
&|A_i(t)-\tilde{A}_i(t)|\\
&\quad=(|A_i(t)|\vee|\tilde{A}_i(t)|)\left|\int_0^t(\gamma^{\mathrm{T}}(X_i(s),V(s))-\gamma^{\mathrm{T}}(\tilde{X}_i(s),\tilde{V}(s)))\,\mathrm{d}B_i(s)\right.\\
&\qquad+\int_{U\times[0,t]}(\beta(X_i(s),V(s),u)-\beta(\tilde{X}_i(s),\tilde{V}(s),u))W(\mathrm{d}u\,\mathrm{d}s)\\
&\qquad+\int_0^t(d(X_i(s),V(s))-d(\tilde{X}_i(s),\tilde{V}(s)))\,\mathrm{d}s\\
&\qquad-\frac{1}{2}\int_0^t(|\gamma(X_i(s),V(s))|^2-|\gamma(\tilde{X}_i(s),\tilde{V}(s))|^2)\,\mathrm{d}s\\
&\qquad\left.-\frac{1}{2}\int_{U\times[0,t]}(\beta(X_i(s),V(s),u)^2-\beta(\tilde{X}_i(s),\tilde{V}(s),u))^2)\mu(\mathrm{d}u)\,\mathrm{d}s\right|.
\end{aligned}$$

Hence, for $t\leqslant\eta_m$,

$$\begin{aligned}
&\left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}|A_i(t)-\tilde{A}_i(t)|\right)^2\\
&\quad\leqslant\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}A_i^2(t)\vee\tilde{A}_i^2(t)\\
&\qquad\times\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left|\int_0^t(\gamma^{\mathrm{T}}(X_i(s),V(s))-\gamma^{\mathrm{T}}(\tilde{X}_i(s),\tilde{V}(s)))\,\mathrm{d}B_i(s)\right.\\
&\qquad+\int_{U\times[0,t]}(\beta(X_i(s),V(s),u)-\beta(\tilde{X}_i(s),\tilde{V}(s),u))W(\mathrm{d}u\,\mathrm{d}s)\\
&\qquad+\int_0^t(d(X_i(s),V(s))-d(\tilde{X}_i(s),\tilde{V}(s)))\,\mathrm{d}s
\end{aligned}$$

$$-\frac{1}{2}\int_0^t (|\gamma(X_i(s),V(s))|^2 - |\gamma(\tilde{X}_i(s),\tilde{V}(s))|^2)\,\mathrm{d}s$$

$$-\frac{1}{2}\int_{U\times[0,t]} (\beta(X_i(s),V(s),u)^2 - \beta(\tilde{X}_i(s),\tilde{V}(s),u))^2)\mu(\mathrm{d}u)\,\mathrm{d}s\Bigg|^2$$

$$\leqslant \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n (A_i(t)^2 + \tilde{A}_i(t)^2)$$

$$\times 5\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n \left(\left|\int_0^t (\gamma^{\mathrm{T}}(X_i(s),V(s)) - \gamma^{\mathrm{T}}(\tilde{X}_i(s),\tilde{V}(s)))\,\mathrm{d}B_i(s)\right|^2\right.$$

$$+\left|\int_{U\times[0,t]} (\beta(X_i(s),V(s),u) - \beta(\tilde{X}_i(s),\tilde{V}(s),u)))W(\mathrm{d}u\,\mathrm{d}s)\right|^2$$

$$+t\int_0^t (d(X_i(s),V(s)) - d(\tilde{X}_i(s),\tilde{V}(s)))^2\,\mathrm{d}s$$

$$+\frac{1}{4}t\int_0^t (|\gamma(X_i(s),V(s))|^2 - |\gamma(\tilde{X}_i(s),\tilde{V}(s))|^2)^2\,\mathrm{d}s$$

$$\left. + \frac{1}{4}t\int_0^t \left(\int_U (\beta(X_i(s),V(s),u)^2 - \beta(\tilde{X}_i(s),\tilde{V}(s),u))^2\mu(\mathrm{d}u)\right)^2 \mathrm{d}s\right)$$

$$\leqslant 2m^2 5\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n \left(\left|\int_0^t (\gamma^{\mathrm{T}}(X_i(s),V(s)) - \gamma^{\mathrm{T}}(\tilde{X}_i(s),\tilde{V}(s)))\,\mathrm{d}B_i(s)\right|^2\right.$$

$$+\left|\int_{U\times[0,t]} (\beta(X_i(s),V(s),u) - \beta(\tilde{X}_i(s),\tilde{V}(s),u)))W(\mathrm{d}u\,\mathrm{d}s)\right|^2$$

$$+K^2 t\int_0^t (|X_i(s) - \tilde{X}_i(s)|^2 + \rho(V^+(s),\tilde{V}^+(s))^2 + \rho(V^-(s),\tilde{V}^-(s))^2)\,\mathrm{d}s$$

$$+\frac{1}{4}t\int_0^t (2K)^2|\gamma(X_i(s),V(s)) - \gamma(\tilde{X}_i(s),\tilde{V}(s))|^2\,\mathrm{d}s$$

$$\left. + \frac{1}{4}t\int_0^t 4K^2\int_U |\beta(X_i(s),V(s),u) - \beta(\tilde{X}_i(s),\tilde{V}(s),u)|^2\mu(\mathrm{d}u)\,\mathrm{d}s\right)$$

$$\leqslant 10m^2\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n \left(\left|\int_0^t (\gamma^{\mathrm{T}}(X_i(s),V(s)) - \gamma^{\mathrm{T}}(\tilde{X}_i(s),\tilde{V}(s))\,\mathrm{d}B_i(s)\right|^2\right.$$

$$+\left|\int_{U\times[0,t]} (\beta(X_i(s),V(s),u) - \beta(\tilde{X}_i(s),\tilde{V}(s),u))W(\mathrm{d}u\,\mathrm{d}s)\right|^2$$

$$+K^2t(1+2K^2)\int_0^t (|X_i(s)-\tilde{X}_i(s)|^2+\rho(V^+(s),\tilde{V}^+(s))^2$$

$$+(\rho(V^-(s),\tilde{V}^-(s))^2)\,\mathrm{d}s).$$

By Fatou's lemma and Doob's inequality, we have

$$\begin{aligned} g_m(t) &\leqslant 10m^2 \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n \mathbb{E}\Bigg(4\int_0^t |\gamma(X_i(s),V(s))-\gamma(\tilde{X}_i(s),\tilde{V}(s))|^2 1_{s\leqslant\eta_m}\,\mathrm{d}s \\ &\quad +4\int_{U\times[0,t]} |\beta(X_i(s),V(s),u)-\beta(\tilde{X}_i(s),\tilde{V}(s),u)|^2\mu(\mathrm{d}u)1_{s\leqslant\eta_m}\,\mathrm{d}s \\ &\quad +K^2t(1+2K^2)\int_0^t (|X_i(s)-\tilde{X}_i(s)|^2+\rho(V^+(s),\tilde{V}^+(s))^2 \\ &\quad +(\rho(V^-(s),\tilde{V}^-(s))^2)1_{s\leqslant\eta_m}\,\mathrm{d}s\Bigg) \\ &\leqslant 10m^2\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n (8K^2+K^2t(1+2K^2)) \\ &\quad \times\int_0^t \mathbb{E}(|X_i(s)-\tilde{X}_i(s)|^2+\rho(V^+(s),\tilde{V}^+(s))^2+\rho(V^-(s),\tilde{V}^-(s))^2)1_{s\leqslant\eta_m}\,\mathrm{d}s \\ &\leqslant 10m^2(8K^2+K^2t(1+2K^2))\int_0^t (f_m(s)+2m^2f_m(s)+2g_m(s))\,\mathrm{d}s. \end{aligned} \tag{2.7}$$

Adding (2.6) and (2.7), for $t\leqslant T$, we have

$$f_m(t)+g_m(t)\leqslant K(m,T)\int_0^t (f_m(s)+g_m(s))\,\mathrm{d}s, \tag{2.8}$$

where $K(m,T)$ is a constant. By Gronwall's inequality, we have $f_m(t)+g_m(t)=0$. Then for each m and $t\in[0,T]$, we have $X_i(t\wedge\eta_m)=\tilde{X}_i(t\wedge\eta_m)$ a.s. and $A_i(t\wedge\eta_m)=\tilde{A}_i(t\wedge\eta_m)$ a.s. By (1.2), we have $V(t\wedge\eta_m)=\tilde{V}(t\wedge\eta_m)$ a.s. Hence $(X,A,V)=(\tilde{X},\tilde{A},\tilde{V})$ for $t\leqslant\eta_m\wedge T$. Taking T, $m\to\infty$, $(X,A,V)=(\tilde{X},\tilde{A},\tilde{V})$ for $t\leqslant\eta_\infty$. By the definition of η_m,

$$\begin{aligned} P(\eta_m\leqslant t) &\leqslant P\left(\sup_{0\leqslant s\leqslant t}\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n A_i(s)^2\geqslant m^2\right) \\ &\leqslant \frac{1}{m^2}\mathbb{E}\sup_{0\leqslant s\leqslant t}\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n A_i(s)^2 \\ &\leqslant \frac{1}{m^2}\liminf_{n\to\infty}\frac{1}{n}\sum_{i=1}^n \mathbb{E}\sup_{0\leqslant s\leqslant t} A_i(s)^2 \\ &= \frac{1}{m^2}\mathbb{E}\sup_{0\leqslant s\leqslant t}(A_1(s)^2), \end{aligned}$$

where the last inequality follows by moving the sup inside the sum and applying Fatou's lemma, and the last equality follows from the exchangeability. Hence, by Proposition 2.1, $P(\eta_\infty \leqslant t) = \lim_{m\to\infty} P\{\eta_m \leqslant t\} = 0$, i.e., $\eta_\infty = \infty$ a.s., and uniqueness follows. □

Finally, we establish the existence of a solution. We will need the following lemma from Kotelenez and Kurtz (1999).

Lemma 2.1. *For* $n = 1,2,\ldots$, *let* $X^n = (X_1^n,\ldots,X_{N_n}^n)$ *be exchangeable families of* $D_E[0,\infty)$*-valued random variables such that* $N_n \Rightarrow \infty$ *and* $X^n \Rightarrow X$ *in* $D_E[0,\infty)^\infty$. *Define* $\Xi_n = (1/N_n)\sum_{i=1}^{N_n} \delta_{X_i^n} \in \mathscr{P}(D_E[0,\infty)), \Xi = \lim_{m\to\infty}(1/m)\sum_{i=}^{m} \delta_{X_i}$, $Z_n(t) = (1/N_n)\sum_{i=1}^{N_n} \delta_{X_i^n(t)} \in \mathscr{P}(E)$, *and* $Z(t) = \lim_{m\to\infty}(1/m)\sum_{i=1}^{m} \delta_{X_i(t)}$, *and set* $D_\Xi = \{t : E[\Xi\{x : x(t) \neq x(t-)\}] > 0\}$. *Then the following hold*:

(a) *For* $t_1,\ldots,t_l \notin D_\Xi$,
$$(\Xi_n, Z_n(t_1),\ldots,Z_n(t_l)) \Rightarrow (\Xi, Z(t_1),\ldots,Z(t_l)).$$

(b) *If* $X^n \Rightarrow X$ *in* $D_{E^\infty}[0,\infty)$, *then* $(X_n,Z_n) \Rightarrow (X,Z)$ *in* $D_{E^\infty\times\mathscr{P}(E)}[0,\infty)$. *If* $X^n \to X$ *in probability in* $D_{E^\infty}[0,\infty)$, *then* $(X_n,Z_n) \to (X,Z)$ *in* $D_{E^\infty\times\mathscr{P}(E)}$ $[0,\infty)$ *in probability.*

Theorem 2.2. *Under Assumptions* (S1), (S2) *and* (2.3), *the system has a solution.*

Proof. Define $B_i^n(t)=B_i([nt]/n)$, $D_n(t)=([nt]/n)$, and $W^n(B\times[0,t])=W(B\times[0,[nt]/n])$, $\forall B \in B(U)$. Consider the discrete time, Euler-type approximation (X^n, A^n) obtained by replacing B_i by B_i^n, dt by d$D_n(t)$, and W by W^n in (1.3) and defining

$$A_i^n(t) = A_i(0)\exp\left\{\int_0^t \gamma^{\mathrm{T}}(X_i^n(s-), V^n(s-))\,\mathrm{d}B_i^n(s) + \int_{U\times[0,t]} \beta(X_i^n(s-), V^n(s-), u)W^n(\mathrm{d}u\times \mathrm{d}s) + \int_0^t D(X_i^n(s-), V^n(s-))\,\mathrm{d}D_n(s)\right\},$$

where

$$D(x,v) = d(x,v) - \frac{1}{2}|\gamma(x,v)|^2 - \frac{1}{2}\int_U \beta(x,v,u)^2\mu(\mathrm{d}u).$$

Note that the exchangeability of $\{(X_i(0), A_i(0))\}$ gives the existence of $V^n(t) = V^n(0)$ for $0 \leqslant t < 1/n$ and the exchangeability of $\{(X_i^n(1/n), A_i^n(1/n))\}$. The exchangeability of $\{(X_i^n(t), A_i^n(t))\}$ and the existence of $V^n(t)$ then follows recursively.

Let

$$C_1 = \sup_{x,v} |c(x,v)|$$

and

$$C_2 = \sup_{x,v}(|\sigma(x,v)\sigma(x,v)^{\mathrm{T}}| + \int_U |\alpha(x,v,u)|^2\mu(\mathrm{d}u)).$$

Then

$$\mathbb{E}[|X_i^n(t+h)-X_i^n(t)|^2|\mathscr{F}_t]\leqslant 2\left(C_1\left(\frac{[n(t+h)]-[nt]}{n}\right)^2+C_2\frac{[n(t+h)]-[nt]}{n}\right),$$

with a similar estimate holding for $\log|A_i^n|$. By Theorem 3.8.6 and Remark 3.8.7 of Ethier and Kurtz (1986), for each i $\{(X_i^n,A_i^n)\}$ is relatively compact for convergence in distribution in $D_{\mathbb{R}^d\times\mathbb{R}}[0,\infty)$. But relative compactness of $\{(X_i^n,A_i^n)\}$ in $D_{\mathbb{R}^d\times\mathbb{R}}[0,\infty)$ implies relative compactness of $\{(X^n,A^n)\}$ in $D_{\mathbb{R}^d\times\mathbb{R}}[0,\infty)^\infty$ (see, for example, Ethier and Kurtz, 1986, Proposition 3.2.4).

Taking a subsequence if necessary, we assume that $(X^n,A^n)\Rightarrow(X,A)$. By the continuity of B_i and W and the boundedness of the coefficients in (1.3) and (1.4), (X_i,A_i) will be continuous for each i, and it follows that the convergence is, in fact, in $D_{(\mathbb{R}^d\times\mathbb{R})^\infty}[0,\infty)$. Define

$$Z^n(t)=\lim_{m\to\infty}\frac{1}{m}\sum_{i=1}^m\delta_{(X_i^n(t),A_i^n(t))},\quad Z=\lim_{m\to\infty}\frac{1}{m}\sum_{i=1}^m\delta_{(X_i(t),A_i(t))}.$$

Then by Lemma 2.1, $Z^n\Rightarrow Z$, or more precisely, $(X^n,A^n,Z^n)\Rightarrow(X,A,Z)$.

For simplicity, assume that $A_i(0)\geqslant 0$ for all i. If, for $\alpha>0$, we define V_α^n by

$$\langle\varphi,V_\alpha^n\rangle=\int_{\mathbb{R}^{d+1}}(a\wedge\alpha)\varphi(x)Z^n(t,\mathrm{d}x\times\mathrm{d}a),$$

and observe that

$$\begin{aligned}||V^n(t)-V_\alpha^n(t)||&=\lim_{m\to\infty}\frac{1}{m}\sum_{i=1}^m(A_i^n(t)-\alpha\wedge A_i^n(t))\\&\leqslant\lim_{m\to\infty}\frac{1}{m}\sum_{i=1}^m\left(\sup_{s\leqslant T}A_i^n(s)-\alpha\wedge\sup_{s\leqslant T}A_i^n(s)\right).\end{aligned}$$

By the same argument as in the proof of Proposition 2.1, $\{\sup_{t\leqslant T}A_1^n(t)\}$ is bounded in L_2 and hence uniformly integrable, so

$$\mathbb{E}\sup_{t\leqslant T}||V^n(t)-V_\alpha^n(t)||\leqslant\mathbb{E}\left(\sup_{t\leqslant T}A_1(t)-\alpha\wedge\sup_{t\leqslant T}A_1(t)\right).\tag{2.9}$$

Since the right-hand side of (2.9) goes to zero as $\alpha\to\infty$, it follows that (X^n,A^n,V^n) is relatively compact, and as in Kurtz and Protter (1996, Proposition 7.4), any limit point will be a distributional solution of (1.3)–(1.4). But as in Yamada and Watanabe (1971), distributional existence and pathwise uniqueness imply strong existence. □

In the classical setting, the limiting empirical process is deterministic and characterized by a McKean–Vlasov equation. Here and, for example, in earlier work by Dawson and Vaillancourt, the limiting equation (1.6) is still stochastic.

The classical McKean–Vlasov limit (no W and no weights) is sometimes described by the equation

$$X(t)=X(0)+\int_0^t\sigma(X(s),Z(s))\,\mathrm{d}B(s)+\int_0^t c(X(s),Z(s))\,\mathrm{d}s,$$

where $Z(t)$ is required to be the distribution of $X(t)$. The analogous formulation in our setting is to consider the system

$$X(t)=X(0)+\int_0^t \sigma(X(s),V(s))\,\mathrm{d}B(s)+\int_0^t c(X(s),V(s))\,\mathrm{d}s$$
$$+\int_{U\times[0,t]} \alpha(X(s),V(s),u)W(\mathrm{d}u\,\mathrm{d}s) \tag{2.10}$$

and

$$A(t)=A(0)+\int_0^t A(s)\gamma^{\mathrm{T}}(X(s),V(s))\,\mathrm{d}B(s)+\int_0^t A(s)\,\mathrm{d}(X(s),V(s))\,\mathrm{d}s$$
$$+\int_{U\times[0,t]} A(s)\beta(X(s),V(s),u)W(\mathrm{d}u\,\mathrm{d}s), \tag{2.11}$$

where, as we will see below, $V(t)$ is the random measure determined by

$$\langle\phi,V(t)\rangle=\mathbb{E}(A(t)\phi(X(t))\,|\,\mathscr{F}_t^W), \tag{2.12}$$

$\{\mathscr{F}_t^W\}$ being the filtration generated by W. (Similar representations have been used by other authors. See, for example, Sowers, 1995.) We require (X,A) to be compatible with (B,W) in the sense that for each time $t\geqslant 0$, the increments of B and W after time t are independent of $\mathscr{F}_t^{X,A,B,W}$. Note that this independence implies

$$\langle\phi,V(t)\rangle=\mathbb{E}(A(t)\phi(X(t))|W). \tag{2.13}$$

As a characterization of V, this system is essentially equivalent to the particle system.

Theorem 2.3. *Let* (X,A,V,B,W) *satisfy* (2.10)–(2.12). *Then there exists a solution*

$$(\{X_i\},\{A_i\},\{B_i\},\tilde{V},\tilde{W})$$

of (1.2)–(1.4) *such that* $(X_1,A_1,\tilde{V},B_1,\tilde{W})$ *has the same distribution as* (X,A,V,B,W). *Conversely, if there exists a pathwise unique solution* $(\{X_i\},\{A_i\},\{B_i\},V,W)$ *of* (1.2)–(1.4), *then* (X_1,A_1,V,B_1,W) *is a solution of* (2.10)–(2.12).

Proof. Since we are not assuming uniqueness, (X,A) may not be uniquely determined by $(X(0),A(0),B,W)$; however, if we let (X,A,V,B,W) be a particular solution of (2.10)–(2.12), then (X,A) will have a regular conditional distribution given $(X(0),A(0),B,W)$. In particular, there will exist a transition function $q(x_0,a_0,b,w,\Gamma)$ such that $P\{(X,A)\in\Gamma|X(0),A(0),B,W\}=q(X(0),A(0),B,W,\Gamma)$, $\Gamma\in\mathscr{B}(D_{\mathbb{R}^d\times\mathbb{R}}[0,\infty))$. Since every probability measure on a complete, separable metric space can be induced by a mapping from the probability space given by Lebesgue measure on $[0,1]$, it follows that there will be a mapping F such that if U is uniformly distributed on the interval $[0,1]$ and $(\tilde{X}(0),\tilde{A}(0),\tilde{B},\tilde{W})$ is independent of U and has the same distribution as $(X(0),A(0),B,W)$, then $(\tilde{X},\tilde{A})=F(\tilde{X}(0),\tilde{A}(0),\tilde{B},\tilde{W},U)$ and $(\tilde{X}(0),\tilde{A}(0),\tilde{B},\tilde{W})$ have the same joint distribution as (X,A) and $(X(0),A(0),B,W)$. Defining $\tilde{V}$ by

$$\langle\phi,\tilde{V}(t)\rangle=\mathbb{E}(\tilde{A}(t)\phi(\tilde{X}(t))|\mathscr{F}_t^{\tilde{W}}), \tag{2.15}$$

$(\tilde{X},\tilde{A},\tilde{V},\tilde{B},\tilde{W})$ will have the same distribution as (X,A,V,B,W). Let W be Gaussian white noise, $\{B_i\}$ be independent standard Brownian motions, $\{(X_i(0),A_i(0))\}$ be iid with the same distribution as $(X(0), A(0))$, and $\{U_i\}$ be independent uniform-$[0,1]$ random variables. Define

$$(X_i, A_i) = F(X_i(0), A_i(0), B_i, W, U_i).$$

Note that V determined by

$$\langle \phi, V(t)\rangle = \mathbb{E}(A_i(t)\phi(X_i(t)) \,|\, \mathscr{F}_t^W),$$

does not depend on i and that $(\{X_i\}, \{A_i\}, V, \{B_i\}, W)$ satisfies (1.3) and (1.4). It remains only to show that V satisfies (1.2).

Note that $\{(X_i, A_i)\}$ is exchangeable so that

$$\langle \phi, \tilde{V}(t)\rangle = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} A_i(t)\phi(X_i(t)) = E(A_1(t)\phi(X_1(t))|\mathscr{I})$$

exists. The second equality holds by the ergodic theorem, and $\mathscr{I}$ is the invariant σ-algebra for the stationary sequence $\{(X_i(0), A_i(0), B_i, U_i, W)\}$. But the independence of $\{(X_i(0), A_i(0), B_i, U_i)\}$ implies $\mathscr{I}$ is contained in the completion of the σ-algebra generated by W. Consequently,

$$\langle \phi, \tilde{V}(t)\rangle = E(A_1(t)\phi(X_1(t))|W) = E(A_1(t)\phi(X_1(t)) \,|\, \mathscr{F}_t^W),$$

where the second equality follows by (2.13), and hence $V(t) = \tilde{V}(t)$.

To obtain the converse, note that pathwise uniqueness implies that the invariant σ-algebra for $\{(X_i, A_i, B_i, W)\}$ is contained in the completion of $\sigma(W)$. Pathwise uniqueness also implies that the solution $\{(X_i, A_i)\}$ is compatible with the $\{B_i\}$ and W, so we have

$$\langle \phi, V(t)\rangle = E(A_1(t)\phi(X_1(t)) \,|\, W) = E(A_1(t)\phi(X_1(t)) \,|\, \mathscr{F}_t^W). \qquad \square$$

3. A nonlinear SPDE

In this section, we establish the existence and uniqueness for the solution to the SPDE (1.6). Our approach is motivated by the second author's uniqueness proof of a nonlinear PDE for the empirical measure (on a conuclear space) of a system of interacting neurons (cf. Xiong, 1999).

We summarize the techniques used in this section. First, by applying Itô's formula, it is shown that V is a solution to (1.6). To prove uniqueness for the solution to (1.6), we assume the existence of another solution V_1 and freeze the nonlinear arguments in (1.6) by V_1 (cf. (3.13) and (3.1)) to obtain a linear SPDE. Similar to the argument in Xiong (1999), the uniqueness for the solution to (1.6) is implied by that of the linear SPDE (3.13) and that of the system (1.2)–(1.4) proved in the previous section (cf. the proof of Theorem 3.5 for this argument).

We actually only prove uniqueness among solutions U such that for each $t \geqslant 0$, $U(t)$ is absolutely continuous with respect to Lebesgue measure and has a density in $L^2(\mathbb{R}^d)$. (We also prove existence of such a solution for all $U(0)$ with this property.) The necessary estimates are obtained by first smoothing the solutions with a Gaussian kernel. As a by product, the estimates (cf. Theorem 3.2) give the existence of a density $v(t,x)$ for the solution to (1.6) under the assumption that $V(0)$ has a density in $L^2(\mathbb{R}^d)$.

Theorem 3.1. *Let V be the weighted empirical measure for the particle system given by Theorems* 2.1 *and* 2.2. *Then V is a solution of* (1.6).

Proof. It is easy to see that

$$\mathbb{E}\sup_{t\leqslant T}\left|\frac{1}{n}\sum_{i=1}^{n}\int_0^t A_i(s)\nabla^{\mathrm{T}}\phi(X_i(s))\sigma(X_i(s),V(s))\,\mathrm{d}B_i(s)\right|^2$$

$$\leqslant 4\frac{1}{n^2}\sum_{i=1}^{n}\mathbb{E}\int_0^{\mathrm{T}} A_i(s)^2|\nabla^{\mathrm{T}}\phi(X_i(s))\sigma(X_i(s),V(s))|^2\,\mathrm{d}s$$

$$\leqslant \frac{4}{n}||\nabla\phi||_\infty^2 K^2 T\,\mathbb{E}\sup_{s\leqslant T}A_1(s)^2 \to 0.$$

By (1.5), it is then easy to prove that V is a solution of (1.6). □

Now we fix an $\mathscr{M}(\mathbb{R}^d)$-valued process V and consider the linear equation

$$\langle\phi,U(t)\rangle = \langle\phi,U(0)\rangle + \int_0^t \langle d_s\phi + L_s\phi, U(s)\rangle\,\mathrm{d}s$$

$$+\int_{U\times[0,t]}\langle\beta_s(\cdot,u)\phi + \alpha_s^{\mathrm{T}}(\cdot,u)\nabla\phi, U(s)\rangle W(\mathrm{d}u\,\mathrm{d}s), \tag{3.1}$$

where

$$L_s\phi = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij,s}(x)\partial_{x_i}\partial_{x_j}\phi(x) + \sum_{i=1}^{d} b_{i,s}(x)\partial_{x_i}\phi(x),$$

$$a_{ij,s}(x) = a_{ij,s}^{(1)}(x) + a_{ij,s}^{(2)}(x),$$

$$a_{ij,s}^{(1)}(x) = \sum_{k=1}^{d}\sigma_{ik}(x,V(s))\sigma_{jk}(x,V(s)),$$

$$a_{ij,s}^{(2)}(x) = \int_U \alpha_i(x,V(s),u)\alpha_j(x,V(s),u)\mu(\mathrm{d}u)$$

and

$$b_{i,s}(x) = b_i(x,V(s)), \qquad d_s(x) = d(x,V(s)),$$

$$\alpha_s(x,u) = \alpha(x,V(s),u), \qquad \beta_s(x,u) = \beta(x,V(s),u).$$

Let $H_0 = L^2(\mathbb{R}^d)$ be the Hilbert space with the usual L^2-norm $||\cdot||$ and inner product $\langle\cdot,\cdot\rangle$ given by $||\phi||_0^2 = \int_{\mathbb{R}^d}|\phi(x)|^2\,\mathrm{d}x$ and $\langle\phi,\psi\rangle_0 = \int_{\mathbb{R}^d}\phi(x)\psi(x)\,\mathrm{d}x$. To obtain good estimates and to derive uniqueness for the solution to (3.1), we transform an $\mathscr{M}(\mathbb{R}^d)$-valued process to a H_0-valued process. A similar idea was employed by Kotelenez (1995).

For any $v\in\mathscr{M}(\mathbb{R}^d)$ and $\delta>0$, let

$$(T_\delta v)(x) = \int_{\mathbb{R}^d} G_\delta(x-y)v(\mathrm{d}y), \tag{3.2}$$

where G_δ is the heat kernel given by $G_\delta(x) = (2\pi\delta)^{-d/2}\exp\big(-|x|^2/2\delta\big)$. We use the same notation for the Brownian motion semigroup on $C_b(\mathbb{R}^d)$, i.e., $T_t\phi(x) = \int_{\mathbb{R}^d} G_t(x-y)\phi(y)\,\mathrm{d}y$, $\forall\phi\in C_b(\mathbb{R}^d)$.

The following facts can be verified easily.

Lemma 3.1. (i) *If* $\phi\in H_0\equiv L^2(\mathbb{R}^d)$ *and* $\delta>0$, *then* $||T_\delta\phi||_0\leqslant||\phi||_0$.

(ii) *If* $v\in\mathscr{M}(\mathbb{R}^d)$ *and* $\delta>0$, *then* $T_\delta v\in H_0$.

(iii) *If* $v\in\mathscr{M}(\mathbb{R}^d)$ *and* $\delta>0$, *then* $||T_{2\delta}|v|||_0\leqslant||T_\delta\,|\,v|||_0$, *where* $|v|$ *is the total variation measure of* v.

Let $Z_\delta(s)=T_\delta U(s)$ where U is an $\mathscr{M}(\mathbb{R}^d)$-valued solution to (3.1). To obtain an estimate for the H_0-norm of the process Z_δ, we adapt Kotelenez's arguments to the present setup. Replacing ϕ by $T_\delta\phi$ in (3.1) and using the fact that differentiation and T_δ commute, we have

$$\begin{aligned}
\langle Z_\delta(t),\phi\rangle_0 &= \langle T_\delta\phi, U(t)\rangle\\
&= \langle T_\delta\phi, U(0)\rangle + \int_0^t \langle T_\delta\phi, d_sU(s)\rangle\,\mathrm{d}s\\
&\quad + \frac{1}{2}\sum_{i,j=1}^d \int_0^t \langle a_{ij,s}\partial_{x_i}\partial_{x_j}T_\delta\phi, U(s)\rangle\,\mathrm{d}s + \sum_{i=1}^d\int_0^t\langle b_{i,s}\partial_{x_i}T_\delta\phi, U(s)\rangle\,\mathrm{d}s\\
&\quad + \int_{U\times[0,t]}\left\langle \sum_{i=1}^d \alpha_{i,s}(\cdot,u)\partial_{x_i}T_\delta\phi + \beta_s(\cdot,u)T_\delta\phi, U(s)\right\rangle W(\mathrm{d}u\,\mathrm{d}s)\\
&= \langle Z_\delta(0),\phi\rangle_0 + \int_0^t\langle T_\delta(d_sU(s)),\phi\rangle_0\,\mathrm{d}s\\
&\quad + \frac{1}{2}\sum_{i,j=1}^d\int_0^t\langle\partial_{x_i}\partial_{x_j}T_\delta(a_{ij,s}U(s)),\phi\rangle_0\,\mathrm{d}s\\
&\quad - \sum_{i=1}^d\int_0^t\langle\partial_{x_i}T_\delta(b_{i,s}U(s)),\phi\rangle_0\,\mathrm{d}s\\
&\quad + \int_{U\times[0,t]}\left\langle -\sum_{i=1}^d\partial_{x_i}T_\delta(\alpha_{i,s}(\cdot,u)U(s)) + T_\delta(\beta_s(\cdot,u)U(s)),\phi\right\rangle_0 W(\mathrm{d}u\,\mathrm{d}s).
\end{aligned}$$

By Itô's formula, we have

$$\begin{aligned}
&\langle Z_\delta(t),\phi\rangle_0^2\\
&\quad=\langle Z_\delta(0),\phi\rangle_0^2 + \int_0^t 2\langle Z_\delta(s),\phi\rangle_0\langle T_\delta(d_sU(s)),\phi\rangle_0\,\mathrm{d}s
\end{aligned}$$

$$+\sum_{i,j=1}^{d}\int_0^t \langle Z_\delta(s),\phi\rangle_0 \langle \partial_{x_i}\partial_{x_j}T_\delta(a_{ij,s}U(s)),\phi\rangle_0\,\mathrm{d}s$$

$$-\sum_{i=1}^{d}\int_0^t 2\langle Z_\delta(s),\phi\rangle_0 \langle \partial_{x_i}T_\delta(b_{i,s}U(s)),\phi\rangle_0\,\mathrm{d}s$$

$$+\int_{U\times[0,t]} 2\langle Z_\delta(s),\phi\rangle_0$$

$$\times\left\langle -\sum_{i=1}^{d}\partial_{x_i}T_\delta(\alpha_{i,s}(\cdot,u)U(s))+T_\delta(\beta_s(\cdot,u)U(s)),\phi\right\rangle_0 W(\mathrm{d}u\,\mathrm{d}s)$$

$$+\int_{U\times[0,t]}\left|\left\langle -\sum_{i=1}^{d}\partial_{x_i}T_\delta(\alpha_{i,s}(\cdot,u)U(s))+T_\delta(\beta_s(\cdot,u)U(s)),\phi\right\rangle_0\right|^2 \mu(\mathrm{d}u)\,\mathrm{d}s.$$

Summing over ϕ in a complete, orthonormal basis of H_0 and taking expectations, we have

$$\begin{aligned}
&\mathbb{E}||Z_\delta(t)||_0^2\\
&=||Z_\delta(0)||_0^2+\mathbb{E}\int_0^t 2\langle Z_\delta(s),T_\delta(d_sU(s))\rangle_0\,\mathrm{d}s\\
&\quad+\sum_{i,j=1}^{d}\mathbb{E}\int_0^t\langle Z_\delta(s),\partial_{x_i}\partial_{x_j}T_\delta(a^{(1)}_{ij,s}U(s))\rangle_0\,\mathrm{d}s\\
&\quad+\sum_{i,j=1}^{d}\mathbb{E}\int_0^t\langle Z_\delta(s),\partial_{x_i}\partial_{x_j}T_\delta(a^{(2)}_{ij,s}U(s))\rangle_0\,\mathrm{d}s\\
&\quad-\sum_{i=1}^{d}\mathbb{E}\int_0^t 2\langle Z_\delta(s),\partial_{x_i}T_\delta(b_{i,s}U(s))\rangle_0\,\mathrm{d}s\\
&\quad+\mathbb{E}\int_{U\times[0,t]}\left|\left|\sum_{i=1}^{d}\partial_{x_i}T_\delta(\alpha_{i,s}(\cdot,u)U(s))\right|\right|_0^2\mu(\mathrm{d}u)\,\mathrm{d}s\\
&\quad-2\,\mathbb{E}\int_{U\times[0,t]}\left\langle\sum_{i=1}^{d}\partial_{x_i}T_\delta(\alpha_{i,s}(\cdot,u)U(s)),T_\delta(\beta_s(\cdot,u)U(s))\right\rangle_0\mu(\mathrm{d}u)\,\mathrm{d}s\\
&\quad+\mathbb{E}\int_{U\times[0,t]}||T_\delta(\beta_s(\cdot,u)U(s))||_0^2\mu(\mathrm{d}u)\,\mathrm{d}s. \qquad (3.3)
\end{aligned}$$

We will show that the integral terms on the right-hand side of (3.3) are bounded by a constant times the integral of $||T_\delta(|U(s)|)||_0$.

Lemma 3.2. *Let $(H,\mathscr{H},\eta)$ be a measure space and $\mathbb{H}=L_2(\eta)$. (We are interested in the cases H a singleton and $H=U$ with $\eta=\mu$.) Let $f_i:\mathbb{R}^d\to\mathbb{H}$, $i=1,2$ satisfy*

$$||f_i(x)-f_i(y)||_{\mathbb{H}}\leqslant K|x-y|,\quad \forall x,y\in\mathbb{R}^d,$$

$$||f_i(x)||_{\mathbb{H}}\leqslant K,\quad \forall x\in\mathbb{R}^d,$$

$g,\partial_{x_i}g\in H_0$, $i=1,\dots,d$, and let $\zeta\in\mathscr{M}(\mathbb{R}^d)$. Then there exist constants K_1, and K_2 depending on f_1 and f_2 but not on ζ such that

$$||\langle g,f_1\partial_{x_i}g\rangle_0||_{\mathbb{H}}\leqslant \tfrac{1}{2}K||g||_0^2, \tag{3.4}$$

$$||||T_\delta(f_1\zeta)||_0||_{\mathbb{H}}\leqslant ||T_\delta(||f_1(\cdot)||_H|\zeta|)||_0\leqslant K||T_\delta(|\zeta|)||_0, \tag{3.5}$$

$$||||f_1\partial_{x_i}T_\delta(\zeta)-\partial_{x_i}T_\delta(f_1\zeta)||_0||_{\mathbb{H}}\leqslant K_1||T_{2\delta}(|\zeta|)||_0, \tag{3.6}$$

$$|\langle T_\delta(f_2\zeta),\partial_{x_i}T_\delta(f_1\zeta)\rangle_{H_0\otimes\mathbb{H}}|\leqslant K_2||T_\delta(|\zeta|)||_0. \tag{3.7}$$

Proof. To prove (3.4), first assume that f_1 and g are continuously differentiable with compact support. Then integrating by parts we have

$$\langle g,f_1(\cdot,u)\partial_{x_i}g\rangle_0=\frac{1}{2}\int_{\mathbb{R}^d}f_1(x,u)\partial_{x_i}(g^2(x))\,\mathrm{d}x=-\frac{1}{2}\int_{\mathbb{R}^d}g^2(x)\partial_{x_i}f_1(x,u)\,\mathrm{d}x,$$

and hence

$$\int_H|\langle g,f_1(\cdot,u)\partial_{x_i}g\rangle_0|^2\eta(\mathrm{d}u)\leqslant\frac{1}{4}\int_H\left|\int_{\mathbb{R}^d}g^2(x)\partial_{x_i}f_1(x,u)\,\mathrm{d}x\right|^2\eta(\mathrm{d}u)\leqslant\frac{1}{4}K^2||g||_0^4.$$

The general result follows by approximation.

Writing

$$\begin{aligned}||||T_\delta(f_1\zeta)||_0||_{\mathbb{H}}^2&=\int_H\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}G_\delta(x-y)f_1(y,u)\zeta(\mathrm{d}y)\right|^2\mathrm{d}x\,\eta(\mathrm{d}u)\\&\leqslant\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}G_\delta(x-y)G_\delta(x-\tilde{y})\\&\quad\times\int_H|f_1(y,u)f_1(\tilde{y},u)|\eta(\mathrm{d}u)|\zeta|(\mathrm{d}y)|\zeta|(\mathrm{d}\tilde{y})\,\mathrm{d}x\\&\leqslant\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}G_\delta(x-y)||f_1(y)||_H|\zeta|(\mathrm{d}y)\right)^2\mathrm{d}x,\end{aligned}$$

(3.5) follows.

Noting that

$$\begin{aligned}&||f_1(x,\cdot)\partial_{x_i}T_\delta(\zeta)(x)-\partial_{x_i}T_\delta(f_1\zeta)(x)||_{\mathbb{H}}\\&\quad=\left|\left|\int_{\mathbb{R}^d}(f_1(x,\cdot)-f_1(y,\cdot))\partial_{x_i}G_\delta(x-y)\zeta(\mathrm{d}y)\right|\right|_{\mathbb{H}}\\&\quad\leqslant\int_{\mathbb{R}^d}K|x-y|\frac{|x_i-y_i|}{\delta}G_\delta(x-y)|\zeta|(\mathrm{d}y)\end{aligned}$$

$$\leqslant K\int_{\mathbb{R}^d}\frac{|x-y|^2}{\delta}\exp\left(-\frac{|x-y|^2}{4\delta}\right)2^{d/2}G_{2\delta}(x-y)|\zeta|(\mathrm{d}y)$$

$$\leqslant 2^{d/2+2}KT_{2\delta}(|\zeta|)(x).$$

Taking the H_0-norm of both sides gives (3.6) with $K_1=2^{d/2+2}K$. (Note that $||\,||h||_{\mathbb{H}}||_0=||\,||h||_0||_{\mathbb{H}}$.)

By (3.4) and (3.6),

$$|\langle T_\delta(f_2\zeta),\partial_{x_i}T_\delta(f_1\zeta)\rangle_{H_0\otimes\mathbb{H}}|$$

$$\leqslant|\langle T_\delta(f_2\zeta),f_1\partial_{x_i}T_\delta(\zeta)\rangle_{H_0\otimes\mathbb{H}}|+|\langle T_\delta(f_2\zeta),\partial_{x_i}T_\delta(f_1\zeta)-f_1\partial_{x_i}T_\delta(\zeta)\rangle_{H_0\otimes\mathbb{H}}|$$

$$\leqslant|\langle T_\delta(f_2\zeta),\partial_{x_i}f_1T_\delta(\zeta)\rangle_{H_0\otimes\mathbb{H}}|+|\langle f_2\partial_{x_i}T_\delta(\zeta),f_1T_\delta(\zeta)\rangle_{H_0\otimes\mathbb{H}}|$$

$$+|\langle(\partial_{x_i}T_\delta(f_2\zeta)-f_2\partial_{x_i}T_\delta(\zeta)),f_1T_\delta(\zeta)\rangle_{H_0\otimes\mathbb{H}}|+K_1||T_\delta(f_2\zeta)||_{H_0\otimes\mathbb{H}}||T_{2\delta}(|\zeta|)||_0$$

$$\leqslant K||\,||T_\delta(f_2\zeta)||_0||_{\mathbb{H}}||T_\delta(\zeta)||_0+K^2||T_\delta(|\zeta|)||_0^2+KK_1||T_\delta(|\zeta|)||_0^2+K_1K||T_\delta(|\zeta|)||_0^2$$

$$\leqslant K_3||T_\delta(|\zeta|)||_0^2,$$

where the next to the last inequality follows by Lemma 3.1(iii) and the previous inequalities. □

Lemma 3.3. *For $i=1,\dots,d$, let $\alpha_i:\mathbb{R}^d\to\mathbb{H}=L_2(\eta)$ satisfy $||\alpha_i(x)||_{\mathbb{H}}\leqslant K$ and $||\alpha_i(x)-\alpha_i(y)||_{\mathbb{H}}\leqslant K|x-y|$. Define $a_{ij}(x)=\int_H\alpha_i(x,u)\alpha_j(x,u)\eta(\mathrm{d}u)$. Then for $\zeta\in\mathscr{M}(\mathbb{R}^d)$*

$$\sum_{i,j=1}^d\langle T_\delta\zeta,\partial_{x_i}\partial_{x_j}T_\delta(a_{ij}\zeta)\rangle_0+\int_U\left|\left|\sum_{i=1}^d\partial_{x_i}T_\delta(\alpha_i(\cdot,u)\zeta)\right|\right|_0^2\eta(\mathrm{d}u)\leqslant K_4||T_\delta(|\zeta|)||_0^2. \quad (3.8)$$

Proof. Note that

$$\sum_{i,j=1}^d\langle T_\delta\zeta,\partial_{x_i}\partial_{x_j}T_\delta(a_{ij}\zeta)\rangle_0$$

$$=\sum_{i,j=1}^d\int_{\mathbb{R}^d}\mathrm{d}x\int_{\mathbb{R}^d}G_\delta(x-y)\zeta(\mathrm{d}y)\int_{\mathbb{R}^d}\partial_{x_i}\partial_{x_j}G_\delta(x-z)a_{ij}(z)\zeta(\mathrm{d}z)$$

$$=\sum_{i,j=1}^d\int_{\mathbb{R}^d}\zeta(\mathrm{d}y)\int_{\mathbb{R}^d}a_{ij}(z)\zeta(\mathrm{d}z)\partial_{z_i}\partial_{z_j}\int_{\mathbb{R}^d}G_\delta(x-y)G_\delta(x-z)\,\mathrm{d}x$$

$$=\sum_{i,j=1}^d\int_{\mathbb{R}^d}\zeta(\mathrm{d}y)\int_{\mathbb{R}^d}\zeta(\mathrm{d}z)\left(\frac{(z_i-y_i)(z_j-y_j)}{4\delta^2}-\frac{1_{i=j}}{2\delta}\right)G_{2\delta}(z-y)a_{ij}(z)$$

$$=\sum_{i,j=1}^d\int_{\mathbb{R}^d}\zeta(\mathrm{d}y)\int_{\mathbb{R}^d}\zeta(dz)\left(\frac{(z_i-y_i)(z_j-y_j)}{4\delta^2}-\frac{1_{i=j}}{2\delta}\right)$$

$$\times G_{2\delta}(z-y)\tfrac{1}{2}(a_{ij}(z)+a_{ij}(y)),$$

where the last equality follows from the symmetry in y, z. Similarly,

$$\int_H \left|\left|\sum_{i=1}^d \partial_{x_i} T_\delta(\alpha_i(\cdot,u)\zeta)\right|\right|_0^2 \eta(\mathrm{d}u)$$

$$= -\int_H \eta(\mathrm{d}u) \sum_{i,j=1}^d \langle T_\delta(\alpha_i(\cdot,u)\zeta), \partial_{x_i}\partial_{x_j} T_\delta(\alpha_j(\cdot,u)\zeta)\rangle_0$$

$$= -\sum_{i,j=1}^d \int_H \eta(\mathrm{d}u) \int_{\mathbb{R}^d} \zeta(\mathrm{d}y) \int_{\mathbb{R}^d} \zeta(dz) \left(\frac{(z_i-y_i)(z_j-y_j)}{4\delta^2} - \frac{1_{i=j}}{2\delta}\right) G_{2\delta}(z-y)$$

$$\times \frac{1}{2}(\alpha_i(y,u)\alpha_j(z,u) + \alpha_i(z,u)\alpha_j(y,u)).$$

Hence

$$\text{LHS of } (3.8) = \sum_{i,j=1}^d \int_{\mathbb{R}^d} \zeta(\mathrm{d}y) \int_{\mathbb{R}^d} \zeta(\mathrm{d}z) \left(\frac{(z_i-y_i)(z_j-y_j)}{4\delta^2} - \frac{1_{i=j}}{2\delta}\right) G_{2\delta}(z-y)$$

$$\times \frac{1}{2} \int_H (\alpha_i(y,u)-\alpha_i(z,u))(\alpha_j(y,u)-\alpha_j(z,u))\eta(\mathrm{d}u)$$

$$\leqslant \frac{1}{2}\sum_{i,j=1}^d \int_{\mathbb{R}^d} |\zeta|(\mathrm{d}y) \int_{\mathbb{R}^d} |\zeta|(\mathrm{d}z) \left(\frac{|z-y|^2}{4\delta^2} + \frac{1}{2\delta}\right) \exp\left(-\frac{|z-y|^2}{4\delta}\right)$$

$$\times 2^{d/2} G_{4\delta}(z-y) K^2 |z-y|^2$$

$$\leqslant 16 d^2 K^2 \sum_{i,j=1}^d \int_{\mathbb{R}^d} |\zeta|(\mathrm{d}y) \int_{\mathbb{R}^d} |\zeta|(\mathrm{d}z) 2^{d/2} G_{4\delta}(z-y)$$

$$= 2^{d/2+4} d^2 K^2 ||T_{2\delta}(|\zeta|)||_0^2$$

$$\leqslant 2^{d/2+4} d^2 K^2 ||T_\delta(|\zeta|)||_0^2,$$

where the second inequality follows by bounding $(v^2+v)\mathrm{e}^{-v/4}$. The lemma follows with $K_4 = 2^{d/2+4} d^2 K^2$. □

The estimates in Lemmas 3.2 and 3.3 give the following.

Theorem 3.2. *If U is an $\mathscr{M}(\mathbb{R}^d)$-valued solution of* (3.1) *and $Z_\delta = T_\delta U$, then*

$$\mathbb{E}||Z_\delta(t)||_0^2 \leqslant ||Z_\delta(0)||_0^2 + K_6 \int_0^t \mathbb{E}||T_\delta(|U(s)|)||_0^2 \, \mathrm{d}s, \tag{3.9}$$

where K_6 is a constant.

Proof. The second and last terms of (3.3) are bounded by a constant times $||T_\delta(|U(s)|)||_0^2$ by (3.5). The bound for the third term follows from Lemma 3.3. (Take H to be a singleton.) The bound for the sum of the fourth and sixth terms also follows from Lemma 3.3. (Take $\mathbb{H}=L_2(\mu)$.) The bound for the fifth and seventh terms follows by (3.7). □

Corollary 3.1. *If U is an $\mathcal{M}_+(\mathbb{R}^d)$-valued solution of* (3.1) *and $U(0)\in H_0$, then $U(t)\in H_0$ a.s. and $\mathbb{E}\|U(t)\|_0^2<\infty$, $\forall t\geqslant 0$.*

Proof. It follows from (3.9) that

$$\mathbb{E}\|Z_\delta(t)\|_0^2\leqslant\|Z_\delta(0)\|_0^2+K_6\int_0^t\mathbb{E}\|Z_\delta(s)\|_0^2\,\mathrm{d}s.$$

By Gronwall's inequality, we have

$$\mathbb{E}\|Z_\delta(t)\|_0^2\leqslant\|Z_\delta(0)\|_0^2\mathrm{e}^{K_6t}.$$

Let $\{\phi_j\}$ be a complete, orthonormal system of H_0 such that $\phi_j\in C_b(\mathbb{R}^d)$. Then, by Fatou's lemma,

$$\mathbb{E}\left[\sum_j\langle\phi_j,U(t)\rangle^2\right]=\mathbb{E}\left[\sum_j\lim_{\delta\to0}\langle\phi_j,U(t)\rangle^2\right]\leqslant\liminf_{\delta\to0}\mathbb{E}\|Z_\delta(t)\|_0^2\leqslant\|U(0)\|_0^2\mathrm{e}^{K_6t}.$$

Hence $U(t)\in H_0$ and $\mathbb{E}\|U(t)\|_0^2<\infty$. □

These estimates give uniqueness of $\mathcal{M}_+(\mathbb{R}^d)$-valued solutions with $U(0)\in H_0$.

Theorem 3.3. *Suppose that $U(0)\in H_0$, $U(0)\geqslant 0$. Then* (3.1) *has at most one $\mathcal{M}_+(\mathbb{R}^d)$-valued solution.*

Proof. Let $U_1(t)$ and $U_2(t)$ be two $\mathcal{M}_+(\mathbb{R}^d)$-valued solutions with the same initial value $U(0)$. By Corollary 3.1, $U_1(t)$, $U_2(t)\in H_0$ a.s. Let $U(t)=U_1(t)-U_2(t)$. Then $U(t)\in H_0$ and

$$\mathbb{E}\|T_\delta U(t)\|_0^2\leqslant K_6\int_0^t\mathbb{E}\|T_\delta(|U(s)|)\|_0^2\,\mathrm{d}s.$$

As before, taking $\delta\to0$, we have

$$\mathbb{E}\|U(t)\|_0^2\leqslant K_6\int_0^t\mathbb{E}\||U(s)|\|_0^2\,\mathrm{d}s=K_6\int_0^t\mathbb{E}\|U(s)\|_0^2\,\mathrm{d}s,$$

and by Gronwall's inequality, we have $U(t)\equiv 0$. □

By exactly the same argument we have the following theorem.

Theorem 3.4. *Suppose that $U(0)\in H_0$. Then* (3.1) *has at most one H_0-valued solution.*

Proof. Let $U_1(t)$ and $U_2(t)$ be two H_0-valued solutions with the same initial value $U(0)$. Let $U(t)=U_1(t)-U_2(t)$. Then $U(t)\in H_0$ and

$$\mathbb{E}\|T_\delta U(t)\|_0^2\leqslant K_6\int_0^t\mathbb{E}\|T_\delta(|U(s)|)\|_0^2\,\mathrm{d}s.$$

Taking $\delta \to 0$, we have

$$\mathbb{E}||U(t)||_0^2 \leqslant K_6 \int_0^t \mathbb{E}|||U(s)|||_0^2 \, \mathrm{d}s = K_6 \int_0^t \mathbb{E}||U(s)||_0^2 \, \mathrm{d}s.$$

By Gronwall's inequality, we have $U(t) \equiv 0$. □

Finally, we consider the uniqueness of the solution of the nonlinear SPDE (1.6).

Theorem 3.5. *Suppose that $V(0) \in H_0$, then there exists a unique H_0-valued solution of* (1.6).

Proof. Let V be the solution of (1.6) given by Theorem 3.1. Then by Corollary 3.1, V^+ and V^- (and hence V) have values in H_0.

Let $V_1(t)$ be another H_0-valued solution of (1.6). Consider the system

$$X_i(t) = X_i(0) + \int_0^t \sigma(X_i(s), V_1(s)) \, \mathrm{d}B_i(s) + \int_0^t c(X_i(s), V_1(s)) \, \mathrm{d}s + \int_{U \times [0,t]} \alpha(X_i(s), V_1(s), u) W(\mathrm{d}u \, \mathrm{d}s) \tag{3.10}$$

and

$$A_i(t) = A_i(0) + \int_0^t A_i(s) \gamma^{\mathrm{T}}(X_i(s), V_1(s)) \, \mathrm{d}B_i(s) + \int_0^t A_i(s) d(X_i(s), V_1(s)) \, \mathrm{d}s + \int_{U \times [0,t]} A_i(s) \beta(X_i(s), V_1(s), u) W(\mathrm{d}u \, \mathrm{d}s). \tag{3.11}$$

Let $V_2^{\pm}(t)$ be given by

$$V_2^{\pm}(t) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{\infty} A_i^{\pm}(t) \delta_{X_i(t)}. \tag{3.12}$$

As in Theorem 3.1, V_2^+ and V_2^- are solutions of

$$\langle \phi, U(t) \rangle = \langle \phi, U(0) \rangle + \int_0^t \langle d(\cdot, V_1(s))\phi + L(V_1(s))\phi, U(s) \rangle \, \mathrm{d}s + \int_{U \times [0,t]} \langle \beta(\cdot, V_1(s), u)\phi + \alpha^{\mathrm{T}}(\cdot, V_1(s), u) \nabla \phi, U(s) \rangle W(\mathrm{d}u \, \mathrm{d}s). \tag{3.13}$$

By Corollary 3.1, V_2^+ and V_2^- (and hence $V_2 = V_2^+ - V_2^-$) are H_0-valued. In particular, V_2 is an H_0-valued solution of (3.13). Since V_1 is also an H_0-valued solution of (3.13), it follows from Theorem 3.4 that $V_2 = V_1$. Hence, V_1 corresponds to a solution of the system (1.2)–(1.4). By the uniqueness of the solution of this system we see that $V(t) = V_1(t)$. □

4. Relationship to filtering equations

Consider a filtering model with a signal given by

$$X(t)=X(0)+\int_0^t \sigma(X(s))\,\mathrm{d}B(s)+\int_0^t b(X(s))\,\mathrm{d}s$$

and the observations given as a set indexed process

$$Y(A,t)=\int_0^t\int_A h(X(s),u)\mu(\mathrm{d}u)+W(A,t).$$

Intuitively, think of Y as representing observations of spatially distributed information. The corresponding Zakai equation, in weak form, is

$$\langle\phi,V(t)\rangle=\langle\phi,V(0)\rangle+\int_0^t\langle L\phi,V(s)\rangle\,\mathrm{d}s+\int_{U\times[0,t]}\langle\phi h(\cdot,u),V(s)\rangle Y(\mathrm{d}u\,\mathrm{d}s),$$

where

$$L\phi(x)=\frac{1}{2}\sum_{ij}a_{ij}(x)\partial_{x_i}\partial_{x_j}\phi(x)+\sum_i b_i(x)\partial_{x_i}\phi(x)$$

with $a(x)=\sigma(x)\sigma(x)^{\mathrm{T}}$. The Kushner–Stratonovich equation is

$$\begin{aligned}\langle\phi,\pi(t)\rangle=\langle\phi,\pi(0)\rangle&+\int_0^t\langle L\phi,\pi(s)\rangle\,\mathrm{d}s\\&-\int_{U\times[0,t]}(\langle\phi h(\cdot,u),\pi(s)\rangle\langle h(\cdot,u),\pi(s)\rangle-\langle\phi,\pi(s)\rangle\langle h(\cdot,u)\pi(s)\rangle^2)\mu(\mathrm{d}u)\,\mathrm{d}s\\&+\int_{U\times[0,t]}(\langle\phi h(\cdot,u),\pi(s)\rangle-\langle\phi,\pi(s)\rangle\langle h(\cdot,u),\pi(s)\rangle)Y(\mathrm{d}u\,\mathrm{d}s),\end{aligned}$$

and V and π are related by

$$\langle\phi,\pi\rangle=\frac{\langle\phi,V(t)\rangle}{\langle\phi,1\rangle}.$$

Following the standard reference measure approach to filtering, we can think of Y as being Gaussian white noise defined on the Girsanov-transformed probability space, and with that interpretation, both equations are of the form (1.6).

In particular, for the Zakai equation, we have

$$\begin{aligned}&d=0,\\&\beta(x,v,u)=h(x,u),\\&\alpha=0,\\&\gamma=0,\end{aligned}$$

and the system is

$$X_i(t)=X_i(0)+\int_0^t\sigma(X_i(s))\,\mathrm{d}B_i(s)+\int_0^t b(X_i(s))\,\mathrm{d}s$$

$$A_i(t)=A_i(0)+\int_{U\times[0,t]}A_i(s)h(X_i(s),u)Y(\mathrm{d}u\,\mathrm{d}s).$$

For the Kushner–Stratonovich equation,

$$d(x,\pi) = -\int_U (\langle h(\cdot,u),\pi\rangle(h(x,u) - \langle h(\cdot,u),\pi\rangle))\mu(\mathrm{d}u),$$

$$\beta(x,\pi,u) = h(x,u) - \langle h(\cdot,u),\pi\rangle,$$

$$\alpha = 0,$$

$$\gamma = 0,$$

and the system becomes

$$X_i(t) = X_i(0) + \int_0^t \sigma(X_i(s),\mathrm{d}B_i(s) + \int_0^t b(X_i(s))\,\mathrm{d}s,$$

$$A_i(t) = A_i(0) - \int_0^t \int_U A_i(s)(\langle h(\cdot,u),\pi(s)\rangle(h(X_i(s),u) - \langle h(\cdot,u),\pi(s)\rangle)\mu(\mathrm{d}u)\,\mathrm{d}s$$
$$+ \int_{U\times[0,t]} A_i(s)(h(X_i(s),u) - \langle h(\cdot,u),\pi(s)\rangle)Y(\mathrm{d}u\,\mathrm{d}s)$$

with

$$\pi(t) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n A_i(t)\delta_{X_i(t)}.$$

Define

$$P(t) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n A_i(t).$$

Then

$$P(t) = P(0) - \int_0^t (1 - P(s)) \int_U \langle h(\cdot,u),\pi(s)\rangle^2 \mu(\mathrm{d}u)\,\mathrm{d}s$$
$$+ \int_{U\times[0,t]} (1 - P(s))\langle h(\cdot,u),\pi(s)\rangle Y(\mathrm{d}u\,\mathrm{d}s),$$

so it follows that if $P(0) = 1$, then $P(t) = 1$, $t \geqslant 0$, and $\pi(t)$ must be a probability measure.

Note that the representation for the Zakai equation is just Monte Carlo integration of the Kallianpur–Striebel formula which was studied in Del Moral (1995). These representations are also closely related to the branching particle methods considered in Crisan and Lyons (1997) and Crisan et al. (1998).

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