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Spatial Point Processes and the Projection Method

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Abstract. The projection method obtains non-trivial point processes from higher-dimensional Poisson point processes by constructing a random subset of the higher-dimensional space and projecting the points of the Poisson process lying in that set onto the lower-dimensional region. This paper presents a review of this method related to spatial point processes as well as some examples of its applications. The results presented here are known for sometime but were not published before. Also, we present a backward construction of general spatial pure-birth processes and spatial birth and death processes based on the projection method that leads to a perfect simulation scheme for some Gibbs distributions in compact regions.

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1. Introduction

A point process is a model of indistinguishable points distributed randomly in some space. The simplest assumption that there are no multiple points and events occurring in disjoint regions of space are independent leads to the well-known Poisson point process. However, it is obvious that not all phenomena can be modelled by a process with independent increments. In $[0, \infty)$, any simple point process N with continuous compensator Λ, that is, $N([0, t]) - \Lambda(t)$ is a local martingale, can be obtained as a random time change $N([0, t]) = Y(\Lambda(t))$, where Y is a unit rate Poisson process. (See, for example, Proposition 13.4.III, Daley and Vere-Jones, 1988.) More generally, multivariate counting processes with continuous compensators and without simultaneous jumps can be obtained as multiple random time

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changes of independent unit Poisson processes (Meyer (1971), Aalen and Hoem (1978), Kurtz (1980b), Kurtz (1982)).

The projection method introduced by Kurtz (1989) can be seen as a generalization of the random time change representations. It constructs point processes through projections of underlying Poisson processes using stopping sets. These projections are made carefully in order that the projected process inherits many of the good properties of the Poisson processes. Garcia (1995a and 1995b) used this method to construct birth and death processes with variable birth and death rates and to study large population behavior for epidemic models. However, Kurtz (1989) is an unpublished manuscript and the generality of Garcia (1995a) hides the beauty of the ideas and methods behind technicalities. The goal of this paper is to present the projection method in detail as well as some examples. Another form of stochastic equation for spatial birth processes is considered by Massoulié (1998) and Garcia and Kurtz (2006). The latter paper also considers birth and death processes.

This paper is organized as follows:

Section 3 is based on Kurtz(1989) and provides a proper presentation of the projection method and states the basic theorems regarding martingale properties and moment inequalities for the projected processes. Simple examples, such as inhomogeneous Poisson processes, $M/G/\infty$ queues, and Cox processes, are presented.

Section 4 characterizes some of the processes that can be obtained by the projection method, the result obtained here is more general than the similar one in Garcia(1995a).

Section 5 presents spatial pure birth processes as projections of Poisson processes and derives some consequences, such as ergodicity. Although these results can be seen as particular cases from Garcia (1995a) the proofs in this case are simpler and provide much better insight into the use of the projection method.

Section 6 deals with birth and death processes in the special case when the stationary distribution is a Gibbs distribution, and for the finite case, a backward scheme provides perfect simulation.

2. Basic definitions

In this work, we are going to identify a point process with the counting measure N given by assigning unit mass to each point, that is, $N(A)$ is the number of points in a set A. With this identification in mind, let (S, r) be a complete, separable metric space, and let $\mathcal{N}(S)$ be the collection of σ-finite, Radon counting measures on S. $\mathcal{B}(S)$ will denote the Borel subsets of S. Typically, $S \subset \mathbb{R}^d$.

For counting measures, the Radon condition is simply the requirement that for each compact $K \subset S$, there exists an open set $G \supset K$ such that $N(G) < \infty$. For a general discussion, see Daley and Vere-Jones (1988). We topologize $\mathcal{N}(S)$ with the vague topology.

Definition 2.1. A sequence $\{\xi_k\} \subset \mathcal{N}(S)$ *converges vaguely* to $\xi \in \mathcal{N}(S)$ if and only if for each compact K there exists an open $G \supset K$ such that

$$\lim_{k\to\infty} \int f d\xi_k = \int f d\xi$$

for all $f \in C_b(S)$ with $\mathrm{supp}(f) \subset G$.

If ξ is simple, that is $\xi(\{x\}) = 0$ or 1 for all $x \in S$, then $\{\xi_k\}$ converges vaguely to ξ if and only if there exist representations

$$\xi_k = \sum_{i=1}^{\infty} \delta_{x_{ki}}, \quad \xi = \sum_{i=1}^{\infty} \delta_{x_i}$$

such that for each i, $\lim_{k\to\infty} x_{ki} = x_i$.

Definition 2.2. Let μ be a σ-finite, Radon measure on S. A point process N on S is a *Poisson process* with mean measure μ if the following conditions hold:

(i) For $A_1, A_2, \ldots, A_k \in \mathcal{B}(S)$ disjoint sets, $N(A_1), N(A_2), \ldots, N(A_k)$ are independent random variables.

(ii) For each $A \in \mathcal{B}(S)$ and $k \geq 0$,

$$\mathbb{P}[N(A) = k] = e^{-\mu(A)} \frac{\mu(A)^k}{k!}.$$

Assuming that μ is diffuse, that is, $\mu\{x\} = 0$ for all $x \in S$, the strong independence properties of a Poisson process imply that N conditional on n points of N being sited at $x_1, x_2, \ldots, x_n$ has the properties of $N + \sum_{k=1}^{n} \delta_{x_k}$. Thus, the process "forgets" where it had the n points and behaves as if it were N with the n points adjoined. The notion of conditioning in this case is not straightforward since the the event "having a point at x" has probability zero. Assuming that $\mu(B_\epsilon(x_k)) > 0$ for all $\epsilon > 0$ and $k = 1, \ldots, n$,

$$\lim_{\epsilon\to 0+} \mathbb{E}[F(N)|N(B_\epsilon(x_k)) > 0, k = 1, \ldots, n] = \mathbb{E}\left[F\left(N + \sum_{k=1}^{n} \delta_{x_k}\right)\right]$$

for all $F \in C_b(\mathcal{N}(S))$. As a consequence, we have the following basic identity for Poisson processes.

Proposition 2.3. *Let N be a Poisson process on S with diffuse mean measure μ. Then*

$$\mathbb{E}\left[\int_S f(z, N) N(dz)\right] = \mathbb{E}\left[\int_S f(z, N + \delta_z)\mu(dz)\right]. \tag{2.1}$$

For example, let $c : S \to [0, \infty)$, $\rho : S \times S \to [0, \infty)$ and $\phi : [0, \infty) \to [0, \infty)$ be Borel measurable functions. Then (*cf.* Garcia (1995a))

$$\mathbb{E}\left[\int_S c(z)\phi\left(\int_S \rho(x, z)N(dx)\right) N(dz)\right]$$
$$= \int_S c(z)\mathbb{E}\left[\phi(\rho(z, z) + \int_S \rho(x, z)N(dx))\right] \mu(dz).$$

Alternatively, we may condition in a formal way in terms of Palm probabilities and Palm distributions (see Karr (1986, Section 1.7) or Daley and Vere-Jones (1988, Chapter 12)).

If we have a sequence of Poisson processes N_n with mean measures $n\mu$, defining the random signed measure W_n by $W_n(A) = n^{-1/2}(N_n(A) - n\,\mu(A))$, for $A_1, \ldots, A_m$ Borel sets with $\mu(A_i) < \infty$, the central limit theorem gives

$$(W_n(A_1), \ldots, W_n(A_m)) \xrightarrow{\mathcal{D}} (W(A_1), \ldots, W(A_m)), \tag{2.2}$$

where W is the mean zero Gaussian process indexed by Borel sets with covariance $\mathbb{E}[W(A)W(B)] = \mu(A \cap B)$.

3. Projection method

The basic idea of the projection method is to obtain point processes from higher-dimensional Poisson processes by constructing a random subset of the higher-dimensional space and projecting the points of the Poisson process lying in that set onto the lower-dimensional subspace. This general approach can be used to construct, for example, Cox processes (the random set is independent of the Poisson process), a large class of Gibbs distributions, and birth and death processes with variable birth and death rates.

This construction gives a natural coupling among point processes and hence a method to compare results and prove limit theorems. The law of large numbers and central limit theorem for Poisson processes can be exploited to obtain corresponding results for the point processes under study. Garcia (1995b) studied large population behavior for epidemic models using the underlying limit theorems for Poisson processes. Ferrari and Garcia (1998) applied the projection method to the study of loss networks.

Even though the basic concepts and ideas described in the remainder of this section were introduced in Kurtz (1989) and were used in Garcia (1995a and 1995b), a number of results appear here for the first time.

3.1. Representation of inhomogeneous Poisson processes as projections of higher-dimensional homogeneous Poisson processes

Let N be a Poisson random measure on $\mathbb{R}^{d+1}$ with Lebesgue mean measure m. Let $C \in \mathcal{B}(\mathbb{R}^{d+1})$ and define N_C, a point process on $\mathbb{R}^d$, by

$$N_C(A) = N(C \cap A \times \mathbb{R}), \quad A \in \mathcal{B}(\mathbb{R}^d). \tag{3.1}$$

Note that N_C is a random measure corresponding to the point process obtained by taking the points of N that lie in C and projecting them onto $\mathbb{R}^d$. Clearly, $N_C(A)$ is Poisson distributed with mean

$$\mu_C(A) = m(C \cap A \times \mathbb{R}).$$

If μ is an absolutely continuous measure (with respect to Lebesgue measure) in $\mathbb{R}^d$ with density f and $C = \{(x, y); x \in \mathbb{R}^d, 0 \le y \le f(x)\}$, then N_C is a Poisson random measure on $\mathbb{R}^d$ with mean measure μ given by

$$\mu_C(A) = m(C \cap A \times \mathbb{R}) = \int_A \int_0^{f(x)} dy\, dx = \int_A f(x) dx = \mu(A).$$

3.2. $M/G/\infty$ **queues**

A particular application of the projection method for inhomogeneous Poisson processes is the $M/G/\infty$ queue. In fact, Foley (1986) exploits precisely this observation. Consider a process where clients arrive according to a λ-homogeneous Poisson process and are served according to a service distribution ν. For simplicity, assume that there are no clients at time $t = 0$. Let $\mu = \lambda m \times \nu$ and $S = [0, \infty) \times [0, \infty)$, and let N be the Poisson process on S with mean measure μ. Define

$$B(t) = \{(x, y) \in S; x \le t\} \tag{3.2}$$

and

$$A(t) = \{(x, y) \in S; y \le t - x\}. \tag{3.3}$$

We can identify the points of N in $B(t)$ with customers that arrive by time t (note that the distribution of $N(B(t))$ is Poisson with parameter λt) and the points in $A(t)$ are identified with customers that complete service by time t. Therefore, the points of N in

$$C(t) = B(t) - A(t) = \{(x, y) : x \le t, y > t - x\}$$

correspond to the customers in the queue and hence the queue length is

$$Q(t) = N(C(t)). \tag{3.4}$$

Notice that we can construct this process starting at $-T$ instead of 0. Therefore, the system at time 0 in this new construction has the same distribution as the system at time T in the old construction. In fact, defining

$$C_T(t) = \{(x, y) \in [-T, \infty) \times [0, \infty); -T \le x \le t, y \ge t - x\}, \tag{3.5}$$

and

$$Q_T(t) = N(C_T(t)), \tag{3.6}$$

we have that

$$Q_T(0) \stackrel{\mathcal{D}}{=} Q(T). \tag{3.7}$$

Letting $T \to \infty$, we have

$$C_T(0) \to \{(x, y); x \le 0, y \ge -x\}$$

and

$$Q_T(0) \to N(\{(x, y) \in \mathbb{R} \times [0, \infty); y \ge -x \ge 0\}) \tag{3.8}$$

which implies

$$Q(T) \Rightarrow N(\{(x, y) \in \mathbb{R} \times [0, \infty); y \ge -x \ge 0\}) \tag{3.9}$$

in the original $M/G/\infty$ queue. Even though this result is well known by other arguments, this “backward construction” can be used for other non trivial cases. (See Section 6.3.)

3.3. Projections through random sets

The projected process N_C defined by (3.1) was a Poisson process due to the fact that the projection set was deterministic. The construction, however, still makes sense if the projection set C is random. Let N denote a Poisson process on $S = S_1 \times S_2$ with mean measure μ, where S_1 and S_2 are complete separable metric spaces. Let Γ be a random subset of S (in general, not independent of N). Define a point process N_Γ on S_1 by $N_\Gamma(B) = N(\Gamma \cap B \times S_2)$.

For example, if the set Γ is independent of the process N, then N_Γ will be a *Cox process* (or doubly stochastic Poisson process). In fact, conditioned on Γ, in the independent case, N_Γ is a Poisson process with mean measure $\mu(\Gamma \cap \cdot \times S_2)$ and the mean measure, $\mu_\Gamma(B) \equiv \mathbb{E}[N_\Gamma(B)]$, for N_Γ is

$$\mu_\Gamma(B) = \mathbb{E}[\mu(\Gamma \cap B \times S_2)]. \tag{3.10}$$

It is tempting to conjecture that (3.10) holds for other random sets, but in general, that is not true (e.g., let Γ be the support of N). However, there is a class of sets for which the identity does hold, the class of *stopping sets*.

Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is complete and that $\{\mathcal{F}_A\}$ is a family of complete sub-σ-algebras indexed by Borel subsets, $A \in \mathcal{B}(S)$, having the property that if $A \subset B$ then $\mathcal{F}_A \subset \mathcal{F}_B$. A Poisson point process N on S is *compatible* with $\{\mathcal{F}_A\}$, if for each $A \in \mathcal{B}(S)$, $N(A)$ is $\mathcal{F}_A$-measurable and, for each $C \in \mathcal{B}(S)$ such that $C \cap A = \emptyset$, $N(C)$ is independent of $\mathcal{F}_A$.

For technical reasons, we will restrict attention to Γ with values in the closed sets $\mathcal{C}(S)$, and we will assume that $\{\mathcal{F}_A, A \in \mathcal{C}(S)\}$ is right continuous in the sense that if $\{A_k\} \subset \mathcal{C}(S)$, $A_1 \supset A_2 \supset \cdots$, then $\cap_k \mathcal{F}_{A_k} = \mathcal{F}_{\cap A_k}$. If $\mathcal{F}_A$ is the completion of the σ-algebra $\sigma(N(B) : B \in \mathcal{B}(S), B \subset A)$, then $\{\mathcal{F}_A, A \in \mathcal{C}(S)\}$ is right continuous.

Definition 3.1. A $\mathcal{C}(S)$-valued random variable Γ is a $\{\mathcal{F}_A\}$-*stopping set*, if $\{\Gamma \subset A\} \in \mathcal{F}_A$ for each $A \in \mathcal{C}(S)$.

Γ is *separable* if there exists a countable set $\{H_n\} \subset \mathcal{C}(S)$ such that $\Gamma = \cap\{H_n : H_n \supset \Gamma\}$. Then Γ is *separable with respect to* $\{H_n\}$.

The *information σ-algebra*, $\mathcal{F}_\Gamma$, is given by

$$\mathcal{F}_\Gamma = \{G \in \mathcal{F} : G \cap \{\Gamma \subset A\} \in \mathcal{F}_A \text{ for all } A \in \mathcal{C}(S)\}.$$

Remark 3.2. The definition of stopping set used here is from Garcia (1995a). It differs from the definition used by Ivanoff and Merzbach (1995). A stopping set in the sense used here is a special case of a generalized *stopping time* as used in Kurtz (1980a), and is essentially the same as an *adapted set* as used in Balan (2001).

The significance of separability of Γ is that we can approximate Γ by a decreasing sequence of discrete stopping sets. Without loss of generality, we can always assume $H_1 = S$.

Lemma 3.3. *Let Γ be separable with respect to $\{H_n\}$, and define $\Gamma_n = \cap\{H_k : k \leq n, H_k \supset \Gamma\}$. Then Γ_n is a stopping set with $\Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma$ and $\cap_n \Gamma_n = \Gamma$.*

Proof. Let $\mathcal{S}_n = \{H_{i_1} \cap \cdots \cap H_{i_m} : 1 \leq i_1, \ldots, i_m \leq n\}$. Then

$$\{\Gamma_n \subset A\} = \cup_{C \subset A, C \in \mathcal{S}_n} \{\Gamma \subset C\} \in \mathcal{F}_A,$$

so Γ_n is a stopping set. Separability then implies $\cap_n \Gamma_n = \Gamma$. □

These definitions are clear analogs of the definitions for real-valued stopping times, and stopping sets have many properties in common with stopping times.

Lemma 3.4. *Let $\Gamma, \Gamma_1, \Gamma_2, \ldots$ be $\{\mathcal{F}_A\}$-stopping sets. Then*

(i) *For $A \in \mathcal{C}(S)$, $\{\Gamma \subset A\} \in \mathcal{F}_\Gamma$.*
(ii) *For $n = 1, 2, \ldots$, $\Gamma_1 \cup \cdots \cup \Gamma_n$ is a $\{\mathcal{F}_A\}$-stopping set.*
(iii) *The closure of $\cup_{i=1}^\infty \Gamma_i$ is a stopping set.*
(iv) *If $\Gamma_1 \subset \Gamma_2$, then $\mathcal{F}_{\Gamma_1} \subset \mathcal{F}_{\Gamma_2}$.*
(v) *If the range of Γ is countable, say $\mathcal{R}(\Gamma) = \{C_k\}$, then*

$$\mathcal{F}_\Gamma = \{B \in \mathcal{F} : B = \cup_k \{\Gamma = C_k\} \cap B_k, B_k \in \mathcal{F}_{C_k}\}.$$

(vi) *If Γ is separable and $\{\Gamma_n\}$ are defined as in Lemma 3.3, then for $B \in \mathcal{B}(S)$, $N(\Gamma \cap B)$ is $\cap_n \mathcal{F}_{\Gamma_n}$-measurable.*

Remark 3.5. The intersection of two stopping sets need not be a stopping set.

Proof. $\{\Gamma \subset A\} \cap \{\Gamma \subset B\} = \{\Gamma \subset A \cap B\} \in \mathcal{F}_{A \cap B} \subset \mathcal{F}_B$, for all $B \in \mathcal{C}(S)$, and hence, $\{\Gamma \subset A\} \in \mathcal{F}_\Gamma$.

Since $\{\Gamma_i \subset A\} \in \mathcal{F}_A$, $\{\cup_{i=1}^n \Gamma_i \subset A\} = \cap_{i=1}^n \{\Gamma_i \subset A\} \in \mathcal{F}_A$. Similarly, since A is closed,

$$\{\mathrm{cl} \cup_i \Gamma_i \subset A\} = \{\cup_i \Gamma_i \subset A\} = \cap_i \{\Gamma_i \subset A\} \in \mathcal{F}_A.$$

Suppose $\Gamma_1 \subset \Gamma_2$, and let $G \in \mathcal{F}_{\Gamma_1}$. Then

$$G \cap \{\Gamma_2 \subset A\} = G \cap \{\Gamma_1 \subset A\} \cap \{\Gamma_2 \subset A\} \in \mathcal{F}_A.$$

If $\mathcal{R}(\Gamma) = \{C_k\}$, then

$$\{\Gamma = C_k\} = \{\Gamma \subset C_k\} - \cup_{C_j \subsetneq C_k} \{\Gamma \subset C_j\} \in \mathcal{F}_{C_k}.$$

Similarly, if $B \in \mathcal{F}_\Gamma$, $B_k \equiv B \cap \{\Gamma = C_k\} \in \mathcal{F}_{C_k}$, and hence $B = \cup_k \{\Gamma = C_k\} \cap B_k$ with $B_k \in \mathcal{F}_{C_k}$. Conversely, if $B = \cup_k \{\Gamma = C_k\} \cap B_k$ with $B_k \in \mathcal{F}_{C_k}$,

$$B \cap \{\Gamma \subset A\} = \cup_{C_k \subset A} (B_k \cap \{\Gamma = C_k\}) \in \mathcal{F}_A,$$

so $B \in \mathcal{F}_\Gamma$.

Finally, $\mathcal{R}(\Gamma_n) = \{C_k^n\}$ is countable, so

$$\{N(\Gamma_n \cap B) = l\} \cap \{\Gamma_n \subset A\} = \cup_{C_k^n \subset A} \{N(C_k^n \cap B) = l\} \cap \{\Gamma_n = C_k^n\} \in \mathcal{F}_A$$

and $\{N(\Gamma_n \cap B) = l\} \in \mathcal{F}_{\Gamma_n}$. Since $\Gamma_1 \supset \Gamma_2 \supset \cdots$ and $\cap_n \Gamma_n = \Gamma$, $\lim_{n\to\infty} N(\Gamma_n \cap B) = N(\Gamma \cap B)$ and $N(\Gamma \cap B)$ is $\cap_n \mathcal{F}_{\Gamma_n}$-measurable. □

Lemma 3.6. *If $K \subset S$ is compact, then $\{\Gamma \cap K = \emptyset\} \in \mathcal{F}_\Gamma$. In particular, for each $x \in S$, $\mathbf{1}_\Gamma(x)$ is $\mathcal{F}_\Gamma$-measurable.*

Proof. If $\Gamma \cap K = \emptyset$, then $\inf\{r(x,y) : x \in \Gamma, y \in K\} > 0$. Consequently, setting $G_n = \{y : \inf_{x\in K} r(x,y) < n^{-1}\}$,

$$\{\Gamma \cap K = \emptyset\} = \cup_n \{\Gamma \cap G_n = \emptyset\} = \cup_n \{\Gamma \subset G_n^c\} \in \mathcal{F}_\Gamma.$$

For the second statement, note that $\{\mathbf{1}_\Gamma(x) = 0\} = \{\Gamma \cap \{x\} = \emptyset\} \in \mathcal{F}_\Gamma$. □

We will need to know that, in some sense, limits of stopping sets are stopping sets. If we assume that S is locally compact, then this result is simple to formulate.

Lemma 3.7. *Assume that S is locally compact. Suppose $\{\Gamma_k\}$ are stopping sets, and define*

$$\Gamma = \limsup_{k\to\infty} \Gamma_k \equiv \cap_m \mathrm{cl} \cup_{k\geq m} \Gamma_k.$$

Then Γ is a stopping set.

Proof. Let $G_1 \subset G_2 \subset \cdots$ be open sets with compact closure satisfying $\cup_n G_n = S$, and for $A \in \mathcal{C}(S)$, let $A_n = \{y \in S : \inf_{x\in A} r(x,y) \leq n^{-1}\}$. Noting that $A = \cap_n (A_n \cup G_n^c)$, $\Gamma \subset A$ if and only if for each n, there exists m such that $\cup_{k\geq m}\Gamma_k \subset A_n \cup G_n^c$. Otherwise, for some n, there exist $x_m \in \cup_{k\geq m}\Gamma_k$ such that $x_m \in A_n^c \cap G_n$, and by the compactness of $\mathrm{cl}G_n$, a limit point x of $\{x_m\}$ such that $x \in \Gamma \cap \mathrm{cl}(A_n^c \cap G_n) \subset A^c$. Consequently,

$$\{\Gamma \subset A\} = \cap_n \cup_m \{\cup_{k\geq m}\Gamma_k \subset A_n \cup G_n^c\} \in \cap_n \mathcal{F}_{A_n \cup G_n^c} = \mathcal{F}_A. \qquad \square$$

Local compactness also simplifies issues regarding separability of stopping sets. Note that the previous lemma implies that the intersection of a decreasing sequence of stopping sets is a stopping set.

Lemma 3.8. *Let S be locally compact. Then all stopping sets are separable, and if $\{\Gamma_n\}$ is a decreasing sequence of stopping sets with $\Gamma = \cap_n \Gamma_n$, then Γ is a stopping set and $\mathcal{F}_\Gamma = \cap_n \mathcal{F}_{\Gamma_n}$.*

Proof. Let $\{x_i\}$ be dense in S, $\epsilon_j > 0$ with $\lim_{j\to\infty} \epsilon_j = 0$, and $\{G_n\}$ be as in the proof of Lemma 3.7. Let $\{H_k\}$ be some ordering of the countable collection of sets of the form $G_n^c \cup \cup_{l=1}^m \bar{B}_{\epsilon_{j_l}}(x_{i_l})$ with $H_1 = S$. Then for $A \in \mathcal{C}(S)$ and $A_n = \cap\{H_k : k \leq n, A \subset H_k\}$, $A = \cap_n A_n$. If $G \in \cap_n \mathcal{F}_{\Gamma_n}$, then

$$G \cap \{\Gamma \subset A\} = \cap_n G \cap \{\Gamma \subset A_n\} = \cap_n G \cap \{\Gamma_n \subset A_n\} \in \mathcal{F}_A. \qquad \square$$

Theorem 3.9. *Let N be a Poisson process in S with mean measure μ and compatible with $\{\mathcal{F}_A\}$. If Γ is a separable $\{\mathcal{F}_A\}$-stopping set and N_Γ is the point process in S_1 obtained by projecting the points of N that lie in Γ onto S_1, then the mean measure for N_Γ satisfies* (3.10) (*allowing $\infty = \infty$*).

More generally, if $\Gamma^{(1)}$ *and* $\Gamma^{(2)}$ *are stopping sets with* $\Gamma^{(1)} \subset \Gamma^{(2)}$*, then for each* D *satisfying* $\mu(D) < \infty$,

$$E[N(\Gamma^{(2)} \cap D) - \mu(\Gamma^{(2)} \cap D)|\mathcal{F}_{\Gamma^{(1)}}] = N(\Gamma^{(1)} \cap D) - \mu(\Gamma^{(1)} \cap D) \tag{3.11}$$

and

$$E[N_{\Gamma^{(2)}}(B) - N_{\Gamma^{(1)}}(B)] = E[\mu((\Gamma^{(2)} - \Gamma^{(1)}) \cap B \times S_2], \tag{3.12}$$

again allowing $\infty = \infty$.

Proof. The independence properties of the Poisson process imply that for each $D \in \mathcal{B}(S)$ such that $\mu(D) < \infty$,

$$M^D(A) = N(A \cap D) - \mu(A \cap D) \tag{3.13}$$

is a $\{\mathcal{F}_A\}$-martingale. Let $\{\Gamma_n\}$ be the stopping sets with countable range defined in Lemma 3.3. Then the optional sampling theorem (see Kurtz (1980a)) implies $\mathbb{E}[N(\Gamma_n \cap D) - \mu(\Gamma_n \cap D)] = 0$, and since $\Gamma_1 \supset \Gamma_2 \supset \cdots$ and $\Gamma = \cap_n \Gamma_n$,

$$\mathbb{E}[N(\Gamma \cap D) - \mu(\Gamma \cap D)] = 0.$$

(Note that Γ being a $\{\mathcal{F}_A\}$-stopping set does not imply that $\Gamma \cap D$ is a $\{\mathcal{F}_A\}$-stopping set.) For $B \in \mathcal{B}(S_1)$, assume $\{D_k\} \subset \mathcal{B}(S)$ are disjoint, satisfy $\mu(D_k) < \infty$, and $\cup_k D_k = B \times S_2$. Then, by the monotone convergence theorem,

$$\mathbb{E}[N(\Gamma \cap B \times S_2)] = \sum_{k=1}^{\infty} \mathbb{E}[N(\Gamma \cap D_k)] = \sum_{k=1}^{\infty} \mathbb{E}[\mu(\Gamma \cap D_k)] = \mathbb{E}[\mu(\Gamma \cap B \times S_2)],$$

and the same argument gives (3.11) and (3.12). □

Some martingale properties of this process:

Theorem 3.10. *Let* N *and* Γ *be as in Theorem* 3.9.

(a) *If* $\mu(D) < \infty$*, then* L^D *defined by*

$$L^D(A) = (N(A \cap D) - \mu(A \cap D))^2 - \mu(A \cap D) \tag{3.14}$$

is an $\{\mathcal{F}_A\}$*-martingale, and if* $\mu_\Gamma(B) = E[N_\Gamma(B)] < \infty$,

$$\mathbb{E}\big[(N_\Gamma(B) - \mu(\Gamma \cap (B \times S_2)))^2\big] = \mu_\Gamma(B). \tag{3.15}$$

(b) *If* $D_1, D_2 \in \mathcal{B}(S)$ *are disjoint with* $\mu(D_1) + \mu(D_2) < \infty$*, then* M^{D_1} *and* M^{D_2} (*as defined by Equation* (3.13)) *are orthogonal martingales in the sense that their product is a martingale. Consequently, if* $B_1, B_2 \in \mathcal{B}(S_1)$ *are disjoint and* $\mathbb{E}[N_\Gamma(B_1)] + \mathbb{E}[N_\Gamma(B_2)] < \infty$*, then*

$$\mathbb{E}\big[(N_\Gamma(B_1) - \mu(\Gamma \cap (B_1 \times S_2)))(N_\Gamma(B_2) - \mu(\Gamma \cap (B_2 \times S_2)))\big] = 0.$$

(c) *Let* $f : S_1 \times S_2 \to \mathbb{R}_+$*, and let* $\mathcal{I}_f = \{A \subset S_1 \times S_2; \int_A |e^{f(z)} - 1|\, \mu(dz) < \infty\}$*. Then,*

$$M_f(A) = \exp\left\{\int_A f(z)\, N(dz) - \int_A (e^{f(z)} - 1)\, \mu(dz)\right\} \tag{3.16}$$

is an $\{\mathcal{F}_A\}$-martingale for $A \in \mathcal{I}_f$, and therefore, for $g : S_1 \to \mathbb{R}_+$ satisfying $\mathbb{E}[\exp\{\int g(x) N_\Gamma(dx)\}] < \infty$,

$$\mathbb{E}\left[\exp\left\{\int g(x)\, N_\Gamma(dx) - \int_\Gamma (e^{g(x)} - 1)\, \mu(dx \times dy)\right\}\right] = 1. \tag{3.17}$$

Proof. Let $\{D_k\}$ be as in the proof of Theorem 3.9. Then $\{N(\Gamma \cap (B \times S_1) \cap D_k) - \mu(\Gamma \cap (B \times S_1) \cap D_k)\}$ are orthogonal random variables in L^2, with

$$\mathbb{E}[(N(\Gamma \cap (B \times S_1) \cap D_k) - \mu(\Gamma \cap (B \times S_1) \cap D_k))^2] = \mathbb{E}[\mu(\Gamma \cap (B \times S_1) \cap D_k)].$$

Consequently, if $\mathbb{E}[\mu(\Gamma \cap (B \times S_1))] < \infty$, the series converges in L^2, giving (3.15).

Part (b) follows similarly.

The moment generating functional of N is given by

$$\mathbb{E}[e^{\int f\, dN}] = \exp\left\{\int (e^{f(z)} - 1)\, \mu(dz)\right\},$$

which shows that (3.16) is a martingale. Observing that $\{M_f(\cdot \cap D_k)\}$ are orthogonal martingales, and hence that $\mathbb{E}[M_f(\Gamma \cap \cup_{k=1}^l D_k)] = 1$, (3.17) follows by the dominated convergence theorem. □

Definition 3.11. Assume that $S = S_1 \times [0, \infty)$. A random function $\phi : S_1 \to [0, \infty)$ is a $\{\mathcal{F}_A\}$-*stopping surface* if the set $\Gamma_\phi = \{(x, y); x \in S_1, 0 \le y \le \phi(x)\}$ is a $\{\mathcal{F}_A\}$-stopping set. (Note that the requirement that Γ_ϕ be closed implies ϕ is upper semicontinuous.)

For simplicity, we will write $\mathcal{F}_\phi$ in place of $\mathcal{F}_{\Gamma_\phi}$. Furthermore, since $\Gamma_\phi \subset A$ if and only if for each $x \in S_1$,

$$\phi(x) \le f(x) = \sup\{z \ge 0 : \{x\} \times [0, z] \subset A\},$$

we only need to consider A of the form $A_f = \{(x, y) : y \le f(x)\}$ for nonnegative, upper semicontinuous f, that is, we only need to verify that $\{\phi \le f\} \in \mathcal{F}_f$ for each nonnegative, upper-semicontinuous f. Again we write $\mathcal{F}_f$ rather than $\mathcal{F}_{A_f}$. Furthermore, since an upper-semicontinuous f is the limit of a decreasing sequence of continuous f_n and $\{\phi \le f\} = \cap_n \{\phi \le f_n\}$ and $\mathcal{F}_f = \cap_n \mathcal{F}_{f_n}$, it is enough to verify $\{\phi \le f\} \in \mathcal{F}_f$ for continuous f.

Lemma 3.12. *Let ϕ be a stopping surface. Then*

(i) *For each $x \in S_1$, $\phi(x)$ is $\mathcal{F}_\phi$- measurable.*
(ii) *For each compact $K \subset S_1$, $\sup_{x \in K} \phi(x)$ is $\mathcal{F}_\phi$-measurable.*
(iii) *If $a \ge 0$ is deterministic and upper semicontinuous, then $\phi + a$ is a stopping surface.*

Proof. Since $\{\phi(x) < y\} = \{\Gamma_\phi \cap \{(x, y)\} = \emptyset\} \in \mathcal{F}_\phi$, by Lemma 3.6, $\phi(x)$ is $\mathcal{F}_\phi$-measurable.

Since ϕ assumes its supremum over a compact set, for $y \ge 0$,

$$\{\sup_{x \in K} \phi(x) < y\} = \{\Gamma_\phi \cap K \times \{y\}\} \in \mathcal{F}_\phi.$$

For Part (iii), first assume that a is continuous. Then for f continuous, $\{\phi + a \le f\} = \{\phi \le f - a\} \in \mathcal{F}_f$, and for f upper semicontinuous, there exists a decreasing sequence of continuous f_n converging to f, so

$$\{\phi + a \le f\} = \cap_n \{\phi + a \le f_n\} \in \mathcal{F}_\phi.$$

But this implies that for every nonnegative, upper-semicontinuous g,

$$\{\phi + a \le f\} \cap \{\phi \le g\} \in \mathcal{F}_g,$$

so in particular, $\{\phi + a \le f\} = \{\phi + a \le f\} \cap \{\phi \le f\} \in \mathcal{F}_f$, and hence, $\phi + a$ is a stopping surface.

Finally, for a upper semicontinuous, there is a decreasing sequence of continuous functions a_n converging to a, so $\Gamma_{\phi+a} = \cap_n \Gamma_{\phi+a_n}$, and $\Gamma_{\phi+a}$ is a stopping set by Lemma 3.7. □

For a signed measure γ, let $T(\gamma, B)$ denote the total variation of γ over the set B. Let $\phi_i : S_1 \to [0, \infty)$, $i = 1, 2$, be stopping surfaces.

Corollary 3.13. *Suppose $\mu = \nu \times m$, and write N_{ϕ_i} instead of $N_{\Gamma_{\phi_i}}$. Then,*

$$\mathbb{E}[T(N_{\phi_1} - N_{\phi_2}, B)] = \mathbb{E}\left[\int_B |\phi_1(x) - \phi_2(x)| \nu(dx)\right], \tag{3.18}$$

and consequently,

$$\mathbb{E}\left[\left|\int f(x) N_{\phi_1}(dx) - \int f(x) N_{\phi_2}(dx)\right|\right] \le \mathbb{E}\left[\int |f(x)||\phi_1(x) - \phi_2(x)| \nu(dx)\right].$$

Proof. Note that

$$T(N_{\phi_1} - N_{\phi_2}, B) = 2N_{\phi_1 \vee \phi_2}(B) - N_{\phi_1}(B) - N_{\phi_2}(B),$$

and since the union of two stopping sets is a stopping set, (3.12) gives (3.18). We then have

$$\begin{aligned} \mathbb{E}\left[\left|\int f(x) N_{\phi_1}(dx) - \int f(x) N_{\phi_2}(dx)\right|\right] &\le \mathbb{E}\left[\int |f(x)|\, T(N_{\phi_1} - N_{\phi_2}, dx)\right] \quad (3.19)\\ &\le \mathbb{E}\left[\int |f(x)||\phi_1(x) - \phi_2(x)|\, \nu(dx)\right]. \ \square \end{aligned}$$

4. Characterization of point processes as projections of higher-dimensional Poisson processes

It would be interesting to know which point processes can be obtained as projections of higher-dimensional Poisson point processes. Gaver, Jacobs and Latouche (1984) characterized finite birth and death models in randomly changing environments as Markov processes in a higher-dimensional space. Garcia (1995a) generalizes this idea to other counting processes. We consider processes in Euclidean space, although most results have analogs in more general spaces. We assume that all processes are defined on a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Let η be a point process on $\mathbb{R}^d \times [0,\infty)$, and let $\mathcal{A}(\mathbb{R}^d) = \{B \in \mathcal{B}(\mathbb{R}^d) : m(B) < \infty\}$. Define $\eta_t(B) = \eta(B \times [0,t])$, $B \in \mathcal{A}(\mathbb{R}^d)$. Assume that $\eta_t(B)$ is a counting process for each $B \in \mathcal{A}(\mathbb{R}^d)$ and if $B_1 \cap B_2 = \emptyset$, that $\eta(B_1)$ and $\eta(B_2)$ have no simultaneous jumps.

Let $\{\mathcal{F}_t\}$ be a filtration, and let $\lambda : \mathbb{R}^d \times [0,\infty) \times \Omega \to [0,\infty)$ be progressive in the sense that for each $t \geq 0$, $\lambda : \mathbb{R}^d \times [0,t] \times \Omega$ is $\mathcal{B}(\mathbb{R}^d) \times \mathcal{B}([0,t]) \times \mathcal{F}_t$-measurable. Assume that λ is locally integrable in the sense that $\int_{B\times[0,t]} \lambda(x,s)ds < \infty$ a.s. for each $B \in \mathcal{A}(\mathbb{R}^d)$ and each $t > 0$. Then η has $\{\mathcal{F}_t\}$-*intensity* λ (with respect to Lebesgue measure) if and only if for each $B \in \mathcal{A}(\mathbb{R}^d)$,

$$\eta_t(B) - \int_{B\times[0,t]} \lambda(x,s)dxds$$

is a $\{\mathcal{F}_t\}$-local martingale.

Theorem 4.1. *Suppose that η has $\{\mathcal{F}_t\}$-intensity λ, λ is upper semicontinuous as a function of x, and there exists $\epsilon > 0$ such that $\lambda(x,t) \geq \epsilon$ for all x, t, almost surely. Then there exists a Poisson random measure N on $\mathbb{R}^d \times [0,\infty)$ such that for*

$$\begin{aligned} \Gamma_t &= \left\{(x,s) : x \in \mathbb{R}^d, 0 \leq s \leq \int_0^t \lambda(x,s)ds\right\} \\ \eta_t(B) &= N(\Gamma_t \cap (B \times [0,\infty)), \end{aligned} \tag{4.1}$$

that is, setting $\phi_t(x) = \int_0^t \lambda(x,s)ds$, $\eta_t = N_{\phi_t}$.

Remark 4.2. The condition that λ is bounded away from zero is necessary to ensure that $\gamma(t,x) = \inf\{s; \int_0^s \lambda(x,u)du \geq t\}$ is defined for all $t > 0$. If this condition does not hold, we can define

$$\eta_t^\epsilon(B) = \eta_t(B) + \xi(B \times [0,\epsilon t]),$$

where ξ is a Poisson process on $\mathbb{R}^d \times [0,\infty)$ with Lebesgue mean measure that is independent of $\vee_t \mathcal{F}_t$. Then it is clear that

$$\eta_t^\epsilon(B) - \int_0^t \int_B (\lambda(x,s) + \epsilon)dx\,ds$$

is a martingale for all $B \in \mathcal{A}(\mathbb{R}^d)$. The theorem then gives the representation

$$\eta_t^\epsilon(B) = N^\epsilon(\Gamma_t^\epsilon \cap B \times [0,\infty))$$

where $\Gamma_t^\epsilon = \{(x,s); x \in \mathbb{R}^d, s \leq \int_0^t(\lambda(x,u) + \epsilon)du\}$. Letting $\epsilon \to 0$, gives the result without the lower bound on λ.

Proof. The proof is similar to that of Theorem 2.6 of Garcia (1995a). □

5. Pure birth processes

The primary approach to building a model using a spatial, pure-birth process with points in a set K (which we will take to be a subset of $\mathbb{R}^d$) is to specify the intensity in a functional form, that is, as a function of the desired process η and, perhaps, additional randomness ξ in a space E. Assume we are given a jointly measurable mapping

$$\tilde{\lambda} : (x, z, u, s) \in K \times D_{\mathcal{N}(K)}[0, \infty) \times E \times [0, \infty) \to \tilde{\lambda}(x, z, u, s) \in [0, \infty).$$

Intuitively, in specifying $\tilde{\lambda}$, we are saying that in the next time interval $(t, t + \Delta t]$, the probability of there being a birth in a small region A is approximately $\int_A \tilde{\lambda}(x, \eta, \xi, t) dx \Delta t$. One can make this precise by requiring the process η to have the property that there is a filtration $\{\mathcal{G}_t\}$ such that

$$\eta_t(B) - \int_0^t \int_B \tilde{\lambda}(x, \eta, \xi, s) dx\, ds$$

is a $\{\mathcal{G}_t\}$-martingale for each $B \in \mathcal{A}(K)$. For this *martingale problem* to make sense, $\tilde{\lambda}$ must depend on η in a nonanticipating way, that is, $\tilde{\lambda}(x, \eta, \xi, s)$ depends on η_r only for $r \leq s$.

The representation (4.1) suggests formulating a stochastic equation by requiring η and a "stopping-surface-valued function" τ to satisfy

$$\begin{aligned} \tau(t, x) &= \int_0^t \tilde{\lambda}(x, \eta, \xi, s) ds \\ \eta_t &= N_{\tau(t)}. \end{aligned} \tag{5.1}$$

If $\tilde{\lambda}$ is appropriately nonanticipating,

$$\int_0^t \int_K \tilde{\lambda}(x, z, u, s) dx ds < \infty \text{ for all } z \in D_{\mathcal{N}(K)}[0, \infty), u \in E,$$

and ξ and N are independent, then the solution of (5.1) and verification of the martingale properties are straightforward. (Just "solve" from one birth to the next.) However, we are interested in situations, say with $K = \mathbb{R}^d$, in which

$$\int_0^t \int_{\mathbb{R}^d} \tilde{\lambda}(x, z, u, s) dx = \infty,$$

that is, there will be infinitely many births in a finite amount of time.

5.1. Existence and uniqueness of time-change equation

To keep the development as simple as possible, we will focus on $\tilde{\lambda}$ such that $\tilde{\lambda}(x, z, u, s) = \lambda(x, z_s)$, $\lambda : \mathbb{R}^d \times \mathcal{N}(\mathbb{R}^d) \to [0, \infty)$, that is, the intensity depends only on the current configuration of points.

In this setting, (5.1) becomes

$$\begin{cases} \dot\tau(t,x) = \lambda(x, N_{\tau(t)}) \\ \tau(0,x) = 0 \\ N_{\tau(t)}(B) = N(\Gamma_{\tau(t)} \cap B \times [0,\infty)) \\ \Gamma_{\tau(t)} = \{(x,y); x \in \mathbb{R}^d, 0 \le y \le \tau(t,x)\} \end{cases} \tag{5.2}$$

where we write the system in this form to emphasize the fact that τ is the solution of a random, but autonomous differential equation. For the earlier analysis to work, the solution must also have the property that $\Gamma_{\tau(t)}$ be a stopping set with respect to the filtration $\{\mathcal{F}_A\}$.

We need the following regularity condition.

Condition 5.1. For $\zeta \in \mathcal{N}(\mathbb{R}^d)$, $x \to \lambda(x,\zeta)$ is upper semicontinuous. For $\zeta_1, \zeta_2 \in \mathcal{N}(\mathbb{R}^d)$ and $\{y_i\} \subset \mathbb{R}^d$ satisfying $\zeta_2 = \zeta_1 + \sum_{i=1}^\infty \delta_{y_i}$ and $\lambda(x,\zeta_1) < \infty$,

$$\lambda(x,\zeta_2) = \lim_{n\to\infty} \lambda\left(x, \zeta_1 + \sum_{i=1}^n \delta_{y_i}\right). \tag{5.3}$$

The following theorem extends conditions of Liggett for models on a lattice (Liggett (1972), Kurtz (1980b)) to the present setting. Garcia (1995a) proves a similar theorem for a general case of birth and death processes. However, this theorem is not a particular case of the general case, since the conditions and techniques used there are not directly applicable for the case in which the death rate equals 0.

Theorem 5.2. *Assume that Condition* 5.1 *holds. Let*

$$a(x,y) = \sup_{\zeta \in \mathcal{N}(\mathbb{R}^d)} |\lambda(x, \zeta + \delta_y) - \lambda(x,\zeta)| \tag{5.4}$$

and $\bar a(x) = \sup_{\zeta_1,\zeta_2} |\lambda(x,\zeta_1) - \lambda(x,\zeta_2)|$. *Suppose there exists a positive function* c *such that* $\sup_x c(x)\bar a(x) < \infty$ *and*

$$M = \sup_x \int_{\mathbb{R}^d} \frac{c(x)a(x,y)}{c(y)} dy < \infty.$$

Then, there exists a unique solution of (5.2) *with* $\tau(t,\cdot)$ *a stopping surface for all* $t \ge 0$.

Remark 5.3. For example, suppose $a(x,y) = a(x-y)$ and $\int |y|^p a(y)\,dy < \infty$. Then we can take $c(x) = (1+|x|^p)^{-1}$. Setting $c_1 = \int |y|^p a(y) dy$ and $c_2 = \int a(y)dy$,

$$\begin{aligned} \int (1+|z|^p) a(x-z) dz &= \int a(x-z)dz + \int |z|^p a(x-z)dz \\ &= \int a(y)dy + \int |y-x|^p a(y)dy \\ &\le c_2 + a_p \int |y|^p a(y)dy + a_p|x|^p) \int a(y)dy \\ &\le c_2 + c_1 a_p + c_2 a_p |x|^p \le M(1+|x|^p), \end{aligned}$$

where a_p satisfies $|y-x|^p \leq a_p(|y|^p+|x|^p)$ and $M=\max\{c_2+c_1a_p, c_2a_p\}$. Consequently,

$$\sup_x \int \frac{(1+|z|^p)a(x-z)}{1+|x|^p} < M.$$

Outline of the proof: Let $\mathcal{H}$ be the space of real-valued, measurable processes indexed by $\mathbb{R}^d$, and let $\mathcal{H}_0$ be the subset of $\xi \in \mathcal{H}$ such that

$$\|\xi\| \equiv \sup_{x\in\mathbb{R}^d} c(x)\mathbb{E}[|\xi(x)|] < \infty.$$

Define $\mathcal{H}^+ = \{\xi \in \mathcal{H} : \xi \geq 0\}$, and similarly for $\mathcal{H}_0^+$. Let $F : \phi \in \mathcal{H}_+ \to \tilde{\phi} \in \mathcal{H}_+$, where $\tilde{\phi}(x) = \lambda(x, N_\phi)$. That is,

$$\Gamma_\phi = \{(x,y); x \in \mathbb{R}^d, 0 \leq y \leq \phi(x)\},$$

$$N_\phi(B) = N(\Gamma_\phi \cap B \times [0,\infty)),$$

and

$$F(\phi)(x) = \lambda(x, N_\phi). \tag{5.5}$$

Then the system (5.2) is equivalent to

$$\dot{\tau}(t) = F(\tau(t)). \tag{5.6}$$

We are only interested in solutions for which $\tau(t)$ is a stopping surface. Let τ_1 and τ_2 be two such solutions. We will show that F is Lipschitz, so that we can find estimates of $\|\tau_1(t)-\tau_2(t)\|$ in terms of $\int_0^t \|\tau_1(s)-\tau_2(s)\|ds$ and apply Gronwall's inequality to obtain the uniqueness of the solution. To prove existence, we are going to construct a sequence of stopping surfaces $\tau^{(n)}(t)$ whose limit is a solution of the system.

The proof of the theorem relies on several lemmas.

Lemma 5.4. *Let $\lambda : \mathbb{R}^d \times \mathcal{N}(\mathbb{R}^d) \to \mathbb{R}^+$ satisfy Condition* 5.1, *and define $a(x,y)$ as in* (5.4). *Then*

$$\sup_\zeta \left|\lambda(x, \zeta + \sum_{i=1}^m \delta_{y_i}) - \lambda(x,\zeta)\right| \leq \sum_{i=1}^m a(x,y_i) \tag{5.7}$$

$$\sup_\zeta \left|\lambda(x, \zeta + \sum_{i=1}^\infty \delta_{y_i}) - \lambda(x,\zeta)\right| \leq \sum_{i=1}^\infty a(x,y_i). \tag{5.8}$$

Proof. The definition of $a(x,y)$ and induction give (5.7), and (5.8) then follows by Condition 5.1. □

Lemma 5.5. *For γ_1 and γ_2 stopping surfaces,*

$$\mathbb{E}[|F(\gamma_1)(x) - F(\gamma_2)(x)|] \leq \int_{\mathbb{R}^d} a(x,y)\mathbb{E}[|\gamma_1(y)-\gamma_2(y)|]dy\,.$$

Proof. Since N is a Poisson point processes with a σ-finite mean measure, it has only countably many points almost surely. Let, $\{y_1, y_2, \ldots\}$ and $\{z_1, z_2, \ldots\}$ be such that $y_i \neq z_j$, for all i, j, and

$$N_{\gamma_2} = N_{\gamma_1} + \sum_i \delta_{y_i} - \sum_j \delta_{z_j}.$$

Then

$$T(N_{\gamma_1} - N_{\gamma_2}, B) = \sum_i \delta_{y_i}(B) + \sum_j \delta_{z_j}(B)$$

and

$$\begin{aligned}|\lambda(x,N_{\gamma_1}) - \lambda(x,N_{\gamma_2})| &= \Bigg|\lambda(x,N_{\gamma_1}) - \lambda\Big(x, N_{\gamma_1} + \sum_i \delta_{y_i}\Big) \\ &\quad + \lambda\Big(x, N_{\gamma_1} + \sum_i \delta_{y_i}\Big) - \lambda\Big(x, N_{\gamma_1} + \sum_i \delta_{y_i} - \sum_j \delta_{z_j}\Big)\Bigg| \\ \text{(by Lemma 5.4(b))} &\leq \sum_i a(x,y_i) + \sum_j a(x,z_j) = \int_{\mathbb{R}^d} a(x,y)T(N_{\gamma_1} - N_{\gamma_2}, dy).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{E}[|F(\gamma_1)(x) - F(\gamma_2)(x)|] &= \mathbb{E}[|\lambda(x, N_{\gamma_1}) - \lambda(x, N_{\gamma_2})] \\ &\leq \mathbb{E}\left[\int_{\mathbb{R}^d} a(x,y)\, T(N_{\gamma_1} - N_{\gamma_2}, dy)\right] \\ \text{by Corollary 3.13} &\leq \mathbb{E}\left[\int_{\mathbb{R}^d} a(x,y)\, |\gamma_1(y) - \gamma_2(y)|\, dy\right],\end{aligned}$$

and the result follows by Fubini's theorem. □

Lemma 5.6. *The mapping $F : \mathcal{H}_+ \to \mathcal{H}_+$ given by* (5.5) *is Lipschitz for stopping surfaces in $\mathcal{H}_+$.*

Proof. By Lemma 5.5,

$$\begin{aligned}\|F(\gamma_1) - F(\gamma_2)\| &= \sup_x c(x)\mathbb{E}[|F(\gamma_1)(x) - F(\gamma_2)(x)|] \\ &\leq \sup_x c(x) \int_{\mathbb{R}^d} a(x,y)\, \mathbb{E}[|\gamma_1(y) - \gamma_2(y)|]\, dy \\ &\leq \sup_x \int \frac{c(x)a(x,y)}{c(y)} dy\, \sup_y c(y)\mathbb{E}[|\gamma_1(y) - \gamma_2(y)|] \\ &\leq M\|\gamma_1 - \gamma_2\|.\end{aligned}$$

□

Proof of Theorem 5.2. *Uniqueness.* By Lemma 5.6 we have

$$\begin{aligned}\| \tau_1(t) - \tau_2(t) \| &= \sup_x c(x)\mathbb{E}[|\tau_1(t,x) - \tau_2(t,x)|] \\ &\leq \sup_x c(x)\mathbb{E}[\int_0^t |F(\tau_1(s))(x) - F(\tau_2(s))(x)|] \\ &\leq \int_0^t \|F(\tau_1(s)) - F(\tau_2(s))\| ds \\ &\leq M\int_0^t \|\tau_1(s) - \tau_2(s)\| ds.\end{aligned}$$

Note that $|F(\tau_1(s))(x) - F(\tau_2(s))(x)| \leq \bar{a}(x)$, so $\sup_s \|F(\tau_1(s)) - F(\tau_2(s))\| < \infty$. Consequently, uniqueness follows by Gronwall's inequality.

Existence. Let $\underline{\lambda}(x) = \inf_\zeta \lambda(x,\zeta)$. Then $|\lambda(x,\zeta) - \underline{\lambda}(x)| \leq \bar{a}(x)$. Define

$$\tau^{(n)}(t) = \int_0^t F(s\underline{\lambda})ds, \quad \text{for } 0 \leq t \leq 1/n \tag{5.9}$$

$$\tau^{(n)}(t) = \tau^{(n)}\left(\frac{k}{n}\right) + \int_{\frac{k}{n}}^t F\left(\tau^{(n)}\left(\frac{k}{n}\right) + \left(s - \frac{k}{n}\right)\underline{\lambda}\right) ds, \quad \text{for } \frac{k}{n} < t \leq \frac{k+1}{n}.$$

Set,

$$\gamma^{(n)}(t) = \tau^{(n)}\left(\frac{[nt]}{n}\right) + \left(t - \frac{[nt]}{n}\right)\underline{\lambda}.$$

Then,

$$\tau^{(n)}(t) = \int_0^t F(\gamma^{(n)}(s))ds. \tag{5.10}$$

Note that $\tau^{(n)}(t)$ and $\gamma^{(n)}(t)$ are stopping surfaces (see the proof below). Consequently,

$$\begin{aligned}\| \tau^{(n)}(t) - \gamma^{(n)}(t) \| &= \left\|\tau^{(n)}(t) - \tau^{(n)}([nt]/n) + \tau^{(n)}([nt]/n) - \gamma^{(n)}(t)\right\| \\ &= \left\|\int_{[nt]/n}^t F(\gamma^{(n)}(s))ds - (t - [nt]/n)\underline{\lambda}\right\| \\ &= \left\|\int_{[nt]/n}^t (F(\gamma^{(n)}(s)) - \underline{\lambda})ds\right\| \leq \int_{[nt]/n}^t \| F(\gamma^{(n)}(s)) - \underline{\lambda} \| ds \\ &\leq \int_{[nt]/n}^t \sup_x c(x)\mathbb{E}[|F(\gamma^{(n)}(s))(x) - \underline{\lambda}(x)|]ds \\ &\leq \int_{[nt]/n}^t \sup_x c(x)\bar{a}(x)ds \leq \frac{\sup_x c(x)\bar{a}(x)}{n}.\end{aligned}$$

Also,

$$\begin{aligned} \| \gamma^{(n)}(t) - \gamma^{(m)}(t) \| &\leq \| \tau^{(n)}(t) - \gamma^{(n)}(t) \| + \| \tau^{(m)}(t) - \gamma^{(m)}(t) \| \\ &\quad + \| \int_0^t F(\gamma^{(n)}(s))ds - \int_0^t F(\gamma^{(m)}(s))ds \| \\ &\leq 2\frac{\sup_x c(x)\bar{a}(x)}{n} + M\int_0^t \| \gamma^{(n)}(s) - \gamma^{(m)}(s) \| \, ds \\ \text{(by Gronwall's inequality)} &\leq 2\frac{\sup_x c(x)\bar{a}(x)}{n} e^{Mt}. \end{aligned}$$

Therefore,

$$\begin{aligned} \| \tau^{(n)}(t) - \tau^{(m)}(t) \| &\leq \| \tau^{(n)}(t) - \gamma^{(n)}(t) \| + \| \gamma^{(n)}(t) - \gamma^{(m)}(t) \| \\ &\quad + \| \gamma^{(m)}(t) - \tau^{(m)}(t) \| \\ &\leq 2\frac{\sup_x c(x)\bar{a}(x)}{n} + 2\frac{\sup_x c(x)\bar{a}(x)}{n} e^{Mt}. \end{aligned}$$

Then, fixing l, $\{\tau^{(n)}(t) - \tau^{(l)}(t)\}$ is a Cauchy sequence in $\mathcal{H}_0$. Completeness of $\mathcal{H}_0$ follows by standard arguments, and so there exists $\tau^*(t) \in \mathcal{H}_0$

$$\tau^*(t) = \lim_{n\to\infty} (\tau^{(n)}(t) - \tau^{(l)}(t)).$$

Then, taking $\tau(t) = \tau^*(t) + \tau^{(l)}(t) = \min_m \sup_{n\geq m} \tau^{(n)}(t)$, $\tau(t)$ is a stopping surface by Lemma 3.7,

$$\lim_{n\to\infty} \|\tau^{(n)}(t) - \tau(t)\| = \lim_{n\to\infty} \|\gamma^{(n)}(t) - \tau(t)\| = 0,$$

and since

$$\left\| \int_0^t F(\gamma^{(n)}(s)ds - \int_0^t F(\tau(s))ds \right\| \leq \int_0^t \|F(\gamma^{(n)}(s)) - F(\tau(s))\| ds \to 0,$$

$$\dot{\tau}(t) = F(\tau(t)).$$

Proof that $\tau^{(n)}(t)$ and $\gamma^{(n)}(t)$ are stopping surfaces. By definition, $\tau^{(n)}(t)$ is a stopping surface if and only if $\{\tau^{(n)}(t) \leq f\} \in \mathcal{F}_f$ for each nonnegative, upper-semicontinuous f.

(i) $0 \leq t \leq 1/n$

$$\tau^{(n)}(t) = \int_0^t F(s\underline{\lambda})ds \qquad \tau^{(n)}(t,x) = \int_0^t \lambda(x, N_{s\underline{\lambda}})ds$$

$$N_{s\underline{\lambda}}(B) = N(\Gamma_{s\underline{\lambda}} \cap B \times [0,\infty)) \qquad \Gamma_{s\underline{\lambda}} = \{(x,u); x \in \mathbb{R}^d, 0 \leq u \leq s\underline{\lambda}(x)\}.$$

By the measurability of λ, the mapping

$$(x, s, \omega) \in \mathbb{R}^d \times [0,t] \times \Omega \to \lambda(x, N_{s\underline{\lambda}})$$

is $\mathcal{B}(\mathbb{R}^d) \times \mathcal{B}([0,t]) \times \mathcal{F}_{t\underline{\lambda}}$-measurable, and hence, $(x,\omega) \to \tau^{(n)}(t,x)$ is $\mathcal{B}(\mathbb{R}^d) \times \mathcal{F}_{t\underline{\lambda}}$-measurable. By the completeness of $\mathcal{F}_{t\underline{\lambda}}$,

$$\{\tau^{(n)}(t) \leq f\}^c = \{\omega : \exists x \ni \tau^{(n)}(t,x) > f(x)\} \in \mathcal{F}_{t\underline{\lambda}}.$$

Note that $\tau^{(n)}(t,x) \geq \underline{\lambda}(x)t$ for all x, by the definition of $\underline{\lambda}$. Consequently, if $f \geq t\underline{\lambda}$, $\{\tau^{(n)}(t) \leq f\} \in \mathcal{F}_{t\underline{\lambda}} \subset \mathcal{F}_f$, and if $f(x) < t\underline{\lambda}(x)$ for some x, $\{\tau^{(n)}(t) \leq f\} = \emptyset \in \mathcal{F}_f$.

(ii) $k/n < t \leq (k+1)/n$.

Proceeding by induction, assume that $\tau^{(n)}(k/n)$ is a stopping surface. Then for $k/n \leq s < (k+1)/n$, by Lemma 3.12, $\gamma^{(n)}(s)$ is a stopping surface. By the definition of $\tau^{(n)}(t,x)$ we have

$$\tau^{(n)}(t,x) = \tau^{(n)}(k/n,x) + \int_{k/n}^{t} \lambda(x, N_{\tau^{(n)}(k/n)+(s-k/n)\underline{\lambda}})ds,$$

and by the definition of $\underline{\lambda}$

$$\tau^{(n)}(t) \geq \gamma^{(n)}(t)$$

Therefore, since $\{\tau^{(n)}(t) \leq f\} \in \mathcal{F}_{\gamma^{(n)}(t)}$

$$\{\tau^{(n)}(t) \leq f\} = \{\tau^{(n)}(t) \leq f\} \cap \{\gamma^{(n)}(t) \leq f\} \in \mathcal{F}_f. \qquad \square$$

An immediate consequence of the existence proof is:

Corollary 5.7. *Let $\tau^{(n)}(t)$ be defined by* (5.9) *and let*

$$\eta_t^n(B) = N_{\tau^n(t)} = N(\Gamma_{\tau^{(n)}(t)} \cap B \times [0,\infty)).$$

Then $\eta_t^n \to \eta_t$, in probability as $n \to \infty$, uniformly in $t \leq T$.

Another important characteristic of the solution of the time-change problem is that two births cannot occur at the same time.

Theorem 5.8. *Under the conditions of Theorem* 5.2*, for each $B \in \mathcal{A}(\mathbb{R}^d)$, the process $t \to \eta_t(B)$ is a counting process with intensity $\int_B \lambda(x,\eta_t)dx$. If $B \cap C = \emptyset$, then $\eta_t(B)$ and $\eta_t(C)$ have no simultaneous jumps.*

Proof. Since

$$M_B(D) = N(D \cap B \times [0,\infty)) - \int_{D\cap B\times[0,\infty)} du\,dx$$

and

$$M_C(D) = N(D \cap C \times [0,\infty)) - \int_{D\cap C\times[0,\infty)} du\,dx$$

are orthogonal martingales with respect to $\{\mathcal{F}_A\}$, that is M_B, M_C, and $M_B M_C$ are all martingales, The optional sampling theorem implies $M_B(\Gamma_{\tau(t)})$, $M_C(\Gamma_{\tau(t)})$, and $M_B(\Gamma_{\tau(t)})M_C(\Gamma_{\tau(t)})$ are all martingales with respect to the filtration $\{\mathcal{F}_{\tau(t)}, t \geq 0\}$. Noting that

$$M_B(\tau(t)) = \eta_t(B) - \int_0^t \int_B \tau(x,s)dxds$$

and

$$M_C(\tau(t)) = \eta_t(C) - \int_0^t \int_C \tau(x,s)dxds,$$

the result follows. $\square$

5.2. Equivalence between time change solution and the solution of the martingale problem

Usually, Markov processes are described through their infinitesimal generator. The pure birth process described in the beginning of this section can be characterized as the solution of the martingale problem corresponding to the generator defined by

$$AF(\zeta) = \int_{\mathbb{R}^d} (F(\zeta + \delta_y) - F(\zeta))\lambda(y, \zeta)\, dy, \tag{5.11}$$

for $F \in \mathcal{D}(A) = \{F : \mathcal{N}(\mathbb{R}^d) \to \mathbb{R}; F(\zeta) = \exp\{-\int g\, d\zeta\}, g \geq 0, g \in B_c(\mathbb{R}^d)\}$, where $B_c(\mathbb{R}^d)$ is the set of bounded functions on $\mathbb{R}^d$ with compact support. Our goal in this section is to prove that the two characterizations are equivalent, that is, the solution of the time change problem is the solution of the martingale problem and vice-versa. To avoid issues of integrability and explosion, we assume that for each compact $K \subset \mathbb{R}^d$, $\sup_{x \in K, \zeta \in \mathcal{N}(\mathbb{R}^d)} \lambda(x, \zeta) < \infty$.

Theorem 5.2 gives conditions for existence and uniqueness of random time changes in the strong sense, that is, given the process N. This result will imply the existence and uniqueness of the solution for the martingale problem. However, existence and uniqueness of the solution of the martingale problem implies existence and uniqueness of the time change solution in the weak sense.

Let N be a Poisson point process with Lebesgue mean measure in $\mathbb{R}^d \times [0, \infty)$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\lambda : \mathbb{R}^d \times \mathcal{N}(\mathbb{R}^d) \to \mathbb{R}$ be non-negative Borel measurable function. We are interested in solutions of the system:

$$\begin{cases} \tau(0, x) = 0 \\ \dot{\tau}(t, x) = \lambda(x, N_{\tau(t)}) \\ \Gamma_{\tau(t)} = \{(x, y); x \in \mathbb{R}^d, 0 \leq y \leq \tau(t, x)\} \\ N_{\tau(t)}(B) = N(\Gamma_{\tau(t)} \cap B \times [0, \infty)), \end{cases} \tag{5.12}$$

where $\tau(t, \cdot)$ is a stopping surface with respect to the filtration $\{\mathcal{F}_A, A \in \mathcal{C}(\mathbb{R}^d)\}$ as defined in Section 3.3.

Definition 5.9. An $\mathcal{N}(\mathbb{R}^d)$-valued process η is a *weak solution* of (5.12) if there exists a probability space $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$ on which are defined processes N^* and τ^* such that N^* is a version of N, (5.12) is satisfied with (N, τ) replaced by (N^*, τ^*), and $N^*_{\tau^*}$ has the same distribution as η.

Theorem 5.10. *Let N be Poisson point process in $\mathbb{R}^d \times [0, \infty)$ with Lebesgue mean measure defined in $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\lambda : \mathbb{R}^d \times \mathcal{N}(\mathbb{R}^d) \to \mathbb{R}$ be a non-negative Borel function, and let A be given by* (5.11)*.*

(a) *If τ and N satisfy* (5.12)*, then $\eta(t) = N_{\tau(t)}$ gives a solution of the martingale problem for A.*
(b) *If η is a $\mathcal{N}(\mathbb{R}^d)$-valued process which is a solution of the $D_{\mathcal{N}(\mathbb{R}^d)}[0, \infty)$-martingale problem for A, then η is a weak solution of* (5.12)*.*

Proof. (a) We want to prove that for each $F \in \mathcal{D}(A)$

$$M(t) = F(N_{\tau(t)}) - \int_0^t AF(N_{\tau(s)})ds$$

is a martingale with respect to the filtration $\{\mathcal{G}_t^\tau\} = \{\mathcal{F}_{\Gamma_{\tau(t)}}, t \geq 0\}$. We have as a direct consequence of Theorem 3.9 that for $B \in \mathcal{B}(\mathbb{R}^d)$ with compact closure,

$$M(B \times [0,t]) = M_t(B) = N_{\tau(t)}(B) - \int_B \tau(t,x)\,dx$$

is a martingale. Hence, for $g \in B_c(\mathbb{R}^d)$, that is, $g \geq 0$, bounded and with compact support, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d \times [0,t]} e^{-\int g(y)N_{\tau(s-)}(dy)}(e^{-g(x)} - 1)M(dx \times ds) \\
&= e^{-\int g(y)N_{\tau(t)}(dy)} - 1 \\
&\quad - \int_0^t \int_{\mathbb{R}^d} \lambda(x, N_{\tau(s)})(e^{-\int g(y)N_{\tau(s)}(dy)+g(x)} - e^{-\int g(y)N_{\tau(s)}(dy)})dx \\
&= F(N_{\tau(t)}) - 1 - \int_0^t AF(N_{\tau(s)})ds
\end{aligned}$$

is a martingale with respect to the filtration $\{\mathcal{G}_t^\tau\}$.

(b) Conversely, if η is a solution of the martingale problem, then there exists a filtration $\{\mathcal{G}_t\}$ such that for $B \in \mathcal{B}(\mathbb{R}^d)$ with compact closure,

$$\eta_t(B) - \int_0^t \int_B \lambda(x, \eta_s)ds$$

is a martingale with respect to $\{\mathcal{G}_t\}$.

Therefore, by the characterization theorem (Theorem 4.1) there exists a Poisson random measure N such that

$$\eta_t(B) = N(\Gamma_t \cap B \times [0, \infty))$$

where $\Gamma_t = \{(x,y); x \in \mathbb{R}^d, y \leq \int_0^t \lambda(x, \eta_s)ds\}$. □

5.3. Stationarity and ergodicity

In the classical theory of stochastic processes, stationarity and ergodicity play important roles in applications. For example, the ergodic theorem asserts the convergence of averages to a limit which is invariant under measure preserving transformations. In the case that the process is ergodic, the ergodic limit is a constant. Important applications arise in establishing consistency of non-parametric estimates of moment densities, and in discussing the frequency of specialized configurations of points (see Example 10.2(a), Daley and Vere-Jones (1988)).

In our case, we are interested in stationarity and invariance under translations (or shifts) in $\mathbb{R}^d$. Our objective is to prove that the process obtained by the time change transformation is stationary and spatially ergodic if λ is translation invariant.

Definition 5.11. λ is *translation invariant*, if for all $x, z \in \mathbb{R}^d, \zeta \in \mathcal{N}(\mathbb{R}^d)$,

$$\lambda(x+z,\zeta) = \lambda(x, S_z\zeta). \tag{5.13}$$

For a general discussion about stationary point processes, see Chapter 10 of Daley and Vere-Jones (1988).

We have a point process $N_{\tau(t)}$ defined on $\mathbb{R}^d$, and we would like to study invariance properties with respect to translations (or shifts) in $\mathbb{R}^d$. For arbitrary $x, z \in \mathbb{R}^d$ and $A \in \mathcal{B}(\mathbb{R}^d)$, write

$$T_x z = x + z \text{ and } T_x A = A + x = \{z + x; z \in A\}.$$

Then, T_x induces a transformation S_x of $\mathcal{N}(\mathbb{R}^d)$ through the equation

$$(S_x\zeta)(A) = \zeta(T_x A),\ \zeta \in \mathcal{N}(\mathbb{R}^d), A \in \mathcal{B}(\mathbb{R}^d). \tag{5.14}$$

Note that if $\zeta = \sum_i \delta_{x_i}$, then $S_x\zeta = \sum_i \delta_{x_i - x}$.

By Lemma 10.1.I (Daley and Vere-Jones (1988)), if $x \in \mathbb{R}^d$, the mapping $S_x : \mathcal{N}(\mathbb{R}^d) \to \mathcal{N}(\mathbb{R}^d)$ defined at (5.14) is continuous and one-to-one. And we have:

(i) $(S_x\delta_z)(\cdot) = \delta_{z-x}$ (where δ is the Dirac measure);
(ii) $\int f(z)(S_x\mu)(dz) = \int f(z)\mu(d(z+x)) = \int f(z-x)\mu(dz)$.

Definition 5.12. A point process ξ with state space $\mathbb{R}^d$ is *stationary* if, for all $u \in \mathbb{R}^d$, the finite-dimensional distributions of the random measures ξ and $S_u\xi$ coincide.

Let $\psi \in \mathcal{N}(\mathbb{R}^d \times [0,\infty))$, and let $\tau^{(n)}(t,\psi)$ be defined by (5.9) with the sample path of the Poisson process N replaced by the counting measure ψ. That is, with reference to (5.5), $F(\phi)(x)$ is replaced by $\lambda(x, \psi_\phi)$, and (5.10) becomes

$$\tau^{(n)}(t,x,\psi) = \int_0^t \lambda(x, \psi_{(\tau^{(n)}(\frac{[ns]}{n},\psi) + (s - \frac{[ns]}{n})\underline{\lambda})})ds. \tag{5.15}$$

For $n = 1, 2, \ldots$, define ξ_n by $\xi_n = \psi_{\tau^{(n)}(t,\psi)}$, that is,

$$\xi_n(B,t,\psi) = \psi(\Gamma_{\tau^{(n)}(t,\psi)} \cap B \times [0,\infty)). \tag{5.16}$$

Lemma 5.13. *If λ is translation invariant, then for $\psi \in \mathcal{N}(\mathbb{R}^d \times [0,\infty))$,*

$$\tau^{(n)}(t, x+z, \psi) = \tau^{(n)}(t, T_z x, \psi) = \tau^{(n)}(t, x, S_{(z,0)}\psi) \tag{5.17}$$

and

$$S_z\xi_n(\cdot, t, \psi) = \xi_n(\cdot, t, S_{(z,0)}\psi). \tag{5.18}$$

Proof. Since $\psi_\phi(B) = \psi\{(x,y) : x \in B, 0 \le y \le \phi(x)\}$,

$$\begin{aligned} S_z(\psi_\phi)(B) &= \psi\{(x,y) : x \in T_z B, 0 \le y \le \phi(x)\} \\ &= \psi\{(x,y) : x - z \in B, 0 \le y \le \phi(T_z(x-z))\} \\ &= S_{(z,0)}\psi\{(x,y) : x \in B, 0 \le y \le \phi(T_z x)\}, \end{aligned}$$

and (5.17) implies (5.18).

To verify (5.17), note that by (5.13),

$$\lambda(x+z,\psi_\phi)=\lambda(T_zx,\psi_\phi)=\lambda(x,S_z(\psi_\phi))=\lambda(x,[S_{(z,0)}\psi]_{\phi\circ T_z})$$

and that by translation invariance of λ, $\underline{\lambda}$ is constant. Then (5.15) gives

$$\begin{aligned}\tau^{(n)}(t,x+z,\psi) &= \int_0^t \lambda(x,S_z(\psi_{(\tau^{(n)}(\frac{[ns]}{n},\psi)+(s-\frac{[ns]}{n})\underline{\lambda})}))ds\\ &= \int_0^t \lambda(x,[S_{(z,0)}\psi]_{(\tau^{(n)}(\frac{[ns]}{n},T_z\cdot,\psi)+(s-\frac{[ns]}{n})\underline{\lambda})}))ds.\end{aligned}$$

Proceeding by induction, (5.17) trivially holds for $t=0$. Assume that it holds for $s\le k/n$. Then for $k/n\le t\le(k+1)/n$, we have

$$\begin{aligned}\tau^{(n)}(t,x+z,\psi) &= \int_0^t \lambda(x,[S_{(z,0)}\psi]_{(\tau^{(n)}(\frac{[ns]}{n},T_z\cdot,\psi)+(s-\frac{[ns]}{n})\underline{\lambda})}))ds\\ &= \int_0^t \lambda(x,[S_{(z,0)}\psi]_{(\tau^{(n)}(\frac{[ns]}{n},S_{(z,0)}\psi)+(s-\frac{[ns]}{n})\underline{\lambda})}))ds\\ &= \tau^{(n)}(t,x,S_{(z,0)}\psi),\end{aligned}$$

and the conclusion follows. □

Theorem 5.14. *Suppose that λ is translation invariant and that the solution of* (5.2) *is weakly unique. Then for each* $t\ge 0$, $N_{\tau(t)}$ *is stationary under spatial shifts.*

Remark 5.15. Weak uniqueness is the assertion that all solutions have the same distribution.

Under the assumptions of Theorem 5.2, the conclusion of the theorem follows by Lemma 5.13, the fact that the distribution of N is invariant under shifts, and the convergence of $N_{\tau^{(n)}(t)}$ to $N_{\tau(t)}$.

Proof. By Theorem 5.10, weak existence and uniqueness for the stochastic equation is equivalent to existence and uniqueness of solutions of the martingale problem. If λ is translation invariant and η is a solution of the martingale problem, then $S_z\eta$ is also a solution, so η and $S_z\eta$ must have the same distribution. □

In practice, the useful applications of the ergodic theorem are to those situations where the ergodic limit is a constant.

A stationary process is ergodic if and only if the invariant σ-algebra is trivial. Due to the independence properties of the Poisson process on $\mathbb{R}^d\times[0,\infty)$, it is ergodic for the measure preserving transformations $S_{(x,0)}$. That is, let $(\mathcal{N}(\mathbb{R}^d\times[0,\infty)),\mathcal{B}(\mathcal{N}(\mathbb{R}^d\times[0,\infty))),P)$ be the probability space corresponding to the underlying Poisson process N. Let $\mathcal{I}$ be σ-algebra of *invariant sets* under $S_{(x,0)}$,

$$\mathcal{I}=\{E\in\mathcal{B}(\mathcal{N}(\mathbb{R}^d\times[0,\infty)));P(S_{(x,0)}E\triangle E)=0,\forall x\in\mathbb{R}^d\}.$$

Then, $\mathcal{I}$ is a trivial σ-algebra, that is, $P(E)=0$ or 1 if $E\in\mathcal{I}$.

Under any conditions that allow us to write $\eta_t=H(t,N)$ for some $H:[0,\infty)\times\mathcal{N}(\mathbb{R}^d\times[0,\infty))\to\mathcal{N}(\mathbb{R}^d)$ so that $S_z\eta_t=H(t,S_{(z,0)}N)$, the spatial ergodicity of η_t follows from the ergodicity of N. Theorem 5.2 give such conditions.

Theorem 5.16. *Let λ be translation invariant, and suppose that the conditions of Theorem 5.2 are satisfied. Then for each $t \geq 0$, $N_{\tau(t)}$ is spatially ergodic.*

Proof. For $\tau^{(n)}(t, x, \psi)$ given by (5.15), $\tau^{(n)}(t)$ in (5.10) is given by $\tau^{(n)}(t, x, N)$, and

$$S_z N_{\tau^{(n)}(t)} = S_{(z,0)} N_{\tau^{(n)}(t, S_{(z,0)}N)}. \tag{5.19}$$

At least along a subsequence $\{n_k\}$, $\tau^{(n_k)}(t)$ converges almost surely to $\tau(t)$. Define $G(t, \psi) = \lim_{k\to\infty} \tau^{(n_k)}(t, \psi)$ if the limit exists, and define $G(t, \psi) \equiv 0$ otherwise. (Note that the collection of ψ for which the limit exists is closed under $S_{(z,0)}$, $z \in \mathbb{R}^d$.) Then $\tau(t) = G(t, N)$ almost surely. Define $H(t, \psi) = \psi_{G(t,\psi)}$, and by (5.19), $S_z H(t, \psi) = H(t, S_{(z,0)}\psi)$. Finally, $N_{\tau(t)} = H(t, N)$, and the theorem follows. □

6. Birth and death processes – constant birth rate, variable death rate

6.1. Gibbs distribution

Consider a spatial point process on $K \subset \mathbb{R}^d$ given by a Gibbs distribution corresponding to a pairwise interaction potential $\rho(x_1, x_2) \geq 0$. The process in which we are interested has a distribution that is absolutely continuous with respect to the spatial Poisson process with mean measure λm on K, with Radon-Nikodym derivative

$$\begin{aligned} L(\zeta) &= C \exp\left\{ -\frac{1}{2} \left[\int\int \rho(x,y)\zeta(dx)\zeta(dy) - \int \rho(x,x)\zeta(dx) \right] \right\} \\ &= C \exp\left\{ -\sum_{i<j} \rho(x_i, x_j) \right\}, \end{aligned}$$

where $x_1, x_2, \ldots$ are the locations of the point masses in $\zeta \in \mathcal{N}(K)$ and C is a normalizing constant depending only on λ and ρ.

The usual approach to simulating this process is to first identify a spatial birth-death process for which the desired Gibbs distribution is the stationary distribution and then to simulate the birth-death process over a "sufficiently long" time. A significant difficulty with this approach is the need to know what "sufficiently long" is. It is desirable to find a new approach for determining when to terminate the simulation or, if possible, to design a perfect simulation scheme.

There are a variety of birth-death processes which give the same stationary distribution. Consider the process in which points are "born" at a rate λ uniformly over the region, that is, the probability of a birth occurring in a region of area ΔA in a time interval of length Δt is approximately $\lambda\,\Delta A\,\Delta t$. The intensity for the death of a point at x is $\exp\{\sum_i \rho(x, x_i)\}$, where the sum is over all points other than the one at x.

The generator for the process takes the form

$$Af(\zeta) = \int_K (f(\zeta + \delta_y) - f(\zeta))\lambda\, dy + \sum_i (f(\zeta - \delta_{x_i}) - f(\zeta))e^{\sum_{j\neq i}\rho(x_i,x_j)}.$$

To see that the Gibbs distribution is the stationary distribution of this process, let ξ be a Poisson process on K with mean measure λm and apply (2.1) to obtain

$$\begin{aligned}\mathbb{E}&\left[\int_K (f(\xi - \delta_x) - f(\xi))e^{\int \rho(x,y)\xi(dy)-\rho(x,x)}\xi(dx)L(\xi)\right]\\ &= \mathbb{E}\left[\int_K (f(\xi) - f(\xi + \delta_x))e^{\int \rho(x,y)\xi(dy)}L(\xi+\delta_x)\lambda dx\right]\\ &= \mathbb{E}\left[\int_K (f(\xi) - f(\xi + \delta_x))\lambda dx L(\xi)\right]\end{aligned}$$

which implies

$$\int Af(\zeta)L(\zeta)\eta_\lambda(d\zeta) = 0$$

where η_λ is the distribution of the Poisson process with mean measure λm on K. It follows by Echeverria's theorem (Ethier and Kurtz (1986), Theorem 4.9.17) that $L(\zeta)\eta_\lambda(d\zeta)$ is a stationary distribution for the birth-death process. Uniqueness follows by a regeneration argument (see Lotwick and Silverman (1981)).

The following problem was proposed and solved by Kurtz (1989) for the case when K is compact. One motivation for this work was to solve the following problem for infinite regions K, particularly $K = \mathbb{R}^d$. However, we could not obtain the desired result and a slightly different problem was considered. Our hope is that the same technique can be applied to obtain the general result.

6.2. Embedding of birth and death process in Poisson process

Given a Poisson process on $K \times [0,\infty)^2$, we want to construct a family of random sets Γ_t in such way that the birth-death process is obtained by projecting the points of a Poisson process lying in Γ_t onto $\mathbb{R}^d$. We use a Poisson process N on $K\times[0,\infty)\times[0,\infty)$ with mean measure $m^d\times m\times e$ (m^d is Lebesgue measure on $\mathbb{R}^d$, m is Lebesgue measure on $[0,\infty)$, e is the exponential distribution on $[0,\infty)$). For $f : \mathbb{R}^d \times [0,\infty) \to [0,\infty)$ and $t \geq 0$, let $\mathcal{F}_{(f,t)}$ be the completion of the σ-algebra generated by $N(A)$, where either $A \in \mathcal{B}(K \times [0,\infty)^2)$ and $A \subset \{(x,y,s) : s \leq f(x,y), y \leq \lambda t\}$ or $A = A_1 \times [0,\infty)$ with $A_1 \in \mathcal{B}(K \times [0,\lambda t])$.

Let

$$\alpha(x,\zeta) = e^{\int \rho(x,z)\,\zeta(dz)} \tag{6.1}$$

with $\rho(x,x) = 0$, and consider the following system:

$$\begin{cases} \tau(x,y,t) = 0, & \text{if } \lambda t < y\\ \dot\tau(x,y,t) = \alpha(x,N_t), & \text{if } \lambda t \geq y\\ \Gamma_t = \{(x,y,s); x\in K, 0\leq y\leq \lambda t, s > \tau(x,y,t)\} & \\ N_t(B) = N(\Gamma_t \cap B\times\mathbb{R}\times[0,\infty)) & \end{cases} \tag{6.2}$$

where $(\tau(\cdot,t),t)$ is a stopping time with respect to the filtration $\{\mathcal{F}_{(f,t)}\}$ in the sense that $\{\tau(\cdot,t) \leq f, t \leq r\} \in \mathcal{F}_{(f,r)}$.

Interpretation. Each point (x,y,s) of N corresponds to an "individual" who is born at time y/λ and is located at x. The individual dies at time t satisfying $\tau(x,y,t) = s$. Note that in N, conditioned on the (x,y)-coordinates, the s-coordinates are independent and exponentially distributed random variables. Consequently, the probability that a point at x which was born at time y/λ and is still alive at time t, dying in the interval $(t, t+\Delta t)$ is approximately $\alpha(x, N_t)\Delta t$.

6.3. Backwards simulation of Gibbs processes

Fix $T > 0$, and consider the following modification of the system (6.2):

$$\begin{cases} \tau_T(x,y,t) = 0, & \text{if } \lambda(t-T) < y \\ \dot{\tau}_T(x,y,t) = \alpha(x, N_t^T), & \text{if } \lambda(t-T) \geq y \\ \Gamma_t^T = \{(x,y,s); x \in K, -\lambda T \leq y \leq \lambda(t-T), s > \tau_T(x,y,t)\} & \\ N_t^T(B) = N(\Gamma_t^T \cap B \times \mathbb{R} \times [0,\infty)) & \end{cases} \tag{6.3}$$

Note that the system (6.3) is essentially the same as (6.2), except that it is defined using the Poisson process on $K \times [-\lambda T, \infty) \times [0,\infty)$ while the system in (6.2) uses the Poisson process on $K \times [0,\infty) \times [0,\infty)$. That is, the construction is simply shifted to the left by λT. In particular, N_t^T has the same distribution as N_t. A regeneration argument shows that N_T converges in distribution as $T \to \infty$ to the desired Gibbs process. The process N_T^T converges almost surely as $T \to \infty$, since for almost every ω there exists a T_0 such that N_T^T is fixed for $T > T_0$. To see this, let $H_t = \{(x,y,s); x \in K, y \leq -\lambda t, s \geq t - y/\lambda\}$, and let T_0 be the smallest $t > 0$ such that $N(H_t) = 0$, ($T_0 < \infty$ with probability 1). Note that $\Gamma_t^T \subset H_{T-t}$ so that for $T > T_0$, $N_{T-T_0}^T = 0$.

Notice that "backward simulation" is the key idea behind the original CFTP (Coupling from the Past) algorithm proposed by Propp and Wilson (1996) and all related work on perfect simulation. However, the basic CFTP algorithm, sometimes called *vertical CFTP*, is in general not applicable to processes with infinite state space. To deal with this situation, Kendall (1997 and 1998) introduced *dominated CFTP* (also called *horizontal CFTP* and *coupling into and from the past*). This extension also requires the state space to have a partial order, as well as the existence of a monotone coupling among the target process and two reversible *sandwiching processes*, which must be easy to sample. Algorithms of this type are available for attractive point processes and, through a minor modification, also for repulsive point processes (Kendall, 1998). Similarly, Häggström, van Lieshout and Møller (1999) combined ideas from CFTP and the two-component Gibbs sampler to perfectly simulate from processes in infinite spaces which do not have maximal (or minimal) elements.

In our case, we sample directly from a time stationary realization of the process. There is no coalescence criterion, either between coupled realizations or between sandwiching processes. The scheme neither requires nor takes advantage of monotonicity properties. Our construction has the same spirit as the clan of ancestors algorithm proposed by Fernández, Ferrari and Garcia (2002) where the stopping time T_0 at which we know that the invariant measure is achieved is a regeneration time for the process.

The existence and uniqueness of the process for $\mathbb{R}^d$ is obtained by refining the arguments of Section 5 (see Garcia, 1995a), but convergence to the invariant measure is not at all clear. The above backward argument does not work for the infinite case since $T_0 = \infty$ with probability 1. At this point we have no general results for $\mathbb{R}^d$; however, we hope that some of the theory developed here may be useful in giving such results.

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