# Martingale problems and linear programs for singular control *

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**Abstract**

Using a controlled Jackson network as the primary example, stochastic control problems are formulated as martingale problems and the optimal solutions are characterized as the solutions of infinite dimensional linear programs. Assuming a heavy traffic scaling, the analogous linear program for a singular control problem is derived, and the solution of the linear program is shown to give the stochastic model minimizing the long-run average cost.

## 1 The model.

We consider a controlled Jackson network with $m$ stations, external arrival rates $v_1, \ldots, v_m$ service rates $u_1, \ldots, u_m$, and routing probabilities $p_{ij}$. We assume that the routing probabilities are fixed but that we can control the arrival and service rates subject, perhaps, to certain constraints.

We can formulate the model as the solution of a system of stochastic equations:

$$Q(t) = Q(0) + Y_i(\int_0^t V_i(s)ds) + \sum_{j=1}^m Y_{ji}(p_{ji}\int_0^t I_{\{Q_j(s)>0\}}U_j(s)ds) - \sum_{j=1}^m Y_{ij}(p_{ij}\int_0^t I_{\{Q_i(s)>0\}}U_i(s)ds),$$

where $\{Y_i\}$ and $\{Y_{ij}\}$ are independent, unit Poisson processes, $\{V_i\}$ are the controlled arrival rates, and $\{U_i\}$ are the controlled service rates.

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The generator of the model has the form

$$
\begin{aligned}
Af(q,u,v) &= \sum_{i=1}^{m} v_i(f(q+e_i)-f(q)) + \sum_{i=1}^{m} u_i p_{i0} I_{\{q_i>0\}}(f(q-e_i)-f(q)) \\
&\quad + \sum_{i,j=1}^{m} u_i p_{ij} I_{\{q_i>0\}}(f(q+e_j-e_i)-f(q)),
\end{aligned}
$$

where, for $i=1,\ldots,m$, $e_i$ denotes the vector with $i$th component 1 and all others 0. Take the domain of $A$, $\mathcal{D}(A)$, to be the collection of bounded functions on $\mathbb{Z}_0^m$, $\mathbb{Z}_0=\{0,1,2,\ldots\}$, and assume that there is a constant such that the controlled rates must satisfy $U_i, V_i \le K$. Then, assuming the $U_i$ and $V_i$ are nonanticipating in an appropriate sense, any solution of the system of equations has the property that for every $f\in\mathcal{D}(A)$,

$$
f(Q(t)) - \int_0^t Af(Q(s),U(s),V(s))ds \tag{1.1}
$$

is a martingale with respect to a filtration independent of $f$.

We require

$$
\begin{gathered}
0 \le v_i \le K, 0 \le u_i \\
\sum_i \alpha_{ki} u_i \le 1, \quad k=1,\ldots,k_0
\end{gathered} \tag{1.2}
$$

and

$$
\sum_i \beta_{li} v_i \ge 1, \quad l=1,\ldots,l_0, \tag{1.3}
$$

where $\alpha_{ki}, \beta_{li} \ge 0$. We assume that for each $i$, there exists at least one $k$ such that $\alpha_{ki}>0$, and, hence, $u_i \le 1/\alpha_{ki}$. Let $P=((p_{ij}))_{1\le i,j\le m}$ be the matrix of transition probabilities where $p_{ij}$ is the probability that a customer of class $i$ moves to class $j$ after completing service. $p_{i0}=1-\sum_{j=1}^m p_{ij}$ is the probability that the customer leaves the system. We assume that the system is open so that

$$
(I-P)^{-1} = \sum_{k=0}^{\infty} P^k
$$

exists.

Let $\mathbf{1}$ be the vector all of whose components are ones. We can write the constraints in (1.2) and (1.3) as $\alpha u \le \mathbf{1}$ and $\beta v \ge \mathbf{1}$, where $\alpha$ is the matrix with rows $\alpha_k=(\alpha_{k1},\ldots,\alpha_{km})$, $k=1,\ldots,k_0$, and $\beta$ is the matrix with rows $\beta_l=(\beta_{l1},\ldots,\beta_{lm})$. Define

$$
\Gamma_\beta = \{v\in[0,K]^m : \beta v \ge \mathbf{1}\}, \quad \Gamma^\alpha = \{u\in[0,\infty)^m : \alpha u \le \mathbf{1}\},
$$

that is, $\Gamma_\beta$ is the collection of admissible external arrival rates and $\Gamma^\alpha$ is the collection of admissible service allocations.

In some examples, it may be more natural to state arrival constraints as equalities rather than inequalities. For example, the external arrival rate into a queue may be fixed. Ordinarily, however, replacing some of the inequalities by equalities will not affect the analysis.

This model includes the Markov version of the model considered by Laws [8]. In his model, job types have fixed routes, so $p_{ij}$ is 0 or 1, but models with more general routing probabilities can be reformulated as models with fixed routes. In his model, a finite collection of servers allocate effort among disjoint collections $J_k$ of job classes, which, in our notation, gives constraints of the form

$$\sum_{i\in J_k} \frac{u_i}{\mu_i} \le 1, \quad k = 1, \ldots, k_0,$$

that is, for $i \in J_k$, $u_i/\mu_i$ is the fraction of the effort of server $k$ that is devoted to job class $i$.

The model also includes special cases of the Markov, continuous time analog of the model considered by Martins and Kushner [10]. Some of their examples allow control of the $\{p_{ij}\}$.

## 1.1 Example: Two queues with alternate routing.

(See [10].) Assume that there are two queues that can serve customers at maximum rates $\mu_1$ and $\mu_2$ respectively. There are three arrival streams with rates $\kappa_1$, $\kappa_2$, and $\kappa_3$. All traffic from the first stream enters queue I, all traffic from the second stream enters queue II, and the traffic from the third stream can be routed to either queue. The constraints become

$$v_1 \ge \kappa_1, \quad v_2 \ge \kappa_2, \quad v_1 + v_2 \ge \kappa_1 + \kappa_2 + \kappa_3$$

and

$$u_1 \le \mu_2, \qquad u_2 \le \mu_1.$$

Of course, these inequalities can be normalized to be of the form above by dividing through by the right side.

## 1.2 Example: Two queues with reassignable server.

(See [1, 4, 12].) Assume that there are two queues with arrival rates $\kappa_1$ and $\kappa_2$ and two servers. The first server can only serve customers in the first queue and serves these customers at rate $\mu_1$. The second server can serve customers in the second queue at rate $\mu_2$ and/or customers in the first queue at rate $\mu_3$. More precisely, if the second server allocates a fraction $\theta$ of its effort to the second queue and the remaining fraction $(1-\theta)$ to the first queue, then it serves customers in the second queue at rate $\theta\mu_2$ and customers in the first queue at rate $(1-\theta)\mu_3$. The constraints then become

$$v_1 \ge \kappa_1, \qquad v_2 \ge \kappa_2$$

and

$$u_1 \le \mu_1 + \mu_3, \quad u_2 \le \mu_2, \quad \frac{u_1}{\mu_3} + \frac{u_2}{\mu_2} \le \frac{\mu_1}{\mu_3} + 1.$$

Again, the inequalities can be normalized by dividing through by the right side.

## 1.3 Example: Two types of work and two servers.

Assume that there are two types of work that arrive in the system at rates $\kappa_1$ and $\kappa_2$. There are two servers. The first server can process Type 1 work at rate $\mu_1$ and Type 2 work at rate

$\mu_2$. The second server can process Type 1 work at rate $\mu_3$ and Type 2 work at rate $\mu_4$. The work can by allocated to the servers in any manner. The system has four customer classes and the corresponding constraints are

$$v_1 + v_3 \geq \kappa_1, \qquad v_2 + v_4 \geq \kappa_2$$

$$\frac{u_1}{\mu_1} + \frac{u_2}{\mu_2} \leq 1, \qquad \frac{u_3}{\mu_3} + \frac{u_4}{\mu_4} \leq 1\ .$$

# 2 Conditions for stability.

Let $\lambda_j(v)$ be the total arrival rate at node $j$ assuming that the external arrival rates are given by the vector $v$. Then

$$\lambda_j(v) = v_j + \sum_i \lambda_i(v) p_{ij}, \tag{2.4}$$

that is

$$\lambda(v) = (I - P^T)^{-1} v,$$

where $P^T$ denotes the transpose of the matrix $P = ((p_{ij}))_{1 \leq i,j \leq m}$. Assume that there exists $v \in \Gamma_\beta$ such that

$$\sum_i \alpha_{ki} \lambda_i(v) < 1, \quad k = 1, \ldots, k_0.$$

Then there is an allocation of effort that stabilizes the system. For example, select $u_i > \lambda_i(v)$ satisfying

$$\sum_i \alpha_{ki} u_i \leq 1, \quad k = 1, \ldots, k_0.$$

Recall that if we fix the arrival rates and the service allocations at $v$ and $u$, then the stationary expectations become

$$E[Q_i] = \frac{\lambda_i(v)}{u_i - \lambda_i(v)}.$$

## 2.1 Linear program for long-run average costs.

Let $c(q, u, v)$ be a nonnegative, lower semicontinuous function. The long-run average cost problem is to select $U$ and $V$ to minimize

$$J(U, V) = \limsup_{t \to \infty} E\left[\frac{1}{t}\int_0^t c(Q(s), U(s), V(s)) ds\right].$$

Note that for bounded $f$, the fact that (1.1) is a martingale implies

$$\lim_{t \to \infty} E\left[\frac{1}{t}\int_0^t Af(Q(s), U(s), V(s)) ds\right] = 0.$$

Define

$$\mu_t(C \times D) = E\left[\frac{1}{t}\int_0^t I_C(Q(s)) I_D(U(s), V(s)) ds\right].$$

If finiteness of $J(U,V)$ implies tightness of $\{\mu_t\}$, for example, if $c(q,u,v)=\sum_{i=1}^m q_i$ (recall that $\Gamma^\alpha$ and $\Gamma_\beta$ are compact), then any limit point $\mu$ satisfies

$$\int_{\mathbb{Z}_0^m\times\Gamma^\alpha\times\Gamma_\beta} Af(q,u,v)\mu(dq\times du\times dv)=0,\quad f\in\mathcal{D}(A), \tag{2.5}$$

and

$$\int_{\mathbb{Z}_0^m\times\Gamma^\alpha\times\Gamma_\beta} c(q,u,v)\mu(dq\times du\times dv)\le J(U,V).$$

Consequently, we formulate the following linear programming problem: Minimize

$$\int_{\mathbb{Z}_0^m\times\Gamma^\alpha\times\Gamma_\beta} c(q,u,v)\mu(dq\times du\times dv) \tag{2.6}$$

subject to (2.5).

The relationship between controlled Markov processes and linear programming was recognized by Manne [9] in the context of finite Markov chains and developed by Stockbridge [11] for general Markov processes.

**Theorem 2.1** *Suppose that for each $K>0$,*

$$\{\mu:\int_{E\times F} c(q,u,v)\mu(dq\times du\times dv)\le\bar{c},\mu \text{ satisfies (2.5)}\} \tag{2.7}$$

*is compact and is nonempty for $\bar{c}$ sufficiently large. Then there exists a solution of the linear programming problem.*

It is not immediately obvious that the solution of the linear programming problem corresponds to the optimal solution of the stochastic control problem. The next result, which is a consequence of more general theorems in [6] or [2], ensures that it does.

**Theorem 2.2** *Let $\mu\in\mathcal{P}(\mathbb{Z}_0^m\times\Gamma^\alpha\times\Gamma_\beta)$ satisfy (2.5), and suppose $\mu(dq\times du\times dv)=\mu_0(dq)\eta(q,du\times dv)$. Then there exists a stationary solution of the controlled martingale problem for $A$ such that $U(s)=\int_{\Gamma^\alpha}u\eta(Q(s),du\times\Gamma_\beta)$, $V(s)=\int_{\Gamma_\beta}v\eta(Q(s),\Gamma^\alpha\times dv)$, and*

$$E[\int h(Q(s),U(s),V(s))]=\int h(q,u,v)\mu(dq\times du\times dv),$$

*for every bounded, measurable $h$. In particular, the long-run average cost is given by (2.6).*

**Remark 2.3** *The general results in [11, 2, 6] require the use of relaxed controls. Relaxed controls are not needed here because the control enters the generator linearly.*

# 3 Heavy traffic limit.

There are two approaches to defining what is meant by a heavy traffic limit for the controlled system. One, which we will refer to as the *perturbed rate model*, is to fix nominal external arrival rates $\gamma_i$ and then to require that the controlled external arrival and service rates be small perturbations of the nominal external arrival and total arrival rates. In terms of a scaling parameter $n$, we write

$$v_i = \gamma_i - \frac{1}{\sqrt{n}} v_i^0, \quad u_i = \lambda_i(\gamma) + \frac{1}{\sqrt{n}} u_i^0, \tag{3.8}$$

where

$$(u^0, v^0) \in F^n \equiv \{(u^0, v^0) : \lambda(\gamma) + \frac{1}{\sqrt{n}} u^0 \in \Gamma^\alpha, \gamma - \frac{1}{\sqrt{n}} v^0 \in \Gamma_\beta\}. \tag{3.9}$$

The magnitudes of $v^0$ and $u^0$ are constrained by penalizing the deviations from $\gamma$ and $\lambda(\gamma)$. For example, take the running cost to be

$$c(q, u, v) = \frac{1}{\sqrt{n}} \sum_{i=1}^m q_i + \sum_{i=1}^m (|v_i^0| + |u_i^0|).$$

This approach is essentially the same as that taken in [10, 3].

The second approach, which we will refer to as the *constrained rate model,* is to assume that the constraints asymptotically determine unique nominal arrival rates. (See, for example, [8, 5, 4, 1, 12].) To formulate a version of this approach, we keep the $p_{ij}$ and $\alpha_{ki}$ fixed and let the $\beta_{li}^n$ vary with a normalizing parameter $n$. We assume that $\beta_{li}^n$ decreases to a limit $\beta_{li}$ such that there is a unique $\gamma \in \Gamma_\beta$ satisfying $\lambda(\gamma) \in \Gamma^\alpha$. Uniqueness implies that for $i = 1, \ldots, m$, we must have

$$\sum_{\{k:\alpha_k \cdot \lambda(\gamma)=1\}} \alpha_k (I - P^T)^{-1} e_i > 0, \tag{3.10}$$

and

$$\gamma_i = 0 \quad \text{or} \quad \sum_{\{l:\beta_l \cdot \gamma=1\}} \beta_{li} > 0. \tag{3.11}$$

In addition, assume that for some $b > 0$,

$$\beta_{li} \leq \beta_{li}^n \leq (1 + \frac{b}{\sqrt{n}})\beta_{li} \tag{3.12}$$

(in particular, $\beta_{li} = 0$ if and only if $\beta_{li}^n = 0$) and

$$\lim_{n\to\infty} \sqrt{n}(\beta_{li}^n - \beta_{li}) = \beta_{li}^0. \tag{3.13}$$

## 3.1 Asymptotic stability.

If we fix $v^0$ and $u^0$ in (3.8), then the stationary expectation satisfies

$$\frac{1}{\sqrt{n}} E[Q_i] = \frac{\lambda_i(\gamma - \frac{1}{\sqrt{n}} v^0)}{\sqrt{n}(\lambda_i(\gamma) + \frac{1}{\sqrt{n}} u_i^0 - \lambda_i(\gamma - \frac{1}{\sqrt{n}} v^0)} \to \frac{\lambda_i(\gamma)}{u_i^0 + \lambda_i(v^0)},$$

where $\lambda(v^0) = (I - P^T)^{-1} v^0$. Consequently, for the perturbed rate model, we have asymptotic stability if we can select $(u^0, v^0) \in F^n$ so that $u^0 + \lambda(v^0) > 0$.

Next, consider the constrained rate model. By the assumptions on $\beta^n_{li}$, $\beta^0_{li} \geq 0$. If $\beta^0 > 0$, then there exists a sequence $\gamma^n \in \Gamma_{\beta^n}$ such that $\gamma^n \to \gamma$ and

$$\sum_i \alpha_{ki}\lambda_i(\gamma^n) < 1, \quad k = 1, \ldots, k_0,$$

for each $n$. More generally, suppose that there exists $c > 0$ and $v^* \in [0, \infty)^m$ such that

$$\beta v^* \leq \beta^0\gamma + c(\beta\gamma - \mathbf{1}), \qquad v^* \leq c\gamma, \tag{3.14}$$

and that there exists $\epsilon > 0$ such that

$$\alpha\lambda(v^*) \geq \epsilon\alpha\lambda(\gamma). \tag{3.15}$$

Without loss of generality, we can assume $\beta^0 v^* \leq c\mathbf{1}$. Note that if $\beta^0 > 0$, then the existence of $v^*$ satisfying (3.14) and (3.15) is immediate. (Take $v^* = \epsilon\gamma$, for $\epsilon > 0$ sufficiently small.)

By (3.13), there exists $\rho_n < 1$ satisfying $\rho_n \to 1$ and

$$\sqrt{n}(\beta^n - \beta) \geq \rho_n\beta^0.$$

For $n > c^2$, define

$$\gamma^n = (1 + \frac{c}{n})\gamma - \frac{\rho_n}{\sqrt{n}}v^*.$$

Then

$$\begin{aligned}
\beta^n\gamma^n &= (\beta_n - \beta)((1 + \frac{c}{n})\gamma - \frac{\rho_n}{\sqrt{n}}v^*) + (1 + \frac{c}{n})\beta\gamma - \frac{\rho_n}{\sqrt{n}}\beta v^* \\
&\geq \frac{\rho_n}{\sqrt{n}}\beta^0((1 + \frac{c}{n})\gamma - \frac{\rho_n}{\sqrt{n}}v^*) - \frac{\rho_n}{\sqrt{n}}\beta^0\gamma^T + (1 + \frac{c}{n} - \frac{\rho_n}{\sqrt{n}}c)\beta\gamma + \frac{\rho_n}{\sqrt{n}}c\mathbf{1} \\
&= \frac{c\rho_n}{n^{3/2}}\beta^0\gamma - \frac{\rho_n^2}{n}\beta^0 v^* + (1 + \frac{c}{n} - \frac{\rho_n}{\sqrt{n}}c)\beta\gamma + \frac{\rho_n}{\sqrt{n}}c\mathbf{1} \\
&\geq (1 - \frac{\rho_n}{\sqrt{n}}c)\beta\gamma + \frac{\rho_n}{\sqrt{n}}c\mathbf{1} \\
&\geq \mathbf{1},
\end{aligned}$$

so $\gamma^n \in \Gamma_{\beta^n}$. We have

$$\lambda(\gamma^n) = (I - P^T)^{-1}((1 + \frac{c}{n})\gamma - \frac{\rho_n}{\sqrt{n}}v^*) = (1 + \frac{c}{n})\lambda(\gamma) - \frac{\rho_n}{\sqrt{n}}\lambda(v^*),$$

and by (3.15), for $n$ sufficiently large,

$$\frac{c}{n}\alpha\lambda(\gamma) < \frac{\rho_n}{\sqrt{n}}\alpha\lambda(v^*).$$

It follows that

$$\alpha\lambda(\gamma^n) \leq \mathbf{1} - \frac{\rho_n}{\sqrt{n}}\alpha\lambda(v^*) + \frac{c}{n}\lambda(\gamma) < \mathbf{1},$$

that is, the $n$th system is stable. In particular, we can take the service allocation to be of the form

$$u^n = \lambda(\gamma^n) + \frac{\delta^n}{\sqrt{n}}\mathbf{1}, \tag{3.16}$$

where

$$\delta^n = \min_k\{\rho_n \alpha_k \cdot \lambda(v^*) - \frac{c}{\sqrt{n}}\alpha_k \cdot \lambda(\gamma)\} \to \delta = \min_k \alpha_k \cdot \lambda(v^*).$$

Furthermore,

$$\lim_{n\to\infty} \sqrt{n}(\lambda(\gamma) - \lambda(\gamma^n)) = \lambda(v^*).$$

Consequently, defining

$$Z^n(t) = \frac{Q^n(nt)}{\sqrt{n}},$$

if we fix the external arrival rates as $\gamma_n$ and the allocation policy as in (3.16), then the stationary means of $Z^n$ satisfy

$$E[Z_i^n] = \frac{\lambda_i(\gamma^n)}{\delta^n} \to \frac{\lambda_i(\gamma)}{\delta}. \tag{3.17}$$

## 3.2 Limiting linear program.

We derive the limiting linear program for the perturbed rate model. The constrained rate model is more delicate and is a subject for future research.

Adding and subtracting partial derivatives where convenient, the generator for $Z^n$ can be written

$$\begin{aligned}
&A^n f(z, u^0, v^0) \\
&= \sum_{i=1}^m (\gamma_i - \frac{1}{\sqrt{n}} v_i^0) n(f(z + \frac{1}{\sqrt{n}} e_i) - f(z) - \frac{1}{\sqrt{n}} \partial_i f(z)) \\
&\quad + \sum_{i=1}^m (\lambda_i(\gamma) + \frac{1}{\sqrt{n}} u_i^0) p_{i0} n(f(z - \frac{1}{\sqrt{n}} e_i) - f(z) + \frac{1}{\sqrt{n}} \partial_i f(z)) \\
&\quad + \sum_{i,j=1}^m (\lambda_i(\gamma) - \frac{1}{\sqrt{n}} u_i^0 p_{ij} n(f(z + \frac{1}{\sqrt{n}}(e_j - e_i)) - f(z) - \frac{1}{\sqrt{n}}(\partial_j f(z) - \partial_i f(z))) \\
&\quad - \sum_{i=1}^m v_i^0 \partial_i f \\
&\quad - \sum_{i=1}^m u_i^0 (\partial_i f(z) - \sum_{j=1}^m p_{ij} \partial_j f(z)) \\
&\quad - \sum_i (\lambda_i(\gamma) + \frac{1}{\sqrt{n}} u_i^0) I_{\{z_i=0\}} p_{i0} n(f(z - \frac{1}{\sqrt{n}} e_i) - f(z)) \\
&\quad - \sum_{i,j} (\lambda_i(\gamma) + \frac{1}{\sqrt{n}} u_i^0) I_{\{z_i=0\}} p_{ij} n(f(z + \frac{1}{\sqrt{n}}(e_j - e_i)) - f(z)) \\
&= L_0^n f(z, u^0, v^0) + (|u^0| + |v^0|) B_0 f(z, u^0/(|u^0| + |v^0|), v^0/(|u^0| + |v^0|)) \\
&\qquad\qquad + \sum_{i=1}^m (\sqrt{n}\lambda_i(\gamma) + u_i^0) I_{\{z_i=0\}} C_{i,n} f(z),
\end{aligned}$$

where $|u^0| = \sum_{i=1}^m |u_i^0|$ and

$$B_0 f(z,u,v) = -(v + (I - P^T)u) \cdot \nabla f(z)$$

and

$$C_{i,n} f(z) = -p_{i0}\sqrt{n}(f(z - \frac{1}{\sqrt{n}}e_i) - f(z)) - \sum_{j=1}^m p_{ij}\sqrt{n}(f(z + \frac{1}{\sqrt{n}}(e_j - e_i)) - f(z)).$$

Note that we have used the fact that $\lambda(\gamma) = (I - P^T)^{-1}\gamma$.

We take $\mathcal{D}(A^n)$ to be $C_c^2(\mathbb{R}_0^m)$, the twice continuously differentiable functions on $\mathbb{R}_0^m = [0,\infty)^m$. Assuming that we can control the magnitude of $u^0$ and $v^0$, $L_0^n f(z,u^0,v^0) \to L_0 f(z)$ and $C_{i,n} f(z) \to C_i f(z)$, where

$$L_0 f(z) = \frac{1}{2} \sum_{1 \le i,j \le m} a_{ij}^0 \partial_i \partial_j f(z),$$

with

$$a_{ii}^0 = \gamma_i + \lambda(\gamma)_i p_{i0} + \sum_{j \ne i} (\lambda(\gamma)_j p_{ji} + \lambda(\gamma)_i p_{ij}), \quad a_{ij} = -\lambda(\gamma)_i p_{ij} - \lambda(\gamma(_j p_{ji},\ i \ne j,$$

and

$$C_i f(z) = \partial_i f(z) - \sum_{j=1}^m p_{ij} \partial_j f(z).$$

Let the running cost function be

$$c(z,u^0,v^0) = \sum_{i=1}^m (z_i + |v_i^0| + |u_i^0|).$$

Noting that $F^n$ defined in (3.9) is an increasing sequence of sets, we define $F$ to be the closure of $\cup_n F^n$.

For each $n$, the optimal long-run average cost is given by the solution of the linear programming problem, minimize

$$\int_{[0,\infty)^m \times F^n} c(z,u^0,v^0)\mu(dz \times du^0 \times dv^0)$$

among $\mu \in \mathcal{P}([0,\infty)^m \times F^n)$ satisfying

$$\int_{[0,\infty)^m \times F^n} A_n f(z,u^0,v^0)\mu(dz \times du^0 \times dv^0) = 0, \quad f \in \mathcal{D}(A_n). \tag{3.18}$$

Let $\mu^n$ be the minimizing probability measure. Define

$$\begin{aligned}
\mu_0^n(dz) &= \mu^n(dz \times F^n) \\
\mu_i^n(dz) &= \int_{F^n} (\sqrt{n}\lambda_i(\gamma) + u_i^0) I_{\{z_i=0\}} \mu^n(dz \times du^0 \times dv^0) \\
\nu_0^n(dz \times H) &= \int_{F^n} I_H(\frac{u^0}{|u^0| + |v^0|}, \frac{v^0}{|u^0| + |v^0|})(|u^0| + |v^0|)\mu^n(dz \times du^0 \times dv^0),
\end{aligned}$$

where $\mu_i^n$ is a measure on $D_i = \{z \in [0,\infty)^m : z_i = 0\}$ and $\nu_0^n$ is a measure on $[0,\infty)^m \times \hat{F}^n$, $\hat{F}^n = \{(u/(|u|+|v|), v/(|u|+|v|)) : (u,v) \in F^n\}$. Since the optimal cost

$$\int_{[0,\infty)^m \times F^n} c(z,u^0,v^0)\mu^n(dz \times u^0 \times v^0) = \int_{[0,\infty)^m} \sum_{i=1}^m z_i \mu_0^n(dz) + \nu_0^n([0,\infty)^m \times \hat{F}^n)$$

is bounded in $n$, it follows that $\{\mu_0^n\}$ is relatively compact in the weak topology and $\{\nu_0^n\}$ and $\{\mu_i^n\}$, $i = 1,\ldots,m$ are relatively compact in the vague topology. If $(\mu_0, \nu_0, \mu_1, \ldots, \mu_m)$ is a limit point of $(\mu_0^n, \nu_0^n, \mu_1^n, \ldots, \mu_m^n)$, for each $f \in C_c^2([0,\infty)^m)$,

$$\int_{[0,\infty)^m} L_0 f(z)\mu_0(dz) + \int_{[0,\infty)^m \times \hat{F}} B_0 f(z,u,v)\nu_0(dz \times du \times dv) + \sum_{i=1}^m \int_{[0,\infty)^m} C_i f(z)\mu_i(dz) = 0,$$

and

$$\liminf_{n\to\infty} \int_{[0,\infty)^m \times F^n} c(z,u^0,v^0)\mu^n(dz \times du^0 \times dv^0) \geq \int_{[0,\infty)^m} \sum_{i=1}^m z_i \mu_0(dz) + \nu_0([0,\infty)^m \times \hat{F}). \tag{3.19}$$

Consequently, the left side of (3.19) is bounded below by the solution of the linear programming problem, minimize

$$\int_{[0,\infty)^m} \sum_{i=1}^m z_i \mu_0(dz) + \nu_0([0,\infty)^m \times \hat{F})$$

subject to

$$\int_{[0,\infty)^m} Lf(z)\mu_0(dz) + \int_{[0,\infty)^m \times F} B_0 f(z,u,v)\nu_0(dz \times du \times dv) + \sum_{i=1}^m \int_{[0,\infty)^m} C_i f(z)\mu_i(dz) = 0. \tag{3.20}$$

The following theorem is a special case of results in [7].

**Theorem 3.1** *Suppose $\mu_0$, $\nu_0$, and $\mu_1,\ldots,\mu_m$ satisfy (3.20). Then there exist $(Z, \Lambda_0, \Lambda_i)$ such that*

$$f(Z(t)) - \int_0^t Lf(Z(s))ds - \int_{[0,\infty)^m \times F \times [0,t]} B_0 f(z,u,v)\Lambda_0(dz \times du \times dv \times ds)$$
$$- \sum_{i-1}^m \int_{D_i \times [0,t]} C_i f(z)\Lambda_i(dz \times ds)$$

*is a martingale for each $f \in C_c^2$, and $Z$ and the increments of $\Lambda_0, \ldots, \Lambda_m$ are stationary with*

$$\begin{aligned} E[f(Z(t))] &= \int f d\mu_0 \\ E[\Lambda_0(G \times H \times [0,t])] &= \nu_0(G \times H)t \\ E[\Lambda_i(G \times [0,t])] &= \mu_i(G)t \end{aligned}$$

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