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Particle representations for stochastic partial differential equations with boundary conditions

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Abstract

In this article, we study weighted particle representations for a class of stochastic partial differential equations (SPDE) with Dirichlet boundary conditions. The locations and weights of the particles satisfy an infinite system of stochastic differential equations. The locations are given by independent, stationary reflecting diffusions in a bounded domain, and the weights evolve according to an infinite system of stochastic differential equations driven by a *common* Gaussian white noise W which is the stochastic input for the SPDE. The weights interact through V, the associated weighted empirical measure, which gives the solution of the SPDE. When a particle hits the boundary its weight jumps to a value given by a function of the location of the particle on the boundary. This function determines the boundary condition for the SPDE. We show existence and uniqueness of a solution of the infinite system of stochastic differential equations giving the locations and weights of the particles and derive two weak forms for the corresponding SPDE depending on the choice of test functions. The weighted empirical measure V is the unique solution for each of the nonlinear stochastic partial differential equations. The work is motivated by and applied to the stochastic Allen-Cahn equation and extends the earlier of work of Kurtz and Xiong in [14, 15].

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1 Introduction

In the following, we study particle representations for a class of nonlinear stochastic partial differential equations that includes the stochastic version of the Allen-Cahn equation [1, 2] and that of the equation governing the stochastic quantization of Φ^4_d Euclidean quantum field theory with quartic interaction [19], that is, the equation

$$dv = \Delta v + G(v)v + W, \tag{1.1}$$

where G is a (possibly) nonlinear function[1] and W is a space-time noise[2]. These particle representations lead naturally to the solution of a weak version of a stochastic partial differential equation similar to (1.1). See equation (1.6) below and Section 5.

The approach taken here has its roots in the study of the McKean-Vlasov problem and its stochastic perturbation. In its simplest form, the problem begins with a finite system of stochastic differential equations

$$X_i^n(t) = X_i^n(0) + \int_0^t \sigma(X_i^n(s), V^n(s))dB_i(s) + \int_0^t c(X_i^n(s), V^n(s))ds, \quad 1 \leq i \leq n, \tag{1.2}$$

where the X_i^n take values in $\mathbb{R}^d$, $V^n(t)$ is the empirical measure $\frac{1}{n}\sum_{i=1}^n \delta_{X_i^n(t)}$, and the B_i are independent, standard Brownian motions in an appropriate Euclidean space. The primary goal in this setting is to prove that the sequence of empirical measures V^n converges in distribution and to characterize the limit V as a measure valued process which solves the following nonlinear partial differential equation, written in weak form,[3]

$$\langle \varphi, V(t)\rangle = \langle \varphi, V(0)\rangle + \int_0^t \langle L(V(s))\varphi, V(s)\rangle ds, \tag{1.3}$$

where $\varphi : \mathbb{R}^d \mapsto \mathbb{R}$ belongs to a suitably chosen class of Borel measurable functions and

$$\langle \varphi, V(t)\rangle = \int \varphi dV(t) = \int \varphi(u)V(t, du).$$

In (1.3), we use $a(x,\nu) = \sigma(x,\nu)\sigma(x,\nu)^T$ and $L(\nu)$ is the differential operator

$$L(\nu)\varphi(x) = \frac{1}{2}\sum_{i,j} a_{ij}(x,\nu)\partial^2_{x_i x_j}\varphi(x) + \sum_i c_i(x,\nu)\partial_{x_i}\varphi(x).$$

There are many approaches to this problem [8, 16, 18]. (See also the recent book [10].) The approach in which we are interested, introduced in [13] and developed further in [4, 11, 14], is simply to let the limit be given by the infinite system

$$X_i(t) = X_i(0) + \int_0^t \sigma(X_i(s), V(s))dB_i(s) + \int_0^t c(X_i(s), V(s))ds, \quad 1 \leq i < \infty. \tag{1.4}$$

To make sense out of this system (in particular, the relationship of V to the X_i), note that we can assume, without loss of generality, that the finite system $\{X_i^n\}$ is exchangeable (randomly permute the index i), so if one shows relative compactness of the sequence, any limit point will be an infinite exchangeable sequence and we can require $V(t)$ to be the de Finetti measure for the sequence $\{X_i(t)\}$, that is,

$$V(t) = \lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^m \delta_{X_i(t)},$$

[1] In the original work of Allen and Cahn (see [1, 2]), $G(v) = 1 - v^2$ for all $v \in \mathbb{R}$ whilst in the case of the Φ^4_d equation of Euclidean quantum field theory, $G(v) = -v^2$ (d represents the state space dimension).

[2] A detailed description of the noise W is given below and in the Appendix.

[3] In (1.3) and thereafter, we use the notation $\langle \varphi, \mu\rangle$ to express the integral of φ with respect to μ, that is, $\langle \varphi, \mu\rangle = \int \varphi(x)\mu(dx)$.

in the sense that $\langle \varphi, V(t)\rangle = \lim_{m\to\infty}\frac{1}{m}\sum_{i=1}^m \varphi(X_i(t))$ for all bounded, measurable φ. More precisely, V is a process with sample paths in $C_{\mathcal{P}(\mathbb{R}^d)}[0,\infty)$. (See Lemma 4.4 of [11].)

Note that while the X_i^n give a *particle approximation* of the solution of (1.3), the X_i give a *particle representation* of the solution, that is, the de Finetti measure of $\{X_i(t)\}$ is the desired $V(t)$.

In the following, we will make use of representations similar to those studied in [14]. They differ from those considered in the other papers mentioned as each particle is given a weight $A_i(t)$ and V is given by

$$V(t) = \lim_{m\to\infty}\frac{1}{m}\sum_{i=1}^m A_i(t)\delta_{X_i(t)}.$$

The models in the current paper differ from those in [14] in two primary ways. First, the X_i will be independent, stationary diffusion processes defined on a domain $D\subset\mathbb{R}^d$ with reflecting boundary. The stationary distribution will be denoted by π. Second, we will place boundary conditions on the solution. Essentially, if $v(t,u)$ is an appropriately defined solution to a stochastic partial differential equation of interest, we would like to have

$$v(t,u) = g(u), \quad u\in\partial D,\ t>0. \tag{1.5}$$

The precise sense in which the boundary conditions hold will be discussed later and will depend on the conditions assumed.

Specifically, in the same vein as equation (1.3), we will consider a class of nonlinear stochastic partial differential equations written in weak form

$$\begin{aligned}\langle \varphi, V(t)\rangle &= \langle \varphi, V(0)\rangle + \int_0^t \langle \mathbb{L}\varphi, V(s)\rangle ds + \int_0^t \langle G(v(s,\cdot),\cdot)\varphi, V(s)\rangle ds\\ &+ \int_0^t\int_{\overline{D}} \varphi(x)b(x)\pi(dx)ds + \int_{\mathbb{U}\times[0,t]}\int_{\overline{D}}\varphi(x)\rho(x,u)\pi(dx)W(du,ds),\end{aligned} \tag{1.6}$$

where D is a bounded, open, connected subset of $\mathbb{R}^d$, for the moment the test functions are $\varphi\in C_c^2(D)$ the twice continuously differentiable functions with compact support in D,

$$\mathbb{L}\varphi(x) = \frac{1}{2}\sum_{i,j} a_{ij}(x)\partial^2_{x_ix_j}\varphi(x) + \sum_i c_i(x)\partial_{x_i}\varphi(x),$$

$v(s,x)$ is the density of $V(s)$ with respect to π, and $\langle\varphi, V(0)\rangle = \int \varphi(x)h(x)\pi(dx)$ for a specified h. Note that if the solution V is adapted to a filtration $\{\mathcal{F}_t\}$, then we can assume that v is a progressively measurable process with values in $L^1(\pi)$. In particular, the mapping $(s,x,\omega)\to v(s,x,\omega)$ is $\mathcal{B}[0,\infty)\times\mathcal{B}(\overline{D})\times\vee_t\mathcal{F}_t$ measurable. ($\mathcal{B}(E)$ denotes the Borel subsets of a metric space E.) The existence of a measurable version of the process of densities follows by a monotone class argument. See Lemma A.1 for a similar argument.

Throughout, we will assume that $\mathbb{U}$ is a complete, separable metric space, μ is a σ-finite Borel measure on $\mathbb{U}$, and ℓ is Lebesgue measure on $[0,\infty)$. W is Gaussian white noise on $\mathbb{U}\times[0,\infty)$ with covariance measure $\mu\times\ell$, that is, $W(C\times[0,t])$ has expectation zero for all $0\le t<\infty$ and $C\in\mathcal{B}(\mathbb{U})$ with $\mu(C)<\infty$, and $\mathbb{E}[W(C_1\times[0,t])W(C_2\times[0,s])= t\wedge s\mu(C_1\cap C_2)$. (See Appendix A.1.)

Formally, equation (1.6) is the weak form of

$$v(t,x) = v(0,x) + \int_0^t \left[\mathbb{L}^* v(s,x) + v(s,x)G(v(s,x),x) + b(x)\right]ds + \int_{\mathbb{U}\times[0,t]}\rho(x,u)W(du,ds), \tag{1.7}$$

where $\mathbb{L}^*$ is the formal adjoint of the operator $\mathbb{L}$ with respect to the inner product $\langle \psi, \varphi \rangle_\pi = \int_{\overline{D}} \psi(x)\varphi(x)\pi(dx)$.

To obtain, for example, the stochastic Allen-Cahn equation (1.1), we can choose $\mathbb{L} = \Delta$.

2 Basic conditions and statement of main theorems

With the results of [14] in mind, we are interested in solutions V of (1.6) that are measures or perhaps signed measures. We emphasize here that we are talking about a *representation* of the solution of the equation, not a limit or approximation theorem (although these representations can be used to prove limit theorems). To obtain the representation of $V(t)$ we desire, we must identify both the locations $X_i(t)$ and the weights $A_i(t)$. The sequence $\{(X_i(t), A_i(t))\}$ is required to be exchangeable, so by de Finetti's theorem, for each t there will exist a random measure $\Xi(t)$, which we will refer to as the de Finetti measure for the sequence $\{(X_i(t), A_i(t))\}$, such that for each bounded, measurable function ψ on $\overline{D} \times \mathbb{R}$,

$$\langle \psi, \Xi(t) \rangle = \lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} \psi(X_i(t), A_i(t)) = \int_{\overline{D}\times\mathbb{R}} \psi(x,a)\Xi(t,dx,da).$$

Then, assuming $\mathbb{E}[|A_i(t)|] < \infty$, $V(t)$ will be given by

$$\langle \varphi, V(t) \rangle = \lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} A_i(t)\varphi(X_i(t)) = \int_{\overline{D}\times\mathbb{R}} a\varphi(x)\Xi(t,dx,da). \tag{2.1}$$

If the $A_i(t)$ are nonnegative, then $V(t)$ will be a measure, but we do not rule out the possibility that the $A_i(t)$ can be negative and $V(t)$ a signed measure. The weights and locations will be solutions of an infinite system of stochastic differential equations that are coupled only through V and common noise terms.

We take the X_i to be independent, stationary solutions of the Skorohod equation

$$X_i(t) = X_i(0) + \int_0^t \sigma(X_i(s))dB_i(s) + \int_0^t c(Xi(s))ds + \int_0^t \eta(X_i(s))dL_i(s),\ i \geq 1, \tag{2.2}$$

where $\eta(x)$ is a vector field defined on the boundary ∂D, and L_i is a local time on ∂D for X_i, that is, L_i is a nondecreasing process that increases only when X_i is in ∂D. To avoid some of the complexities of reflecting diffusions and focus on the new ideas in our representation, we will assume the following condition throughout.

Condition 2.1. *a) $D \subset \mathbb{R}^d$ is bounded, open, connected, and has a C^2 boundary.*

b) σ, c, and η are continuous, σ is nondegenerate on $\overline{D}$, and $\eta(x) \cdot n_D(x) > 0$, $x \in \partial D$, where $n_D(x)$ is the unit inward normal at x.

c) For a standard Brownian motion B and $X(0) \in \overline{D}$ independent of B, the solution of

$$X(t) = X(0) + \int_0^t \sigma(X(s))dB(s) + \int_0^t c(X(s))ds + \int_0^t \eta(X(s))dL(s), \tag{2.3}$$

is (weakly) unique for all initial distributions in $\mathcal{P}(\overline{D})$.

d) The X_i are independent solutions of (2.2) with independent standard Brownian motions B_i and independent and identically distributed $X_i(0) \in \overline{D}$. The distribution π of $X_i(0)$ is a stationary distribution for (2.3).

Remark 2.2. See [5] for conditions implying strong uniqueness for (2.3). Weak uniqueness can be proved employing results from partial differential equations. (See, for example, Theorem 8.1.5 in [6]. In the notation of that theorem, $\eta(x) = -c(x)$.) Weak uniqueness can also be obtained via submartingale problems. (See [20].) These approaches all require different conditions on D and the coefficients, so we simply assume uniqueness.

Lemma 2.3. *The equation (2.3) determines a strong Markov process, and the corresponding semigroup defined by*

$$\mathbb{T}(t)\varphi(x) = \mathbb{E}[\varphi(X(t))|X(0) = x]$$

satisfies

$$\mathbb{T}(t) : C_b(\overline{D}) \to C_b(\overline{D}). \tag{2.4}$$

Proof. Let X^x denote the solution of (2.3) with $X^x(0) = x$. Continuity of the coefficients and the assumption of uniqueness implies that the mapping $x \to X^x$ is continuous in the sense of convergence in distribution in $C_{\overline{D}}[0,\infty)$. The simplest way to see this continuity is to first define the time change

$$\tau^x(t) = \inf\{r; r + L^x(r) > t\}.$$

Then setting $\lambda^x(t) = L^x(\tau^x(t))$ and $Y^x(t) = X^x(\tau^x(t))$, we have

$$Y^x(t) = x + \int_0^t \sigma(Y^x(s))dB \circ \tau^x(s) + \int_0^t c(Y^x(s))ds + \int_0^t \eta(Y^x(s))d\lambda^x(s).$$

Since all the coefficients are continuous and $\tau^x(t) + \lambda^x(t) = t$, τ^x and λ^x are Lipschitz with constant 1, $\{Y^x, x \in \overline{D}\}$ is relatively compact, and any limit point $(Y^{x_0}, \tau^{x_0}, \lambda^{x_0})$ of (Y^x, τ^x, λ^x) with $x \to x_0$ will satisfy

$$Y^{x_0}(t) = x_0 + \int_0^t \sigma(Y^{x_0}(s))dB \circ \tau^{x_0}(s) + \int_0^t c(Y^{x_0}(s))ds + \int_0^t \eta(Y^{x_0}(s))d\lambda^{x_0}(s).$$

Since λ^{x_0} increases only when $Y^{x_0} \in \partial D$, the assumptions on η ensure that τ^{x_0} is strictly increasing (as are the τ^x). Letting γ^{x_0} denote the inverse of τ^{x_0}, $X^{x_0} \equiv Y^{x_0} \circ \gamma^{x_0}$, is a solution of (2.3) with $X(0) = x_0$, and by uniqueness, $X^x \Rightarrow X^{x_0}$ as $x \to x_0$ giving (2.4). $\square$

If $\varphi \in C_b^2(\overline{D})$, then by Itô's formula,

$$\begin{aligned}\varphi(X(t)) = \varphi(X(0)) + \int_0^t \nabla\varphi(X(s))^T \sigma(X(s))dB(s) + \int_0^t \mathbb{L}\varphi(X(s))ds \\ + \int_0^t \nabla\varphi(X(s)) \cdot \eta(X(s))dL(s),\end{aligned} \tag{2.5}$$

where

$$\mathbb{L}\varphi(x) = \frac{1}{2}\sum_{i,j} a_{ij}(x)\partial^2_{x_i x_j}\varphi(x) + \sum_i c_i(x)\partial_{x_i}\varphi(x), \tag{2.6}$$

with $a(x) = \sigma(x)\sigma(x)^T$, where σ^T is the transpose of σ.

Lemma 2.4. *The infinitesimal generator* $\mathbb{A}$ *for the semigroup* $\{\mathbb{T}(t)\}$ *is an extension of*

$$\{(\varphi, \mathbb{L}\varphi) : \varphi \in C_b^2(\overline{D}),\ \eta(x) \cdot \nabla\varphi|_{\partial D} = 0\}. \tag{2.7}$$

Proof. The boundary condition and the martingale property of the stochastic integral implies

$$\frac{\mathbb{T}(t)\varphi(x)-\varphi(x)}{t}=\frac{1}{t}\int_0^t\mathbb{E}[\mathbb{L}\varphi(X^x(s))]ds. \tag{2.8}$$

The continuity in distribution of $x\to X^x$ and the continuity of $\mathbb{L}\varphi$ assure that if $x\to x_0$ and $t\to 0$, the right side of (2.8) converges to $\mathbb{L}\varphi(x_0)$. The compactness of $\overline{D}$ then assures that the convergence of the left side of (2.8) to $\mathbb{L}\varphi(x)$ is uniform in x. □

Lemma 2.5. *Let X,L be a solution of (2.3). Then for each $t>0$,*

$$\int_0^t\mathbf{1}_{\partial D}(X(s))ds=0\quad a.s.,\quad t\geq 0,$$

and $\mathbb{E}[L(t)]<\infty$.

Remark 2.6. The result is a special case of Proposition 6.1 of [9]. We give a short proof in our simpler setting.

Proof. We can obtain D as $D=\{x:\psi_D(x)>0\}$ where ψ_D is C^2 on $\mathbb{R}^d$ and $\nabla\psi_D(x)=\epsilon(x)n_D(x)$, $x\in\partial D$, where $\inf_{x\in\partial D}\epsilon(x)>0$. Observe that

$$\begin{aligned}\psi_D(X(t))=\psi_D(X(0))+\int_0^t\nabla\psi_D(X(s))^T\sigma(X(s))dB(s)+\int_0^t\mathbb{L}\psi_D(X(s))ds \\ +\int_0^t\nabla\psi_D(X(s))\cdot\eta(X(s))dL(s).\end{aligned} \tag{2.9}$$

Every term in (2.9) has finite expectation, except possibly the last, but then the last must also. Since $\kappa_1=\inf_{x\in\partial D}\nabla\psi_D(x)\cdot\eta(x)>0$, we have

$$\mathbb{E}[L(t)]<\frac{1}{\kappa_1}\mathbb{E}[\int_0^t\nabla\psi_D(X(s))\cdot\eta(X(s))dL(s)]<\infty.$$

Setting $q_\epsilon(z)=\int_0^z\int_0^y\mathbf{1}_{[0,\epsilon]}(u)dudy$,

$$\begin{aligned}q_\epsilon(\psi_D(X(t)))\quad=\quad & q_\epsilon(\psi_D(X(0)))+\int_0 q'_\epsilon(\psi_D(X(s)))\nabla\psi_D(X(s))^T\sigma(X(s))dB(s)\\ &+\int_0^t q'_\epsilon(\psi_D(X(s)))\mathbb{L}\psi_D(X(s))ds\\ &+\int_0^t q'_\epsilon(\psi_D(X(s)))\nabla\psi_D(X(s))\cdot\eta(X(s))dL(s)\\ &+\int_0^t\mathbf{1}_{[0,\epsilon]}(\psi_D(X(s)))\nabla\psi_D(X(s))^T\sigma(X(s))\sigma^T(X(s))\nabla\psi(X(s))ds.\end{aligned} \tag{2.10}$$

Since $\kappa_2=\inf_{x\in\overline{D}}\nabla\psi_D(x)^T\sigma(x)\sigma^T(x)\nabla\psi(x)>0$, $\int_0^t\mathbf{1}_{\partial D}(X(s))ds$ is bounded by κ_2^{-1} times the last term on the right of (2.10). Since every term in (2.10) converges to zero except, possibly, the last term, the last term also converges to zero giving the lemma. □

Lemma 2.7. *Let X^x be as in the proof of Lemma 2.3, and let $\gamma^x=\inf\{t>0:X^x(t)\in\partial D\}$. Then each $x_0\in\partial D$ is regular in the sense that*

$$\lim_{x\to x_0}\mathbb{E}[\gamma^x]=0\text{ and }\lim_{x\to x_0}\mathbb{E}[|X^x(\gamma^x)-x_0|]=0.$$

In particular,

$$\mathbb{E}[|X^{x_0}(\gamma^{x_0})-x_0|]=\mathbb{E}[\gamma^{x_0}]=0.$$

Proof. With reference to Section 6.2 of [7], a function $w_{x_0} \in C^2(\overline{D})$ is a *barrier* at x_0 if $w_{x_0} \geq 0$, $w_{x_0}(x) = 0$ only if $x = x_0$, and $\mathbb{L}w_{x_0}(x) \leq -1$, and under Condition 2.1, a barrier exists for each $x_0 \in \partial D$. By Itô's formula,

$$w_{x_0}(X^x(t \wedge \gamma^x)) - w_{x_0}(x) - \int_0^{t\wedge\gamma^x} \mathbb{L}w_{x_0}(X^x(s))ds.$$

Taking expectations and letting $t \to \infty$, we have

$$w_{x_0}(x) \geq \mathbb{E}[w_{x_0}(X(\gamma^x))] + \mathbb{E}[\gamma^x].$$

Since $\lim_{x\to x_0} w_{x_0}(x) = 0$, the lemma follows. □

Lemma 2.8. *The Markov process corresponding to (2.3) has a stationary distribution denoted by π, and the support of π is $\overline{D}$.*

Proof. The compactness of $\overline{D}$ and Lemma 2.3 imply existence of a stationary distribution by Theorem 4.9.3 of [6]. To show that π charges every open set, first observe that $\pi(\partial D) = 0$, since the process spends zero real time in ∂D. Let P_x^X be the distribution of the solution X with $X(0) = x$. It then suffices to show that for any $x \in D$ and any ball $B \subset D$, $P_x^X(\tau_B < \tau_{\partial D}) > 0$, where τ_B denotes the first hitting time of B and $\tau_{\partial D}$ denotes the first hitting time of the boundary. By connectedness, we can find a differentiable path $\mathcal{P}$ starting at x and ending at the center of the ball B without hitting the boundary. Let $\epsilon = \inf\{|x - y| : x \in \mathcal{P}, y \in \partial D\}$. Since σ is nondegenerate, by Girsanov's theorem, we can construct a distribution Q_x^X equivalent to P_x^X such that under Q_x^X, with high probability X stays within ϵ of $\mathcal{P}$ until τ_B. □

Since we are assuming that the X_i are independent and stationary (obtained by assuming that the $X_i(0)$ are independent with distribution π), an immediate consequence of our assumptions is that $V(t)$ given by (2.1) will be absolutely continuous with respect to π.

Lemma 2.9. *The measure $V(t)$ is absolutely continuous with respect to π.*

Proof. The de Finetti measure for $\{(A_i(t), X_i(t))\}$ is given by the regular conditional distribution $\Xi(t, da, dx)$ of $(A_1(t), X_1(t))$ given the tail σ-algebra $\mathcal{T} = \cap_n \sigma((A_i(t), X_i(t)) : i \geq n)$. Then the measure $V(t)$ can be written as

$$V(t, B) = \int_{\mathbb{R}\times\overline{D}} \mathbf{1}_B(x) a \Xi(t, da, dx).$$

But the $\overline{D}$-marginal of $\Xi(t, da, dx)$ is π, so there is a transition function $\mathcal{V}(t, x, da)$ satisfying

$$V(t, B) = \int_B \int_{\mathbb{R}} a\mathcal{V}(t, x, da)\pi(dx),$$

and hence $V(t)$ is absolutely continuous with respect to π. □

Consequently, we can write

$$V(t, dx) = v(t, x)\pi(dx),$$

with

$$v(t, x) = \int_{\mathbb{R}} a\mathcal{V}(t, x, da). \tag{2.11}$$

As noted above, we can assume that v is $\mathcal{B}[0,\infty) \times \mathcal{B}(\overline{D}) \times \vee_t \mathcal{F}_t$-measurable.

To construct a particle representation for a solution of (1.6), we still need to define our weights, A_i. Set $\tau_i(t) = 0 \vee \sup\{s \le t : X_i(s) \in \partial D\}$, that is $\tau_i(t)$ is the most recent time that X_i has been on the boundary, or if X_i has not hit the boundary by time t, $\tau_i(t) = 0$. Of course, $\tau_i(t)$ is not a stopping time; however, it is independent of W, so the stochastic integral in the following equation is well-defined. We take A_i to satisfy

$$\begin{aligned} A_i(t) &= g(X_i(\tau_i(t)))\mathbf{1}_{\{\tau_i(t)>0\}} + h(X_i(0))\mathbf{1}_{\{\tau_i(t)=0\}} + \int_{\tau_i(t)}^t b(X_i(s))ds \\ &\quad + \int_{\tau_i(t)}^t G(v(s,X_i(s)),X_i(s))A_i(s)ds + \int_{\mathbb{U}\times(\tau_i(t),t]} \rho(X_i(s),u)W(du,ds), \end{aligned} \tag{2.12}$$

where $v(t,x)$ is the density with respect to π determined by (2.11).

As we will see in Section 5, A_i does not appear to be a semimartingale, but the difficulties only occur when X_i is at the boundary. Consequently, we can define the integrals in the following lemma directly as limits of Riemann-like sums.

Lemma 2.10. *For φ in $C_c^2(D)$,*

$$\begin{aligned} \varphi(X_i(t))A_i(t) &= \varphi(X_i(0))A_i(0) + \int_0^t \varphi(X_i(s))dA_i(s) \\ &\quad + \int_0^t A_i(s)\nabla\varphi(X_i(s))^T\sigma(X_i(s))dB_i(s) + \int_0^t \mathbb{L}\varphi(X_i(s))A_i(s)ds \\ &= \varphi(X_i(0))A_i(0) + \int_0^t \varphi(X_i(s))G(v(s,X_i(s)),X_i(s))A_i(s)ds \\ &\quad + \int_0^t \varphi(X_i(s))b(X_i(s))ds + \int_{\mathbb{U}\times[0,t]} \varphi(X_i(s))\rho(X_i(s),u)W(du\times ds) \\ &\quad + \int_0^t A_i(s)\nabla\varphi(X_i(s))^T\sigma(X_i(s))dB_i(s) + \int_0^t \mathbb{L}\varphi(X_i(s))A_i(s)ds, \end{aligned} \tag{2.13}$$

where for partitions $\{t_i\}$ of $[0,t]$,

$$\int_0^t \varphi(X_i(s))dA_i(s) = \lim_{\max_k |t_{k+1}-t_k|\to 0} \sum_k \varphi(X_i(t_k))(A_i(t_{k+1}) - A_i(t_k)). \tag{2.14}$$

Remark 2.11. Since φ vanishes in a neighborhood of the ∂D, the local time integral in (2.5) does not appear in this identity.

Proof. The convergence of the limit in (2.14) follows by observing that for $\max_k |t_{k+1} - t_k|$ sufficiently small, $t_k \le \tau_i(s) \le t_{k+1}$ implies $\varphi(X_i(t_k)) = 0$. We also note that by this observation and the fact that the independence of B_i and W implies that the covariation of the two stochastic integral terms is zero,

$$\lim_{\max_k |t_{k+1}-t_k|\to 0} \sum_k (\varphi(X_i(t_{k+1})) - \varphi(X_i(t_k)))(A_i(t_{k+1}) - A_i(t_k)) = 0.$$

□

To see that the weights A_i determined by (2.12) should give a solution of (1.6), we want to average (2.13). The next to the last term in the right side of (2.13) is a martingale, and these martingales are orthogonal for different values of i. Consequently, they will average to zero. Assuming exchangeability of $\{(A_i, X_i)\}$, which will follow from the exchangeability of $\{(A_i(0), X_i)\}$ provided we can show existence and uniqueness for the system (2.12), averaging gives (1.6).

We also note that the weights, at least plausibly, capture the desired boundary conditions. Intuitively, for x close to the boundary ∂D, the value of $v(t,x)$ is determined by the particles with locations $X_i(t)$ close to the boundary, but if $X_i(t)$ is close to the boundary it should have recently hit the boundary and $A_i(t)$ should be close to $g(X_i(\tau_i(t)))$.

That intuition leads to the following interpretation of the boundary condition. Assume that g is continuous, and let $\overline{g} : D \cup \partial D \mapsto \mathbb{R}$ be a continuous function such that $\overline{g}|_{\partial D} = g$. For $\epsilon > 0$, define

$$\partial_\epsilon D = \{x \in \overline{D} | \mathrm{dist}(x, \partial D) < \epsilon\}.$$

Then,

$$\int_{\partial_\epsilon D} |v(t,x) - \overline{g}(x)| \pi(dx) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\partial_\epsilon D}(X_i(t) |v(t, X_i(t)) - \overline{g}(X_i(t))|.$$

Once we have existence for (2.12), it will follow from Lemma 3.2 that $\mathbb{E}[A_i(t)|W, X_i(t)] = v(t, X_i(t))$, and by the intuition and the continuity of $\overline{g}$, if $X_i(t)$ is close to the boundary, we should have $v(t, X_i(t)) \approx \overline{g}(X_i(t))$ and

$$\lim_{m\to\infty} \frac{1}{\pi(\partial_\epsilon D)} \int_{\partial_\epsilon D} |v(t,x) - \overline{g}(x)| \pi(dx) = 0. \tag{2.15}$$

The intuition is made precise under regularity conditions on the time-reversal of the X_i. See Lemma 4.7 for details.

Our first step will be to prove uniqueness for the system (2.12) under the following condition which we assume throughout the paper.

Condition 2.12. *The coefficients in (2.12) satisfy*

1. g *and* h *are bounded with sup norms* $\|g\|$ *and* $\|h\|$.
2. $K_1 \equiv \sup_{x\in\overline{D}} |b(x)| < \infty$.
3. $K_2 \equiv \sup_{x\in\overline{D}} \int \rho(x,u)^2 \mu(du) < \infty$.
4. $K_3 \equiv \sup_{v\in\mathbb{R}, x\in\overline{D}} G(v,x) < \infty$.
5. $L_1 \equiv \sup_{v\in\mathbb{R}, x\in\overline{D}} \frac{|G(v,x)|}{1+|v|^2} < \infty$.
6. $L_2 \equiv \sup_{v_1\neq v_2\in\mathbb{R}, x\in\overline{D}} \frac{|G(v_1,x) - G(v_2,x)|}{|v_1 - v_2|(1+|v_1|+|v_2|)} < \infty$.

Observe that Condition 2.12.4 does not imply that G has a lower bound, but only an upper bound. For example, $G(v,x) = 1 - v^2$ gives the classical Allen-Cahn equation, whilst $G(v,x) = -v^2$ gives the Φ^4_d equation.

Theorem 2.13. *The solution of (2.12) with* $v(t,x)$ *the density of* V *given by (2.1) exists and is unique.*

Proof. Uniqueness is proved in Section 3.1 and existence in Section 3.2. □

Theorem 2.13 ensures the existence of a (signed) measure-valued process satisfying (1.6) for $\varphi \in C^2_c(D)$. Unfortunately, even coupled with some interpretation of the boundary condition, (1.6) with this space of test functions does not, in general, uniquely determine a measure-valued process.[4] Consequently, we need to enlarge the space of test functions. We have two ways of doing that, first by taking the test functions to be $C^2_0(\overline{D})$, the space of twice continuously differentiable functions that vanish on the boundary, Theorem 2.16, and second by taking the test functions to be $\mathcal{D}(\mathbb{A})$, the domain of the generator for the semigroup $\{\mathbb{T}(t)\}$ corresponding to the X_i, Theorem 2.18.

We need to identify the space in which the solution will live.

[4] For example, consider X_i reflecting Brownian motion with differing directions of reflection but whose stationary distribution is still normalized Lebesgue measure.

Definition 2.14. *A process Z is* compatible *with a process Y if*

$$\mathbb{E}[f(Y)|\mathcal{F}_t^{Y,Z}] = \mathbb{E}[f(Y)|\mathcal{F}_t^Y],$$

for all bounded, measurable f defined on the range of Y and all t.

Remark 2.15. If Y is a process with independent increments, then Z is compatible with Y if $Y(t+\cdot) - Y(t)$ is independent of $\mathcal{F}_t^{Y,Z}$. See [12], Lemma 2.4.

Let $\mathcal{L}(\pi)$ be the space of processes v compatible with W taking values in $L^1(\pi)$ such that for each $T > 0$ and some $\varepsilon_T > 0$, v satisfies

$$\sup_{t \le T} \mathbb{E}\left[\int_{\overline{D}} e^{\varepsilon_T |v(t,x)|^2} \pi(dx)\right] < \infty.$$

We prove the following theorem in Section 5.1.

Theorem 2.16. *Consider the equation*

$$\begin{aligned}
\langle \varphi(\cdot,t), V(t)\rangle &= \langle \varphi(\cdot,0), h\rangle_\pi + \int_0^t \langle \varphi(\cdot,s) G(v(s,\cdot),\cdot), V(s)\rangle ds \\
&\quad + \int_0^t \int_{\overline{D}} \varphi(x,s) b(x) \pi(dx) ds \\
&\quad + \int_{\mathbb{U}\times[0,t]} \int_{\overline{D}} \varphi(x,s)\rho(x,u)\pi(dx) W(du \times ds) \\
&\quad + \int_0^t \langle \mathbb{L}\varphi(\cdot,s) + \partial\varphi(\cdot,s), V(s)\rangle ds \\
&\quad + \int_0^t \int_{\partial D} g(x)\eta(x)\cdot\nabla\varphi(x,s)\beta(dx) ds,
\end{aligned} \tag{2.16}$$

where the test functions $\varphi(x,t)$ are twice differentiable in x, differentiable in t, and vanish on $\partial D \times [0,\infty)$, π is the stationary distribution for the particle location process, and β is the measure associated with the local time defined in Section 4.

Suppose $\mathcal{D}_0 = \{\varphi \in C^2(\overline{D}) : \varphi(x) = 0$ and $\mathbb{L}\varphi(x) = 0, x \in \partial D\}$ is a core for $\mathbb{A}^0$, the generator of the semigroup $\{\mathbb{T}^0(t)\}$ for the diffusion that absorbs at ∂D, that is, $\mathbb{A}^0$ is the closure of $\{(\varphi, \mathbb{L}\varphi) : \varphi \in \mathcal{D}_0\}$. Then V defined in (2.1) with $\{(X_i, A_i)\}$ given by (2.2) and (2.12) is the unique solution of (2.16) in $\mathcal{L}(\pi)$.

Remark 2.17. Theorem 8.1.4 of [6] gives conditions implying $\mathcal{D}_0$ is a core. Note that any solution of (2.16) is a solution of (1.6).

Now we take the test functions to be $\mathcal{D}(\mathbb{A})$, the domain of the generator for the semigroup $\{\mathbb{T}(t)\}$ corresponding to the location processes. More precisely, let $\mathcal{T}$ be the collection of functions $\varphi(x,t)$ for which there exists $t_\varphi > 0$ such that $\varphi(t,x) = 0$ for $t \ge t_\varphi$, φ is continuously differentiable in t, and $\varphi(\cdot,t) \in \mathcal{D}(\mathbb{A})$, $t \ge 0$, with $\mathbb{A}\varphi$ bounded and continuous.

Let

$$\gamma_i(s) = \inf\{t > s : X_i(t) \in \partial D\}, \tag{2.17}$$

and note that $\mathbf{1}_{\{\tau_i(t)=0\}} = \mathbf{1}_{\{\gamma_i(0)>t\}}$. Let $P(dy, ds|x)$ be the conditional distribution of $(X_i(\gamma_i(0)), \gamma_i(0))$ given $X_i(0) = x$, and let $P\varphi(x) = \int \varphi(y,s)P(dy,ds|x)$. Let X^* be the reversed process and γ^* be the first time that X^* hits the boundary.

To simplify notation in the equation, we extend g to all of $\overline{D}$ by setting $g(x) = h(x)$ for $x \in D$. We do not require that this extension be continuous.

We prove the following theorem in Section 5.2.

Theorem 2.18. *For the equation*

$$
\begin{aligned}
0 \quad = \quad & \int_0^\infty \langle(\varphi(\cdot,s) - P\varphi(\cdot,\cdot+s))G(v(s,\cdot),\cdot), V(s)\rangle ds \\
& + \int_0^\infty \int b(x)(\varphi(x,s) - P\varphi(x,\cdot+s))\pi(dx)ds \\
& + \int_{\mathbb{U}\times[0,\infty)} \int_{\overline{D}} (\varphi(x,s) - P\varphi(x,\cdot+s))\rho(x,u)\pi(dx)W(du\times ds) \\
& + \int_0^\infty \langle \mathbb{A}\varphi(\cdot,s) + \partial\varphi(\cdot,s), V(s)\rangle ds \\
& - \int_0^\infty \int_{\overline{D}} \mathbb{E}[g(X^*(\gamma^*\wedge s))|X^*(0)=x](\mathbb{A}\varphi(x,s)+\partial\varphi(x,s))\pi(dx)ds.
\end{aligned} \tag{2.18}
$$

for all $\varphi \in \mathcal{T}$, V defined in (2.1) with $\{(X_i, A_i)\}$ given by (2.2) and (2.12) is the unique solution of (2.18) in $\mathcal{L}(\pi)$.

Remark 2.19. The form of (2.18) may not be very intuitive; however, this equation and (2.16) determine the same unique solution in $\mathcal{L}(\pi)$.

To see that the restriction of (2.18) to $\varphi \in C_c^2(D)$ gives (1.6), note first that for $\varphi \in C_c^2(D)$, $P\varphi = 0$. To see that the last term in (2.18) is zero for $\varphi \in C_c^2(D)$, note that $u(x,s) = \mathbb{E}[g(X^*(\gamma^*\wedge s))|X^*(0)=x]$ should be a solution of the Dirichlet problem $(\mathbb{A}_0^* - \partial)u(x,s) = 0$ on $\overline{D}\times[0,\infty)$ with boundary conditions $u(x,0) = h(x)$, $x \in D$, and $u(x,s) = g(x)$, $s > 0$, $x \in \partial D$.

3 Existence and uniqueness of the weighted particle system

In this section we prove a number of estimates for particle systems of the type we consider in this paper, leading up to a proof of Theorem 2.13. The system of stochastic differential equations (2.12) must be considered in conjunction with the existence of an empirical distribution

$$V(t) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n A_i(t)\delta_{X_i(t)}$$

required to have a density $v(t,\cdot)$ with respect to π. It is by no means clear that a solution satisfying all these constraints exists.

First we explore the properties that a solution must have by replacing v by an arbitrary, measurable $L^1(\pi)$-valued stochastic process U that is independent of $\{X_i\}$ and compatible with W, (see Definition 2.14). In the current setting, compatibility means that for each $t > 0$, $\sigma(W(C\times(t,t+s]) : C \in \mathcal{B}(\mathbb{U}), \mu(C) < \infty, s > 0)$ is independent of

$$\mathcal{F}_t^{U,W} = \sigma(U(s), W(C\times[0,s]) : 0 \le s \le t, C \in \mathcal{B}(\mathbb{U}), \mu(C) < \infty).$$

Define A_i^U to be the solution of

$$
\begin{aligned}
A_i^U(t) \quad = \quad & g(X_i(\tau_i(t)))\mathbf{1}_{\{\tau_i(t)>0\}} + h(X_i(0))\mathbf{1}_{\{\tau_i(t)=0\}} \\
& + \int_{\tau_i(t)}^t G(U(s,X_i(s)),X_i(s))A_i^U(s)ds + \int_{\tau_i(t)}^t b(X_i(s))ds \\
& + \int_{\mathbb{U}\times(\tau_i(t),t]} \rho(X_i(s),u)W(du\times ds).
\end{aligned} \tag{3.1}
$$

Existence and uniqueness of the solution of (3.1) holds under modest assumptions on the coefficients, in particular, under Condition 2.12.

Lemma 3.1. *Let*

$$H_i(t) = \int_{\mathbb{U}\times[0,t]} \rho(X_i(s),u)W(du,ds).$$

Then H_i is a martingale with respect to the filtration $\{\mathcal{H}_t^i\} \equiv \{\mathcal{F}_t^W \vee \sigma(X_i)\}$ and there exists a standard Brownian motion Z_i such that Z_i is independent of X_i and

$$H_i(t) = Z_i\left(\int_0^t \rho^2(X_i(s),u)\mu(du)\right).$$

For all A_i^U defined as in (3.1) and K_1, K_2, and K_3 defined in Condition 2.12,

$$\begin{aligned} |A_i^U(t)| &\le (\|g\|\vee\|h\| + K_1(t-\tau_i(t)) + \sup_{\tau_i(t)\le r\le t}|H_i(t)-H_i(r)|)e^{K_3(t-\tau_i(t))} \qquad (3.2)\\ &\le (\|g\|\vee\|h\| + K_1 t + \sup_{0\le s\le t}|H_i(t)-H_i(s)|)e^{K_3 t} \equiv \overline{A}_i(t)\\ &\le (\|g\|\vee\|h\| + K_1 t + 2\sup_{0\le s\le t}|Z_i(sK_2)|)e^{K_3 t} \equiv \Gamma_i(t). \end{aligned}$$

For each $T>0$, there exists ε_T such that

$$\mathbb{E}[e^{\varepsilon_T \sup_{t\le T}|A_i^U(t)|^2}] \le \mathbb{E}[e^{\varepsilon_T \Gamma_i(T)^2}] < \infty. \qquad (3.3)$$

Proof. Let $A_i^+(t) = A_i^U(t)\vee 0$ and $A_i^-(t) = (-A_i^U(t))\vee 0$. Define

$$\gamma_i^+(t) = \tau_i(t)\vee\sup\{s<t : A_i^U(s)<0\}.$$

If $\gamma_i^+(t) = \tau_i(t) < t$, then $A_i^U(s)\ge 0$ for all s in (3.1); if $0<\gamma_i^+(t)<t$, $A_i^U(\gamma_i^+(t)) = 0$ and

$$\begin{aligned} A_i^U(t) &= \int_{\tau_i(t)}^t G(U(s,X_i(s)),X_i(s))A_i^U(s)ds + \int_{\tau_i(t)}^t b(X_i(s))ds\\ &\quad + \int_{\mathbb{U}\times(\tau_i(t),t]} \rho(X_i(s),u)W(du\times ds); \end{aligned}$$

if $\gamma_i^+(t) = t$, $A_i^+(t)\le\|g\|\vee\|h\|$. In any of these cases

$$A_i^+(t) \le \|g\|\vee\|h\| + \int_{\gamma_i^+(t)}^t K_3A_i^+(s)ds + K_1(t-\gamma_i^+(t)) + \sup_{\gamma_i^+(t)\le s\le t}|H_i(t)-H_i(s)|,$$

so by Gronwall,

$$A_i^+(t) \le (\|g\|\vee\|h\| + K_1(t-\gamma_i^+(t)) + \sup_{\gamma_i^+(t)\le s\le t}|H_i(t)-H_i(s)|)e^{K_3(t-\gamma_i^+(t))}.$$

Letting $\gamma_i^-(t) = \tau_i(t)\vee\sup\{s<t : A_i^U(s)>0\}$, similar observations give

$$A_i^-(t) \le \|g\|\vee\|h\| + \int_{\gamma_i^-(t)}^t K_3A_i^-(s)ds + K_1(t-\gamma_i^-(t)) + \sup_{\gamma_i^-(t)\le s\le t}|H_i(t)-H_i(s)|,$$

so we have a similar bound on A_i^-. Together the bounds give the first two inequalities in (3.2).

H_i is a continuous martingale with quadratic variation $\int_0^t\int\rho(X_i(s),u)^2\mu(du)ds$. Define

$$\gamma(u) = \inf\{t : \int_0^t\int\rho(X_i(s),u)^2\mu(du)ds \ge u\},$$

and $Z_i(u) = H_i(\gamma(u))$. Then Z_i is a continuous martingale with respect to the filtration $\{\mathcal{H}_{\gamma(u)}^i\}$ and $[Z_i]_u = u$, so Z_i is a standard Brownian motion. Since $\sigma(X_i)\subset\mathcal{H}_0^i$, Z_i is independent of X_i.

The first inequality in (3.3) follows by the monotonicity of Γ_i and the finiteness by standard estimates on the distribution of the supremum of Brownian motion. $\square$

The pairs $\{(A_i^U, X_i)\}$ will be exchangeable, so with reference to Lemma 2.9, we can define $\Phi U(t,x)$ to be the density with respect to π of the signed measure determined by

$$\langle \varphi, \Phi U(t)\rangle = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N A_i^U(t)\varphi(X_i(t)). \tag{3.4}$$

Lemma 3.2. *Suppose that U is compatible with W and that (U,W) is independent of $\{X_i\}$. Then ΦU is compatible with W and for each i,*

$$\mathbb{E}[A_i^U(t)|U,W,X_i(t)] = \Phi U(t, X_i(t)). \tag{3.5}$$

If, moreover, U is $\{\mathcal{F}_t^W\}$-adapted, then ΦU is $\{\mathcal{F}_t^W\}$-adapted.

Proof. It follows from (3.4) that $\Phi U(t)$ is measurable with respect to the shift invariant sigma algebra of the stationary sequence (X_i, U, W). Since the $\{X_i\}$ sequence is i.i.d. and independent of (U,W), this sigma algebra is contained in the completion of the sigma algebra generated by (U,W). It then follows from the ergodic theorem that

$$\langle \varphi, \Phi U(t)\rangle = \mathbb{E}[A_1^U(t)\varphi(X_1(t))|U,W].$$

Since A_i^U is $\{\mathcal{F}_t^{X_i,U,W}\}$-adapted, X_i is $\{\mathcal{F}_t^{X_i}\}$-adapted, and X_i and (U,W) are independent, we may replace conditioning by (U,W) by conditioning by $\mathcal{F}_t^{U,W}$ in the previous expression, which shows the compatibility for ΦU in general and $\{\mathcal{F}_t^W\}$-adaptedness if U is $\{\mathcal{F}_t^W\}$-adapted. By exchangeability,

$$\begin{aligned}\mathbb{E}[A_i^U(t)\varphi(X_i(t))F(U,W)] &= \mathbb{E}[\int \varphi(x)\Phi U(t,x)\pi(dx)F(U,W)]\\ &= \mathbb{E}[\varphi(X_i(t))\Phi U(t,X_i(t))F(U,W)],\end{aligned}$$

where the second equality follows by the independence of $X_i(t)$ and (U,W). The lemma then follows by the definition of conditional expectation. □

We will restrict attention in the following results to the case where U is $\{\mathcal{F}_t^W\}$-adapted to simplify the notation slightly. Analogues of these results hold for compatible U as well. The next result shows existence of a version of the density $\Phi U(t,x)$ with the property that $t\mapsto \Phi U(t, X_i(t))$ is well-behaved pathwise.

Lemma 3.3. *Suppose that U is $\{\mathcal{F}_t^W\}$-adapted, and let $\mathcal{G}_t^{X_i} = \sigma(X_i(r) : r \geq t)$. Then there exists a version of $\Phi U(t,x)$ such that*

$$\Phi U(t, X_i(t)) = \mathbb{E}[A_i^U(t)|W, X_i(t)] = \mathbb{E}[A_i^U(t)|\sigma(W)\vee \mathcal{G}_t^{X_i}], \tag{3.6}$$

where we interpret the right side as the optional projection, and for this version

$$\mathbb{E}[\sup_{0\le t\le T} |\Phi U(t,X_i(t))|^2] \le 4\mathbb{E}[\sup_{0\le t\le T}|A_i^U(t)|^2]. \tag{3.7}$$

Moreover the identity (3.5) holds with t replaced by any nonnegative $\sigma(W)$-measurable random variable τ.

Proof. The first equality in (3.6) is just (3.5), and the second follows from the fact that X_i is Markov. By Lemma A.1, there exists a Borel measurable function g on $[0,\infty)\times\mathbb{R}^d\times\mathbb{W}$ such that

$$g(t, X_i(t), W) = \mathbb{E}[A_i^U(t)|\sigma(W)\vee\mathcal{G}_t^{X_i}].$$

It follows that $g(t,x,W)$ is a version of $\Phi U(t,x)$. (Note that an arbitrary version of $\Phi U(t,x)$ may not have the measurability properties of $g(t,x,W)$.)

Corollary A.2, the properties of reverse martingales, and Doob's inequality give (3.7), and the last statement follows by the definition of the optional projection. □

With (3.5) in mind, given an exchangeable family $\{A_i\}$ such that A_i is adapted to $\{\mathcal{F}_t^{X_i} \vee \mathcal{F}^W\}$, define $\Phi A_i \equiv A_i^U$ taking U to be given by

$$U(t, X_i(t)) = \mathbb{E}[A_i(t)|W, X_i(t)] = \mathbb{E}[A_i(t)|\sigma(W) \vee \mathcal{G}_t^{X_i}]$$

in (3.1).

Lemma 3.4. *Suppose that U is $\{\mathcal{F}_t^W\}$-adapted. Then for all $T > 0$ and $p \geq 1$, there exists a constant $C_{p,T}$ so that for all $t \leq T$*

$$\mathbb{E}[|\Phi U(t, X_i(t))|^p|\mathcal{G}_T^{X_i}] \leq C_{p,T}$$

and

$$\mathbb{E}[|A_i^U(t)|^p|\mathcal{G}_T^{X_i}] \leq C_{p,T}.$$

Proof. Recall that we have the bound

$$|A_i^U(t)| \leq \left(\|g\| \vee \|h\| + K_1 t + 2 \sup_{0 \leq r \leq t} | \int_{\mathbb{U} \times (0,r]} \rho(X_i(s), u) W(du \times ds)| \right) e^{K_3 t}$$

and that

$$\Phi U(t, X_i(t)) = \mathbb{E}[A_i^U(t)|W, \mathcal{G}_t^{X_i}].$$

Notice that Jensen's inequality gives

$$\begin{aligned} \mathbb{E}[|\Phi U(t, X_i(t))|^p|\mathcal{G}_T^{X_i}] &= \mathbb{E}\left[\left| \mathbb{E}[A_i^U(t)|W, \mathcal{G}_t^{X_i}] \right|^p \middle| \mathcal{G}_T^{X_i} \right] \\ &\leq \mathbb{E}\left[\mathbb{E}\left[|A_i^U(t)|^p \middle| W, \mathcal{G}_t^{X_i} \right] \middle| \mathcal{G}_T^{X_i} \right] \end{aligned}$$

and that $t \leq T$ implies $\mathcal{G}_T^{X_i} \subset \mathcal{G}_t^{X_i} \vee \sigma(W)$. It follows that

$$\mathbb{E}\left[\mathbb{E}\left[|A_i^U(t)|^p \middle| W, \mathcal{G}_t^{X_i} \right] \middle| \mathcal{G}_T^{X_i} \right] = \mathbb{E}\left[|A_i^U(t)|^p \middle| \mathcal{G}_T^{X_i} \right] \leq \mathbb{E}\left[\Gamma_i^p(t) |\, \mathcal{G}_T^{X_i} \right].$$

Fix $S \in \mathcal{G}_T^{X_i}$ with $P(S) > 0$, so that W is an $\{\mathcal{F}_t^W \vee \sigma(X_i)\}$-martingale measure under $P(\cdot|S)$. By the Burkholder-Davis-Gundy inequality, we find

$$\begin{aligned} &\mathbb{E}\left[\left| \sup_{0 \leq r \leq t} | \int_{\mathbb{U} \times (0,r]} \rho(X_i(s), u) W(du \times ds) \right|^p \middle| S \right] \\ &\leq C_p \mathbb{E}\left[\left(\int_0^t \int_{\mathbb{U}} \rho(X_i(s), u)^2 \mu(du) ds \right)^{\frac{p}{2}} \middle| S \right] \\ &\leq C_p (K_2 t)^{\frac{p}{2}}. \end{aligned}$$

S was arbitrary, so the result follows. □

Lemma 3.5. *Suppose that U is $\{\mathcal{F}_t^W\}$-adapted. Then for each $T \geq 0$, there exists $\varepsilon_T > 0$ such that*

$$\mathbb{E}\left[e^{\varepsilon_T \sup_{t \leq T} |\Phi U(t, X_i(t))|^2} \right] < \infty. \tag{3.8}$$

Proof. As in Lemma 3.1, for each $T > 0$, there exists $\varepsilon_T > 0$ such that $\mathbb{E}[e^{\varepsilon_T \Gamma_i(T)^2}] < \infty$. Recalling that

$$|\Phi U(t, X_i(t))| = |\mathbb{E}[A_i^U(t)|\sigma(W) \vee \mathcal{G}_t^{X_i}]| \leq \mathbb{E}[\Gamma_i(T)|\sigma(W) \vee \mathcal{G}_t^{X_i}],$$

Jensen's and Doob's inequalities give

$$\begin{aligned}\mathbb{E}[e^{\varepsilon_T \sup_{t\leq T}|\Phi U(t,X_i(t)|^2}] &\leq \mathbb{E}[\sup_{t\leq T}\mathbb{E}[e^{\varepsilon_T \Gamma_i(T)^2}|\sigma(W)\vee\mathcal{G}_t^{X_i}]] \\ &\leq 4\mathbb{E}[e^{\varepsilon_T\Gamma_i(T)^2}].\end{aligned}$$

□

3.1 Proof of uniqueness in Theorem 2.13

Truncations based on the moment estimates given above allow us to apply Gronwall's inequality to prove uniqueness for the system (2.12).

Let U_1 and U_2 satisfy $|U_k(t,X_i(t))|\leq \mathbb{E}[\Gamma_i(t)|W,X_i(t)]$, $k=1,2$. Then there exists a constant $L_3>0$ such that

$$\begin{aligned}&|A_i^{U_1}(t)-A_i^{U_2}(t)| \\ &\quad\leq \int_{\tau_i(t)}^t |G(U_1(s,X_i(s)),X_i(s))A_i^{U_1}(s)-G(U_2(s,X_i(s)),X_i(s))A_i^{U_2}(s)|ds \\ &\quad\leq \int_{\tau_i(t)}^t L_1(1+\mathbb{E}[\Gamma_i(s)|W,X_i(s)]^2)|A_i^{U_1}(s)-A_i^{U_2}(s)|ds \\ &\qquad+\int_{\tau_i(t)}^t L_2(1+2\mathbb{E}[\Gamma_i(s)|W,X_i(s)]\Gamma_i(s))|U_1(s,X_i(s))-U_2(s,X_i(s))|ds \\ &\quad\leq \int_0^t L_1(1+C^2)|A_i^{U_1}(s)-A_i^{U_2}(s)|ds \\ &\qquad+\int_0^t L_2(1+2C^2)|U_1(s,X_i(s))-U_2(s,X_i(s))|ds \\ &\qquad+\int_0^t \mathbf{1}_{\{\Gamma_i(s)>C\}\cup\{\mathbb{E}[\Gamma_i(s)|W,X_i(s)]>C\}}\Gamma_i(s)L_3(1+\mathbb{E}[\Gamma_i(s)|W,X_i(s)]^2)ds.\end{aligned}$$

Suppose $U_k=\Phi U_k$, $k=1,2$, that is, we have two solutions. Then conditioning both sides of the above inequality on W and observing

$$\mathbb{E}[|U_1(s,X_i(s))-U_2(s,X_i(s))||W]\leq \mathbb{E}[|A_i^{U_1}(s)-A_i^{U_2}(s)||W],$$

we have

$$\begin{aligned}\mathbb{E}[|A_i^{U_1}(t)-A_i^{U_2}(t)||W]\leq e^{(L_1+2L_2)(1+C^2)t}\int_0^t &\mathbb{E}[\mathbf{1}_{\{\Gamma_i(s)>C\}\cup\{\mathbb{E}[\Gamma_i(s)|W,X_i(s)]>C\}} \\ &\times\Gamma_i(s)L_3(1+\mathbb{E}[\Gamma_i(s)|W,X_i(s)]^2)|W]ds.\end{aligned}$$

Taking expectations of both sides and applying Hölder's inequality,

$$\begin{aligned}&\mathbb{E}[|A_i^{U_1}(t)-A_i^{U_2}(t)|] \\ &\quad\leq e^{(L_1+2L_2)(1+C^2)t}L_3\int_0^t (P\{\Gamma_i(s)>C\}^{1/3}+P\{\mathbb{E}[\Gamma_i(s)|W,X_i(s)]>C\}^{1/3}) \\ &\qquad\qquad\times\mathbb{E}[\Gamma_i(s)^3]^{1/3}\mathbb{E}[(1+\mathbb{E}[\Gamma_i(s)|W,X_i(s)]^2)^3]^{1/3}ds \\ &\quad\leq e^{(L_1+2L_2)(1+C^2)t}e^{-\varepsilon_T C^2/3}2L_3\int_0^t \mathbb{E}[e^{\frac{\varepsilon_T}{3}\Gamma_i(s)^2}\mathbb{E}[\Gamma_i(s)^3]^{1/3} \\ &\qquad\qquad\times\mathbb{E}[(1+\mathbb{E}[\Gamma_i(s)|W,X_i(s)]^2)^3]^{1/3}ds,\end{aligned}$$

so for $t<\frac{\varepsilon_T}{3(L_1+2L_2)}$, the right side goes to zero as $C\to\infty$ implying $U_1=U_2$ on $[0,t]$. The same argument and induction extends uniqueness to any interval.

3.2 Proof of existence in Theorem 2.13

The estimates of the previous section can be applied to give convergence of an iterative sequence proving existence. For an exchangeable family $\{A_i^{(1)}\}$ with $A_i^{(1)}$ $\{\mathcal{F}_t^{X_i,W}\}$-adapted and satisfying $|A_i^{(1)}(t)| \leq \Gamma_i(t)$, recursively define $A_i^{(n+1)} = \Phi A_i^{(n)}$. By the estimates of the previous section, for $n, m \geq 1$

$$\begin{aligned}
&|A_i^{(n+1)}(t) - A_i^{(n+m+1)}(t)| \\
&\quad \leq \int_{\tau_i(t)}^t |G(\mathbb{E}[A_i^{(n)}(s)|W, X_i(s)], X_i(s))A_i^{(n+1)}(s) \\
&\qquad\qquad\qquad -G(\mathbb{E}[A_i^{(n+m)}(s)|W, X_i(s)], X_i(s))A_i^{(n+m+1)}(s)|ds \\
&\quad \leq \int_{\tau_i(t)}^t L_1(1+\mathbb{E}[\Gamma_i(s)|W, X_i(s)]^2)|A_i^{(n+1)}(s) - A_i^{(n+m+1)}(s)|ds \\
&\qquad + \int_{\tau_i(t)}^t L_2(1+2\mathbb{E}[\Gamma_i(s)|W, X_i(s)]\Gamma_i(s))|\mathbb{E}[A_i^{(n)}(s) - A_i^{(n+m)}(s)|W, X_i(s)]|ds \\
&\quad \leq \int_0^t L_1(1+C^2)|A_i^{(n+1)}(s) - A_i^{(n+m+1)}(s)|ds \\
&\qquad + \int_0^t L_2(1+2C^2)|\mathbb{E}[A_i^{(n)}(s) - A_i^{(n+m)}(s)|W, X_i(s)]|ds \\
&\qquad + \int_0^t \mathbf{1}_{\{\Gamma_i(s)>C\}\cup\{\mathbb{E}[\Gamma_i(s)|W,X_i(s)]>C\}}\Gamma_i(s)L_3(1+\mathbb{E}[\Gamma_i(s)|W, X_i(s)]^2)ds.
\end{aligned}$$

Setting

$$H_C(t) = \mathbb{E}[\mathbf{1}_{\{\Gamma_i(s)>C\}\cup\{\mathbb{E}[\Gamma_i(s)|W,X_i(s)]>C\}}\Gamma_i(s)L_3(1+\mathbb{E}[\Gamma_i(s)|W, X_i(s)]^2)|W],$$

we have

$$\begin{aligned}
&\mathbb{E}[\sup_{r\leq t}|A_i^{(n+1)}(r) - A_i^{(n+m+1)}(r)||W] \\
&\quad \leq \int_0^t L_1(1+C^2)\mathbb{E}[\sup_{r\leq s}|A_i^{(n+1)}(r) - A_i^{(n+m+1)}(r)||W]ds \\
&\qquad + \int_0^t L_2(1+2C^2)\mathbb{E}[\sup_{r\leq s}|A_i^{(n)}(r) - A_i^{(n+m)}(r)||W]ds \\
&\qquad + \int_0^t H_C(s)ds.
\end{aligned}$$

It follows that

$$\limsup_{n\to\infty}\sup_{m\geq 1}\mathbb{E}[\sup_{r\leq t}|A_i^{(n)}(r) - A_i^{(n+m)}(r)||W] \leq e^{t(L_1+2L_2)(1+C^2)}\int_0^t H_C(s)ds,$$

and as in the previous section, for $t < \frac{\varepsilon_T}{3(L_1+2L_2)}$, the expectation of the right side goes to zero as C goes to infinity. Consequently, the sequence $\{A_i^{(n)}\}$ is Cauchy on $[0,T]$ for any fixed $T < \frac{\varepsilon_T}{3(L_1+2L_2)}$. Then there exists A_i such that $\lim_{n\to\infty}\mathbb{E}[\sup_{r\leq t}|A_i^{(n)}(r) - A_i(r)|] = 0$ giving existence of a solution on the interval $[0,T]$. The same argument gives existence on $[T/2, 3T/2]$ and uniqueness shows that these solutions coincide on $[T/2, T]$. Using induction, we deduce the global existence of the solution of (2.12).

4 Boundary behavior

In this section, we present two senses in which the particle representation satisfies the boundary condition (1.5). These results depend on the boundary regularity of the stationary diffusions $\{X_i\}$ run forward or backward in time. We begin with the result coming from regularity of the forward process proved in Lemma 2.7, which leads to the weak formulation of the stochastic PDE including the boundary condition in Theorem 2.16.

Let X satisfy (2.3), and assume $X(0)$ has distribution π so that X is stationary. By stationarity, for $t \in \mathbb{R}_+$, the process $X(t+\cdot)$ has the same distribution as $X(\cdot)$. Therefore

$$\mathbb{E}\left[\int_s^t \varphi(X(r))dL(r)\right] \quad = \quad \mathbb{E}\left[\int_0^{t-s} \varphi(X(r))dL(r)\right].$$

For bounded and continuous φ, define

$$Q(t,\varphi) = \mathbb{E}\left[\int_0^t \varphi(X(s))dL(s)\right]$$

which, by the above discussion, satisfies $Q(t+s,\varphi) = Q(t,\varphi) + Q(s,\varphi)$ and $|Q(t,\varphi)| \leq tC\|\varphi\|_\infty$ for some constant C. Therefore, since Q is additive in its first coordinate, there exists a constant C_φ so that $Q(t,\varphi) = tC_\varphi$. Since Q is also linear in its second coordinate, it then follows from the Riesz representation theorem that there exists a measure β on ∂D which satisfies

$$\varphi \mapsto \frac{1}{t}\mathbb{E}\left[\int_0^t \varphi(X(s))dL(s)\right] \quad = \quad \int_{\partial D} \varphi(x)\beta(dx).$$

By considering test functions of product form which are step functions in time, we can see that for sufficiently regular space-time functions φ, we have

$$\int_0^t \int_{\partial D} \varphi(x,s)\beta(dx)ds = \mathbb{E}\left[\int_0^t \varphi(X(s),s)dL(s)\right]. \tag{4.1}$$

Denote partial derivatives with respect to the time variable by ∂. Applying Ito's lemma to $\varphi(X(t),t)$ for sufficiently smooth φ and taking expectations, we also have the following relation between π and β:

$$\int_0^t \int_D (\partial + \mathbb{L})\,\varphi(x,s)\pi(dx)ds \quad = \quad \int_0^t \int_{\partial D} \nabla\varphi(x,s)\cdot\eta(x)\beta(dx)ds.$$

Before the next result, we recall some definitions from analysis. Given a set $A \subset \mathbb{R}$ and $a \in A$, we say that a is an *isolated point* if there exists $\epsilon > 0$ such that $A \cap (a-\epsilon, a+\epsilon) = \{a\}$. We say that $a \in A$ is *left-isolated* if there exists $\epsilon > 0$ such that $A \cap (a-\epsilon, a) = \emptyset$.

Lemma 4.1. *Almost surely, the set $\{t \geq 0 : X(t) \in \partial D\}$ is a closed set with no isolated points and the collection of left-isolated points of this set is countable.*

The proof of Lemma 4.1 is the same as the proof of the analogous property for the zero set of one dimensional Brownian motion and uses the results in Lemma 2.7. See, for example, the proof of Theorem 2.28 in [17].

Lemma 4.2. *Let $\tau(t) = 0 \vee \sup\{s \leq t : X(s) \in \partial D\}$. Then, almost surely,*

$$\int_{\{t:\tau(t)\neq t\}} dL(s) = \int_{\{t:\tau(t-)\neq t\}} dL(s) = 0.$$

Proof. Local time is a continuous measure supported on the set $\{t \geq 0 : X(t) \in \partial D\}$ and therefore assigns measure zero to the (countable) set of left isolated points of $\{t \geq 0 : X(t) \in \partial D\}$. If $t_0 \in \{t \geq 0 : X(t) \in \partial D\}$ is not left-isolated, then $t_0 = \tau(t_0) = \tau(t_0-)$. □

By (4.1) and Lemma 4.2, we have the following theorem.

Theorem 4.3. *Almost surely, for dL_i almost every t, $A_i(t) = A_i(t-) = g(X_i(t))$ and therefore*

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}\int_0^t A_i(s-)\eta(X_i(s))\cdot\nabla\varphi(X_i(s),s)dL_i(s) \\
= &\ \mathbb{E}\left[\int_0^t A_i(s-)\eta(X_i(s))\cdot\nabla\varphi(X_i(s),s)dL_i(s)|U,W\right] \\
= &\ \int_0^t\int_{\partial D} g(x)\eta(x)\cdot\nabla\varphi(x,s)\beta(dx)ds.
\end{aligned}$$

In the next section, we show how Theorem 4.3 leads to a weak formulation of the stochastic partial differential equation with a broader class of test functions than those considered above. We now turn to the form of the boundary condition mentioned in the introduction at (2.15), which depends on regularity of the time-reversed process.

For each t and $s \le t$, define the time reversal of X_i by $X^*_{i,t}(s) = X_i(t-s)$. For notational convenience, when it is clear from context what the value of t is, we will suppress the subscript and take the convention that $X^*_{i,t}(s) \equiv X^*_i(s)$. Since X_i is stationary, the time reversal X^*_i is a Markov process whose generator $\mathbb{A}^*$ satisfies

$$\int_{\overline{D}} g\mathbb{A}f d\pi = \int_{\overline{D}} f\mathbb{A}^* g d\pi, \quad f\in\mathcal{D}(\mathbb{A}).$$

Remark 4.4. If D is sufficiently smooth and $\mathbb{A} = \Delta$ with normally reflecting boundary conditions, then π is proportional to Lebesgue measure, β is proportional to the surface measure, and $\mathbb{A}^* = \mathbb{A}$.

Define the hitting time of the boundary for the reversed process by $\sigma_i = \inf\{s : X^*_i(s) \in \partial D\}$, so if the reversal is from time t, $\sigma_i = t - \tau_i(t)$. Showing that (2.15) is satisfied will depend on the following condition.

Condition 4.5. *The boundary ∂D is regular for X^*_i in the sense that for each $\delta > 0$ and $x \in \partial D$,*

$$\lim_{y\in D\to x} P\left(\sigma_i > \delta | X^*_i(0) = y\right) = 0, \tag{4.2}$$

and

$$\lim_{y\in D\to x} \mathbb{E}\left[|X^*_i(\sigma_i) - x| \wedge 1 | X^*_i(0) = y\right] = 0. \tag{4.3}$$

Remark 4.6. Condition 4.5 is difficult to verify in general. A natural sufficient condition for Condition 4.5 to hold is that the time-reversal is again a regular diffusion; see [3] for results in this direction. The motivating examples discussed in the introduction all have $\mathbb{L} = \Delta$ and are naturally considered with respect to π proportional to the Lebesgue measure, β proportional to the surface measure of ∂D, and reflection along the unit normal vector of ∂D. For sufficiently smooth D, these cases correspond to choosing X_i distributed as normally reflecting Brownian motion, which is reversible, as in Remark 4.4. For these cases, the desired regularity is then standard.

Proposition 4.7. *Suppose that Condition 4.5 is satisfied. With reference to Lemma 3.1, assume U is a $\{\mathcal{F}^W_t\}$-adapted, $L^1(\pi)$-valued process and satisfies*

$$U(t, X_i(t)) \le \mathbb{E}[\Gamma_i(t)|W, X_i(t)], \quad t \ge 0.$$

Let $\overline{g}$ be any continuous function on $\overline{D}$ with $\overline{g}|_{\partial D} = g$. Then for any $t > 0$,

$$\lim_{\epsilon\to 0}\frac{\int_{\partial_\epsilon(D)} |\Phi U(t,x) - \overline{g}(x)|\pi(dx)}{\pi(\partial_\epsilon(D))} = 0$$

in $L^1(P)$.

Proof. By Lemma 3.5 and Jensen's inequality, we have

$$\mathbb{E}\left[\pi(\partial_\epsilon(D))^{-1}\int_{\partial_\epsilon(D)}|\Phi U(t,x)-\overline{g}(x)|\pi(dx)\right]\leq\mathbb{E}\left[|A_i^U(t)-\overline{g}(X_i(t))|\Big|X_i(t)\in\partial_\epsilon(D)\right].$$

Recalling the definition of $A_i^U(t)$, we have

$$\begin{aligned}|A_i^U(t)-\overline{g}(X_i(t))| \quad\leq\quad & |g(X_i(\tau_i(t)))-\overline{g}(X_i(t))|\mathbf{1}_{\{\tau_i(t)>0\}}+2\|h\|_\infty\vee\|g\|_\infty\mathbf{1}_{\{\tau_i(t)=0\}}\\ &+\int_{\tau_i(t)}^t|G(U(s,X_i(s),X_i(s))A_i^U(s)|ds \qquad\qquad (4.4)\\ &+\Big|\int_{D\times(\tau_i(t),t]}\rho(X_i(s),u)W(du,ds)\Big|.\end{aligned}$$

Next, observe that

$$\begin{aligned}&\mathbb{E}\left[|g(X_i(\tau_i(t)))-\overline{g}(X_i(t))|\mathbf{1}_{\{\tau_i(t)>0\}}\Big|X_i(t)\in\partial_\epsilon(D)\right]\\ &\quad=\mathbb{E}\left[|g(X_i^*(\sigma_i))-\overline{g}(X_i^*(0))|\mathbf{1}_{\{\sigma_i<t\}}\Big|X_i^*(0)\in\partial_\epsilon(D)\right]\\ &\quad\quad\leq\sup_{x\in\partial_\epsilon K}\mathbb{E}\left[|g(X_i^*(\sigma_i))-\overline{g}(x)|\mathbf{1}_{\{\sigma_i<t\}}\Big|X_i^*(0)=x\right].\end{aligned}$$

Since D is bounded under Condition 2.1, $\overline{D}$ is compact. Then there exists $x_0\in\partial D$ and $x_n\to x_0$ so that

$$\begin{aligned}&\limsup_{\epsilon\to 0}\sup_{x\in\partial_\epsilon(D)}\mathbb{E}\left[|g(X_i^*(\sigma_i))-\overline{g}(x)|\mathbf{1}_{\{\sigma_i<t\}}\Big|X_i^*(0)=x\right]\\ &\qquad=\lim_{n\to\infty}\mathbb{E}\left[|g(X_i^*(\sigma_i))-\overline{g}(x_n)|\mathbf{1}_{\{\sigma_i<t\}}\Big|X_i^*(0)=x_n\right].\end{aligned}$$

Continuity of $\overline{g}$ and (4.3) imply that the limit is zero. A similar argument and (4.2) show that $P\left(\tau_i(t)=0|X_i(t)\in\partial_\epsilon(D)\right)$ tends to zero, so that the conditional expectations of the first two terms on the right of (4.4) go to zero. Next, we recall that

$$|G(U(s,X_i(s),X_i(s))A_i^U(s)|\leq L_1\left(1+|U(s,X_i(s))|^2\right)|A_i^U(s)|.$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned}&\mathbb{E}\left[\int_{\tau_i(t)}^t|G(U(s,X_i(s),X_i(s))A_i^U(s)|ds\Big|X_i(t)\in\partial_\epsilon(D)\right]\\ &\leq L_1\mathbb{E}\left[t-\tau_i(t)|X_i(t)\in\partial_\epsilon(D)\right]^{\frac{1}{2}}\mathbb{E}\left[\int_0^t\left(1+|U(s,X_i(s))|^2\right)^2|A_i^U(s)|^2ds|X_i(t)\in\partial_\epsilon(D)\right]^{\frac{1}{2}}.\end{aligned}$$

The second factor in this inequality is uniformly bounded by Lemma 3.4, and

$$\begin{aligned}&\mathbb{E}\left[t-\tau_i(t)|X_i(t)\in\partial_\epsilon(D)\right]^{\frac{1}{2}}\\ &\qquad=\mathbb{E}\left[\sigma_i\wedge t|X_i^*(0)\in\partial_\epsilon(D)\right]^{\frac{1}{2}},\end{aligned}$$

which tends to zero by 4.2. Notice that W remains white noise when conditioned on X_i. Using Itô's isometry and the fact that the L^2 norm dominates the L^1 norm, we have

$$\begin{aligned}&\mathbb{E}\left[\Big|\int_{\mathbb{U}\times(\tau_i(t),r]}\rho(X_i(s),u)W(du\times ds)\Big||X_i(t)\in\partial_\epsilon(D)\right]\\ &\qquad\leq\mathbb{E}\left[\int_{\mathbb{U}\times(\tau_i(t),r]}\rho^2(X_i(s),u)duds|X_i(t)\in\partial_\epsilon(D)\right]^{\frac{1}{2}}\\ &\qquad\leq K_2\mathbb{E}\left[t-\tau_i(t)|X_i(t)\in\partial_\epsilon(D)\right]^{\frac{1}{2}}.\end{aligned}$$

As above, the last expression tends to zero by (4.2). □

5 Weak equations with boundary terms

At least in general, uniqueness will not hold for the weak-form stochastic partial differential equation (1.6) using test functions with compact support in D. Consequently, to obtain an equation that has a unique solution, we must enlarge the class of test functions. We do this in two different ways, obtaining two formally different weak-form stochastic differential equations; however, for both equations, we give conditions under which the process given by the particle representation constructed above is the unique solution. The first extension, which gives the simplest form for the stochastic partial differential equation, is obtained by taking the test functions to be $C_0^2(D)$, the space of twice continuously differentiable functions that vanish on the boundary. The second extension is obtained by taking the test functions to be $\mathcal{D}(\mathbb{A})$, the domain of the generator of the semigroup corresponding to the reflecting diffusion giving the particle locations. In this section, we prove Theorems 2.16 and 2.18.

5.1 Proof of Theorem 2.16

In this subsection, we apply the results of Section 4 to enlarge the class of test functions in the definition of the equation (1.6) to all φ in $C_0^2(D)$, the class of C^2-functions that vanish on ∂D. More precisely, we consider test functions of the form $\varphi(x, s)$ which are twice continuously differentiable in x, continuously differentiable in s, and vanish on $\partial D \times [0, \infty)$. To simplify notation, extend g to all of D by setting $g(x) = h(x)$ for $x \in D$. We assume that g is continuous on the boundary, but none of the calculations below require this extension to be continuous.

5.1.1 Derivation of (2.16)

We begin by showing that the process V we have constructed solves (2.16). Define

$$\begin{aligned} Z_i(t) \quad &= \quad -g(X_i(t)) + \int_0^t G(v(s, X_i(s)), X_i(s))A_i(s)ds \\ &\qquad + \int_0^t b(X_i(s))ds + \int_{\mathbb{U}\times[0,t]} \rho(X_i(s), u)W(du \times ds). \end{aligned}$$

Then

$$A_i(t) = g(X_i(t)) + Z_i(t) - Z_i(\tau_i(t)) = Y_i(t) - Z_i(\tau_i(t)), \tag{5.1}$$

and note that Y_i is a semimartingale.

Since A_i, or more precisely, $Z_i \circ \tau_i$, does not appear to be a semimartingale, we derive a version of Itô's formula for $A_i\varphi \circ X_i$ from scratch. Specifically, we consider the limit of the telescoping sum

$$\begin{aligned} \varphi(X_i(t), t)A_i(t) \quad &= \quad \varphi(X_i(0), 0)A_i(0) \\ &\qquad + \sum_k (\varphi(X_i(t_{k+1}), t_{k+1})A_i(t_{k+1}) - \varphi(X_i(t_k), t_k)A_i(t_k)) \\ &= \quad \varphi(X_i(0), 0)A_i(0) + \sum A_i(t_k)(\varphi(X_i(t_{k+1}), t_{k+1}) - \varphi(X_i(t_k), t_k)) \\ &\qquad + \sum \varphi(X_i(t_{k+1}), t_{k+1})(A_i(t_{k+1}) - A_i(t_k)) \end{aligned} \tag{5.2}$$

as the mesh size of the partition $\{t_k, 0 \le k \le m\}$ goes to zero. Since X_i is a semimartingale, $\varphi \circ X_i$ is a semimartingale and the second term on the right converges to the usual semimartingale integral. Since (5.2) is an identity, the second sum must also converge. To distinguish limits of expressions of this form from the usual semimartingale integral, we will write

$$\int_0^t \varphi(X_i(s),s)d^+A_i(s) = \lim_{\max(t_{k+1}-t_k)\to 0} \sum \varphi(X_i(t_{k+1}),t_{k+1})(A_i(t_{k+1})-A_i(t_k)),$$

where the $\{t_k\}$ are partitions of $[0,t]$. Note that this integral differs from the usual semimartingale integral in that we evaluate the integrand at the right end point of the interval $(t_k,t_{k+1}]$ rather than the left. Of course the d^+-integral is still bilinear, so we have

$$\int_0^t \varphi(X_i(s),s)d^+A_i(s) = \int_0^t \varphi(X_i(s),s)d^+Y_i(s) - \int_0^t \varphi(X_i(s),s)d^+Z_i(\tau_i(s)).$$

Observing that the covariation of $\varphi\circ X_i$ and Y_i is zero, applying semimartingale integral results, we get limits for everything in (5.2) but

$$-\sum \varphi(X_i(t_{k+1}),t_{k+1})(Z_i(\tau_i(t_{k+1}))-Z_i(\tau_i(t_k))).$$

Note that the summands are zero unless $\tau_i(t_{k+1})>\tau_i(t_k)$ which means $X_i(t)\in\partial D$ for some $t\in[t_k,t_{k+1}]$. Breaking this expression into pieces, we have

$$\begin{aligned}
-\sum &\varphi(X_i(t_{k+1}),t_{k+1})(Z_i(\tau_i(t_{k+1}))-Z_i(\tau_i(t_k)))\\
&=\sum \varphi(X_i(t_{k+1}),t_{k+1})(g(X_i(\tau_i(t_{k+1})))-g(X_i(\tau_i(t_k))))\\
&\quad -\sum \varphi(X_i(t_{k+1}),t_{k+1})\int_{\tau_i(t_k)}^{\tau_i(t_{k+1})} G(v(s,X_i(s)),X_i(s))A_i(s)ds\\
&\quad -\sum \varphi(X_i(t_{k+1}),t_{k+1})\int_{\tau_i(t_k)}^{\tau_i(t_{k+1})} b(X_i(s))ds\\
&\quad -\sum \varphi(X_i(t_{k+1}),t_{k+1})\int_{\mathbb{U}\times(\tau_i(t_k),\tau_i(t_{k+1})]} \rho(X_i(s),u)W(du\times ds),
\end{aligned}$$

and it is clear that the last three sums on the right converge to zero. It is not immediately clear that the first sum converges to zero, but it does.

Lemma 5.1. *The integral* $\int_0^t \varphi(X_i(s),s)d^+g(X_i(\tau(s)))=0$ *for all* $t\geq 0$.

Proof. Let $\gamma_i(t)=\inf\{s\geq t: X_i(s)\in\partial D\}$. “Summing by parts,”

$$\begin{aligned}
\sum_{k=0}^{m-1} &\varphi(X_i(t_{k+1}),t_{k+1})(g(X_i(\tau_i(t_{k+1})))-g(X_i(\tau_i(t_k))))\\
&=-\sum_{k=0}^{m-1}(\varphi(X_i(t_{k+1}),t_{k+1})-\varphi(X_i(t_k),t_k))g(X_i(\tau_i(t_k)))\\
&\qquad +\varphi(X_i(t),t)g(X_i(\tau_i(t))-\varphi(X(0),0)g(X(0))\\
&\to -\int_0^t g(X_i(\tau_i(s-)))d\varphi(X_i(s),s)\\
&\qquad +\varphi(X_i(t),t)g(X_i(\tau_i(t)))-\varphi(X_i(0),0)g(X_i(0))\\
&=-\int_{\gamma_i(0)}^t g(X_i(\tau_i(s-)))d\varphi(X_i(s),s)+\varphi(X_i(t),t)g(X_i(\tau_i(t))),
\end{aligned}$$

where the last equality follows from the fact that $\tau_i(s)=0$ for $s<\gamma_i(0)$ and $\varphi(X_i(\gamma_i(0)),0)$ $=0$. For each n, define

$$U_n(s)=\sum_{k=0}^{\infty} g(X_i(\gamma_i(\frac{k}{n})))\mathbf{1}_{[\gamma_i(\frac{k}{n}),\gamma_i(\frac{k+1}{n}))}(s).$$

Since $\varphi(X_i(\gamma_i(t),\gamma_i(t))=0$,

$$\begin{aligned}&\int_{\gamma_i(0)}^t g(X_i(\gamma_i(\tfrac{k}{n})))\mathbf{1}_{[\gamma_i(\frac{k}{n}),\gamma_i(\frac{k+1}{n}))}d\varphi(X_i(s),s)\\ &=\begin{cases} g(\gamma_i(\frac{k}{n}))\varphi(X_i(t),t) & \gamma_i(\frac{k}{n})\le t<\gamma_i(\frac{k+1}{n})\\ 0 & \text{otherwise}\end{cases},\end{aligned}$$

and we have

$$\int_{\gamma_i(0)}^t U_n(s-)d\varphi(X_i(s),s)=U_n(t)\varphi(X_i(t),t).$$

If $\gamma_i(\frac{k}{n})<\gamma_i(\frac{k+1}{n})$, then $\frac{k}{n}\le\gamma_i(\frac{k}{n})<\frac{k+1}{n}$, and if $\gamma_i(\frac{k}{n})\le s<\gamma_i(\frac{k+1}{n})$, then $\gamma_i(\frac{k}{n})\le\tau_i(\frac{k}{n})<\frac{k+1}{n}$. Consequently, $\lim_{n\to\infty}U_n(s)=g(X_i(\tau_i(s)))$ and

$$g(X_i(\tau_i(t)))\varphi(X_i(t),t)=\lim_{n\to\infty}\int_{\gamma_i(0)}^t U_n(s-)d\varphi(X_i(s),s)=\int_{\gamma_i(0)}^t g(\tau_i(s-))d\varphi(X_i(s),s)$$

proving the lemma. □

Defining

$$M_{\varphi,i}(t)=\int_0^t\nabla\varphi(X_i(s),s)^T\sigma(X_i(s))dB_i(s),$$

we have

$$\begin{aligned}\varphi(X_i(t),t)A_i(t) &= \varphi(X_i(0),0)A_i(0)+\int_0^t\varphi(X_i(s),s)d^+A_i(s)\\ &\quad+\int_0^t A_i(s)dM_{\varphi,i}(s)+\int_0^t A_i(s)(\mathbb{L}\varphi(X_i(s),s)+\partial\varphi(X_i(s),s))ds\\ &\quad+\int_0^t A_i(s-)\nabla\varphi(X_i(s),s)\cdot\eta(X_i(s))dL_i(s)\\ &= \varphi(X_i(0),0)g(X_i(0))+\int_0^t A_i(s)\varphi(X_i(s),s)G(v(s,X_i(s)),X_i(s))ds\\ &\quad+\int_0^t\varphi(X_i(s),s)b(X_i(s))ds\\ &\quad+\int_{\mathbb{U}\times[0,t]}\varphi(X_i(s),s)\rho(X_i(s),u)W(du\times ds)\\ &\quad+\int_0^t A_i(s)dM_{\varphi,i}(s)+\int_0^t A_i(s)(\mathbb{L}\varphi(X_i(s),s)+\partial\varphi(X_i(s),s))ds\\ &\quad+\int_0^t A_i(s-)\nabla\varphi(X_i(s),s)\cdot\eta(X_i(s))dL_i(s).\end{aligned}$$

Applying Theorem 4.3, the averaged identity becomes (2.16).

5.1.2 Proof of uniqueness in Theorem 2.16

Fix any $U\in\mathcal{L}(\pi)$. We begin by proving uniqueness for the equation linearized by replacing V in G by U, that is, we want to solve

$$\begin{aligned}\langle\varphi(\cdot,t),V(t)\rangle &= \langle\varphi(\cdot,0),h\rangle_\pi+\int_0^t\langle\varphi(\cdot,s)G(U(s,\cdot),\cdot),V(s)\rangle ds\\ &\quad+\int_0^t\int_{\overline{D}}\varphi(x,s)b(x)\pi(dx)ds \qquad (5.3)\end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{U}\times[0,t]}\int_{\overline{D}}\varphi(x,s)\rho(x,u)\pi(dx)W(du\times ds)\\
&+\int_0^t\langle\mathbb{L}\varphi(\cdot,s)+\partial\varphi(\cdot,s),V(s)\rangle ds\\
&+\int_0^t\int_{\partial D}g(x)\eta(x)\cdot\nabla\varphi(x,s)\beta(dx)ds.
\end{aligned}
$$

We will refer to this equation as the linearized equation with input U.

Suppose there are two solutions V_1 and V_2 of (5.3), and let $\delta V = V_1 - V_2$. Then by linearity,

$$
\begin{aligned}
\langle\varphi(\cdot,t),\delta V(t)\rangle &= \int_0^t\langle\varphi(\cdot,s)G(U(s,\cdot),\cdot),\delta V(s)\rangle ds \\
&\quad+\int_0^t\langle\mathbb{L}\varphi(\cdot,s)+\partial\varphi(\cdot,s),\delta V(s)\rangle ds.
\end{aligned}
\tag{5.4}
$$

The assumption that $\mathbb{A}^0$ is the closure of $\{(\varphi,\mathbb{L}\varphi):\varphi\in\mathcal{D}_0\}$ implies that we can extend this identity to functions φ which are differentiable in time and satisfy $\varphi(\cdot,s)\in\mathcal{D}(\mathbb{A}^0)$ with $\mathbb{A}^0\varphi$ and $\partial\varphi$ bounded and continuous. We denote by $\mathbb{T}^0$ the semigroup generated by $\mathbb{A}^0$. Then for $\psi\in\mathcal{D}(\mathbb{A}_0)$ and $\varphi(x,s)=r_\epsilon(s)\mathbb{T}^0(t-s)\psi(x)$, where $0\le r_\epsilon\le 1$ is continuously differentiable, $r_\epsilon(s)=0$ for $s\ge t$ and $r_\epsilon(s)=1$, for $s\le t-\epsilon$, (5.4) becomes

$$
0=\int_0^t\langle r_\epsilon(s)\mathbb{T}^0(t-s)\psi G(U(s,\cdot),\cdot),\delta V(s)\rangle ds+\int_0^t\langle\partial r_\epsilon(s)\mathbb{T}^0(t-s)\psi,\delta V(s)\rangle. \tag{5.5}
$$

Assuming $r_\epsilon(s)\to\mathbf{1}_{[0,t)}(s)$ appropriately, the second term on the right of (5.5) converges to $\langle\psi,\delta V(t)\rangle$ and hence

$$
\langle\psi,\delta V(t)\rangle=-\int_0^t\langle\mathbb{T}^0(t-s)\psi(\cdot)G(U(s,\cdot),\cdot),\delta V(s)\rangle ds.
$$

With reference to Condition 2.12, taking the supremum over $\psi\in\mathcal{D}(\mathbb{A}^0)$ with $|\psi|\le 1$,

$$
\begin{aligned}
\int_{\overline{D}}|\delta v(t,x)|\pi(dx) &\le \int_0^t\int_{\overline{D}}|G(U(s,x),x)||\delta v(s,x)|\pi(dx)ds\\
&\le \int_0^t L_1\int_{\overline{D}}(1+|U(s,x)|^2)|\delta v(s,x)|\pi(dx)ds\\
&\le \int_0^t L_1(1+C^2)\int_{\overline{D}}|\delta v(s,x)|\pi(dx)ds\\
&\qquad+\int_0^t L_1\int_{\overline{D}}\mathbf{1}_{\{|U(s,x)|\ge C\}}(1+|U(s,x|^2)|\delta v(s,x)|\pi(dx)ds.
\end{aligned}
$$

Taking expectations of both sides and applying the Hölder inequality, we have

$$
\begin{aligned}
&\int_{\overline{D}}\mathbb{E}[|\delta v(t,x)|]\pi(dx)\\
&\quad\le\int_0^t L_1(1+C^2)\int_{\overline{D}}\mathbb{E}[|\delta v(s,x)|]\pi(dx)ds\\
&\qquad+\int_0^t L_1\left(\int_{\overline{D}}P\{|U(s,x)|\ge C\}\pi(dx)\right)^{1/3}\\
&\qquad\quad\times\left(\int_{\overline{D}}\mathbb{E}[(1+|U(s,x|^2)^3]\pi(dx)\right)^{1/3}\left(\int_{\overline{D}}|\delta v(s,x)|^3\pi(dx)\right)^{1/3}ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq \int_0^t L_1(1+C^2)\int_{\overline{D}}\mathbb{E}[|\delta v(s,x)|]\pi(dx)ds\\
&\quad + e^{-\varepsilon_T C^2/3}\int_0^t L_1\left(\int_{\overline{D}}\mathbb{E}[e^{\varepsilon_T|U(s,x)|^2}]\pi(dx)\right)^{1/3}\\
&\qquad \times\left(\int_{\overline{D}}\mathbb{E}[(1+|U(s,x|^2)^3]\pi(dx)\right)^{1/3}\left(\int_{\overline{D}}|\delta v(s,x)|^3\pi(dx)\right)^{1/3}ds,
\end{aligned}
$$

for $t \leq T$. As in the proof of Theorem 2.13, Gronwall's inequality implies $\delta v(t,\cdot)=0$ for $t \leq T \wedge \frac{\varepsilon_T}{6L_1}$, that is, $t<T$ satisfying $\lim_{C\to\infty}(2L_1(1+C^2)t-\varepsilon_T C^2/3=-\infty$. But local uniqueness implies global uniqueness, so we have uniqueness for the linearized equation.

To complete the proof, we follow an argument used in the proof of Theorem 3.5 of [14]. Let V be the solution constructed in Section 3, and let U be another solution of (2.16) in $\mathcal{L}(\pi)$. Taking this U as the input in (3.1), we can construct a particle system with weights A_i^U and let ΦU be given by (3.4). Recall that $\Phi U \in \mathcal{L}(\pi)$. Averaging, we see that ΦU satisfies

$$
\begin{aligned}
\langle \varphi(\cdot,t),\Phi U(t)\rangle &= \langle \varphi(\cdot,0),h\rangle_\pi + \int_0^t \langle \varphi(\cdot,s)G(U(s,\cdot),\cdot),\Phi U(s)\rangle ds\\
&\quad + \int_0^t\int_{\overline{D}}\varphi(x,s)b(x)\pi(dx)ds\\
&\quad + \int_{\mathbb{U}\times[0,t]}\int_{\overline{D}}\varphi(x,s)\rho(x,u)\pi(dx)W(du\times ds)\\
&\quad + \int_0^t\langle \mathbb{L}\varphi(\cdot,s)+\partial\varphi(\cdot,s),\Phi U(s)\rangle ds\\
&\quad + \int_0^t\int_{\partial D} g(x)\eta(x)\cdot\nabla\varphi(x,s)\beta(dx)ds,
\end{aligned}
$$

that is, ΦU satisfies the linearized equation with input U. But solutions of the linearized equation are unique, and since U is a solution of the nonlinear equation, it is also a solution of the linearized equation with input U and by uniqueness, we have $\Phi U = U$. But that means U has a particle representation which solves the same system of equations as the particle representation for V, and hence Theorem 2.13 implies that $U = V$.

5.2 Proof of Theorem 2.18

In this subsection, we take the set of test functions to be $\mathcal{D}(\mathbb{A})$, the domain of the generator for the semigroup corresponding to the location processes. More precisely, we take $\varphi(x,t)$ that are continuously differentiable in t with $\varphi(x,t)=0$ for $t>t_\varphi$, and $\varphi(\cdot,t)\in\mathcal{D}(\mathbb{A})$, $t\geq 0$, so that $\mathbb{A}\varphi$ is bounded and continuous. As above, we extend g to all of D by setting $g(x)=h(x)$ for $x\in D$. Let

$$\gamma_i(s)=\inf\{t>s : X_i(t)\in\partial D\}, \tag{5.6}$$

and note that $\mathbf{1}_{\{\tau_i(t)=0\}}=\mathbf{1}_{\{\gamma_i(0)>t\}}$. Let $P(dy,ds|x)$ be the conditional distribution of $(X_i(\gamma_i(0)),\gamma_i(0))$ given $X_i(0)=x$, and let $P\varphi(x)=\int\varphi(y,s)P(dy,ds|x)$. Let X^* be the reversed process and γ^* be the first time that X^* hits the boundary.

5.2.1 Derivation of (2.18)

For φ satisfying the conditions stated above,

$$M_{\varphi,i}(t)=\varphi(X_i(t),t)-\int_0^t(\mathbb{A}\varphi(X_i(s),s)+\partial\varphi(X_i(s),s)ds$$

is a $\{\mathcal{F}_t^{W,X_i}\}$-martingale. As before, consider

$$\begin{aligned}\varphi(X_i(t),t)A_i(t) &= \varphi(X_i(0),0)A_i(0) \\ &\quad + \sum_k (\varphi(X_i(t_{k+1}),t_{k+1})A_i(t_{k+1}) - \varphi(X_i(t_k),t_k)A_i(t_k)) \\ &= \varphi(X_i(0),0)A_i(0) + \sum A_i(t_k)(\varphi(X_i(t_{k+1}),t_{k+1}) - \varphi(X_i(t_k),t_k)) \\ &\quad + \sum \varphi(X_i(t_{k+1}),t_{k+1})(A_i(t_{k+1}) - A_i(t_k))\end{aligned} \tag{5.7}$$

as the mesh size of the partition $\{t_k, 0 \le k \le m\}$ goes to zero. Since $M_{\varphi,i}$ is a martingale, $\varphi \circ X_i$ is a semimartingale and the second term on the right converges to the usual semimartingale integral. We must evaluate

$$\int_0^t \varphi(X_i(s),s)d^+A_i(s),$$

Breaking A_i into its components, the difficulty is again

$$-\sum \varphi(X_i(t_{k+1}),t_{k+1})(Z_i(\tau_i(t_{k+1})) - Z_i(\tau_i(t_k))),$$

but the limit is different. Breaking this expression into pieces, we have

$$\begin{aligned}&-\sum \varphi(X_i(t_{k+1}),t_{k+1})(Z_i(\tau_i(t_{k+1})) - Z_i(\tau_i(t_k))) \\ &\quad = \sum \varphi(X_i(t_{k+1}),t_{k+1})(g(X_i(\tau_i(t_{k+1}))) - g(X_i(\tau_i(t_k)))) \\ &\qquad - \sum \varphi(X_i(t_{k+1}),t_{k+1}) \int_{\tau_i(t_k)}^{\tau_i(t_{k+1})} G(v(s,X_i(s)),X_i(s))A_i(s)ds \\ &\qquad - \sum \varphi(X_i(t_{k+1}),t_{k+1}) \int_{\tau_i(t_k)}^{\tau_i(t_{k+1})} b(X_i(s))ds \\ &\qquad - \sum \varphi(X_i(t_{k+1}),t_{k+1}) \int_{\mathbb{U}\times(\tau_i(t_k),\tau_i(t_{k+1})]} \rho(X_i(s),u)W(du\times ds).\end{aligned}$$

Summing by parts, we can write the first expression on the right as

$$\begin{aligned}&\sum_{k=0}^{m-1} \varphi(X_i(t_{k+1}),t_{k+1})(g(X_i(\tau_i(t_{k+1}))) - g(X_i(\tau_i(t_k)))) \\ &\quad = -\sum_{k=0}^{m-1} (\varphi(X_i(t_{k+1}),t_{k+1}) - \varphi(X_i(t_k),t_k))g(X_i(\tau_i(t_k))) \\ &\qquad +\varphi(X_i(t),t)g(X_i(\tau_i(t)) - \varphi(X(0),0)g(X(0)) \\ &\quad \to -\int_0^t g(X_i(\tau_i(s-)))dM_{\varphi,i}(s) - \int_0^t g(X_i(\tau_i(s)))(\mathbb{A}\varphi(X_i(s),s) + \partial\varphi(X_i(s),s)ds \\ &\qquad +\varphi(X_i(t),t)g(X_i(\tau_i(t))) - \varphi(X_i(0),0)g(X_i(0)).\end{aligned}$$

For the remaining three terms, a summand is nonzero only if $\tau_i(t_{k+1}) \neq \tau_i(t_k)$ which implies that X_i hits the boundary between t_k and t_{k+1}. If the mesh size is small, by continuity, $X_i(t_{k+1})$ must be close to $X_i(\tau(t_{k+1}))$. Let $\mathcal{E}_i(t)$ be the set of complete excursions from the boundary in the interval $[0,t]$. Then the last three terms converge to

$$-\sum_{(\alpha,\beta]\in\mathcal{E}_i(t)} \varphi(X_i(\beta),\beta) \int_\alpha^\beta G(v(s,X_i(s)),X_i(s))A_i(s)ds$$

$$-\sum_{(\alpha,\beta]\in\mathcal{E}_i(t)}\varphi(X_i(\beta),\beta)\int_\alpha^\beta b(X_i(s))ds$$
$$-\sum_{(\alpha,\beta]\in\mathcal{E}_i(t)}\varphi(X_i(\beta),\beta)\int_{\mathbb{U}\times(\alpha,\beta]}\rho(X_i(s),u)W(du\times ds).$$

Note that for γ_i given by (5.6) and $s\in(\alpha,\beta)$, $\gamma_i(s)=\beta$, so we can write

$$\begin{aligned}
-\int_0^\infty &\varphi(X_i(s),s)d^+Z_i(\tau_i(s))\\
&=-\int_0^\infty g(X_i(\tau_i(s-)))dM_{\varphi,i}(s)-\int_0^\infty g(X_i(\tau_i(s)))(\mathbb{A}\varphi(X_i(s),s)+\partial\varphi(X_i(s),s))ds\\
&\quad-\varphi(X_i(0),0)g(X_i(0))-\int_0^\infty\varphi(X_i(\gamma_i(s)),\gamma_i(s))G(v(s,X_i(s)),X_i(s))A_i(s)ds\\
&\quad-\int_0^\infty\varphi(X_i(\gamma_i(s)),\gamma_i(s))b(X_i(s))ds\\
&\quad-\int_{\mathbb{U}\times(0,\infty)}\varphi(X_i(\gamma_i(s)),\gamma_i(s))\rho(X_i(s),u)W(du\times ds),
\end{aligned}$$

and we have

$$\begin{aligned}
\varphi(X_i(t),t)A_i(t) &= \varphi(X_i(0),0)A_i(0)+\int_0^t\varphi(X_i(s),s)d^+A_i(s)\\
&\quad+\int_0^t A_i(s)dM_{\varphi,i}(s)+\int_0^t(\mathbb{A}\varphi(X_i(s),s)+\partial\varphi(X_i(s),s))A_i(s)ds\\
&= \varphi(X_i(0),0)g(X_i(0))+\int_0^t\varphi(X_i(s),s)G(v(s,X_i(s)),X_i(s))A_i(s)ds\\
&\quad+\int_0^t\varphi(X_i(s),s)b(X_i(s))ds\\
&\quad+\int_{\mathbb{U}\times[0,t]}\varphi(X_i(s),s)\rho(X_i(s),u)W(du\times ds)\\
&\quad+\int_0^t A_i(s)dM_{\varphi,i}(s)+\int_0^t(\mathbb{A}\varphi(X_i(s),s)+\partial\varphi(X_i(s),s))A_i(s)ds\\
&\quad-\int_0^t\varphi(X_i(s),s)d^+Z_i(\tau_i(s))\\
&= \int_0^t(\varphi(X_i(s),s)-\mathbf{1}_{\{\gamma_i(s)\le t\}}\varphi(X_i(\gamma_i(s))),\gamma_i(s)))G(v(s,X_i(s)),X_i(s))A_i(s)ds\\
&\quad+\int_0^t(\varphi(X_i(s),s)-\mathbf{1}_{\{\gamma_i(s)\le t\}}\varphi(X_i(\gamma_i(s))),\gamma_i(s)))b(X_i(s))ds\\
&\quad+\int_{\mathbb{U}\times[0,t]}(\varphi(X_i(s),s)-\mathbf{1}_{\{\gamma_i(s)\le t\}}\varphi(X_i(\gamma_i(s))),\gamma_i(s)))\rho(X_i(s),u)W(du\times ds)\\
&\quad+\int_0^t(A_i(s)-g(X_i(\tau_i(s))))dM_{\varphi,i}(s)\\
&\quad+\int_0^t(\mathbb{A}\varphi(X_i(s),s)+\partial\varphi(X_i(s),s))(A_i(s)-g(X_i(\tau_i(s))))ds\\
&\quad+\varphi(X_i(t),t)g(X_i(\tau_i(t)))
\end{aligned}$$

Recalling that φ is nonzero only on a finite time interval and letting $t=\infty$, averaging gives (2.18).

5.2.2 Proof of uniqueness in Theorem 2.18

The proof of uniqueness is essentially the same as the corresponding proof for Theorem 2.16.

A Appendix

A.1 Gaussian white noise

Let μ be a σ-finite Borel measure on a complete, separable metric space $(\mathbb{U}, d)$, and let ℓ be Lebesgue measure on $[0,\infty)$. Define $\mathcal{A}(\mathbb{U}) = \{A \in \mathcal{B}(\mathbb{U}) : \mu(A) < \infty\}$ and $\mathcal{A}(\mathbb{U} \times [0,\infty) = \{C \in \mathcal{B}(\mathbb{U} \times [0,\infty)) : \mu \times \ell(C) < \infty\}$ with $\mathcal{A}(\mathbb{U} \times [0,t])$ defined similarly. Then $W = \{W(C) : C \in \mathcal{A}(\mathbb{U} \times [0,\infty))\}$ is space-time Gaussian white noise with covariance measure μ if the $W(C)$ are jointly Gaussian, with $\mathbb{E}[W(C)] = 0$ and $\mathbb{E}[W(C_1)W(C_2)] = \mu \times \ell(C_1 \cap C_2)$. In particular, for each $A \in \mathcal{A}(\mathbb{U})$, $W(A \times [0,t])$ is Gaussian with $\mathbb{E}[W(A \times [0,t])] = 0$ and $Var(W(A \times [0,t])) = \mu(A)t$, that is, $W(A \times [0,\cdot])$ is a Brownian motion with parameter $\mu(A)$, and

$$\mathbb{E}[W(A \times [0,t])W(B \times [0,s])] = \mu(A \cap B)t \wedge s.$$

It follows that for disjoint C_i satisfying $\sum_{i=1}^{\infty} \mu \times \ell(C_i) < \infty$,

$$W(\cup_i^\infty C_i) = \sum_{i=1}^{\infty} W(C_i)$$

almost surely, but the exceptional event of probability zero, will in general depend on the sequence $\{C_i\}$, that is, while W acts in some ways like a signed measure, it is not a random signed measure.

Define the filtration $\{\mathcal{F}_t^W\}$ by

$$\mathcal{F}_t^W = \sigma(W(C) : C \in \mathcal{A}(\mathbb{U} \times [0,t]).$$

For $i = 1, \ldots, m$, let $A_i \in \mathcal{A}(\mathbb{U})$ and let ξ_i be a process adapted to $\{\mathcal{F}_t^W\}$ satisfying $\mathbb{E}[\int_0^t \xi_i(s)^2 ds] < \infty$ for each $t > 0$. Define

$$Y(u,t) = \sum_{i=1}^{m} \mathbf{1}_{A_i}(u)\xi_i(t)$$

and

$$Z(t) = \int_{U \times [0,t]} Y(u,s)W(du \times ds) = \sum_{i=1}^{m} \int_0^t \xi_i(s) dW(A_i \times [0,s]).$$

Then, from properties of the Itô integral,

$$\mathbb{E}[Z(t)] = 0 \qquad [Z]_t = \int_0^t \int_{\mathbb{U}} Y^2(u,s)\mu(du)ds$$
$$\mathbb{E}[Z^2(t)] = \mathbb{E}[[Z]_t] = \int_0^t \int_{\mathbb{U}} \mathbb{E}[Y^2(u,s)]\mu(du)ds.$$

By this last identity, the integral can be extended to all measurable processes Y that are adapted to $\{\mathcal{F}_t^W\}$ in the sense that Y restricted to $[0,t] \times \mathbb{U}$ is $\mathcal{B}([0,t]) \times \mathcal{B}(\mathbb{U}) \times \mathcal{F}_t^W$-measurable and satisfy

$$\mathbb{E}[\int_0^t \int_{\mathbb{U}} Y^2(u,s)\mu(du)ds] < \infty, \quad t > 0.$$

By a truncation argument, the integral can be extended to adapted Y satisfying

$$\int_0^t \int_{\mathbb{U}} Y^2(u,s)\mu(du)ds < \infty \quad a.s., \quad t > 0.$$

Note that under this last extension, the integral is a locally square integrable martingale with

$$[Z]_t = \int_0^t \int_{\mathbb{U}} Y^2(u,s)\mu(du)ds.$$

A.2 Measurability of density process

Lemma A.1. *Let E and $\mathbb{W}$ be complete separable metric spaces, let X be a stationary Markov process in E with no fixed points of discontinuity, and let W be a $\mathbb{W}$-valued random variable that is independent of X. Let $f : [0,\infty) \times D_E[0,\infty) \times \mathbb{W} \to \mathbb{R}$ be bounded and Borel measurable and be nonanticipating in the sense that*

$$f(t,x,w) = f(t,x(\cdot \wedge t-),w), \quad (t,x,w) \in [0,\infty) \times D_E[0,\infty) \times \mathbb{W}.$$

Letting $\{\mathcal{G}_t^X\}$ denote the reverse filtration, $\mathcal{G}_t^X = \sigma(X(s-), s \geq t)$, there exists a Borel measurable $g : [0,\infty) \times E \times \mathbb{W} \to \mathbb{R}$ such that $\{g(t,X(t-),W), t \geq 0\}$ gives the optional projection of $\{f(t,X,W), t \geq 0\}$, that is, for each reverse stopping time τ, ($\{\tau \geq t\} \in \sigma(W) \vee \mathcal{G}_t^X, t \geq 0$),

$$\mathbb{E}[f(\tau,X,W)|\sigma(W) \vee \mathcal{G}_\tau^X] = g(\tau, X(\tau-), W). \tag{A.1}$$

Proof. Let R be the collection of bounded, Borel measurable functions f for which the assumptions and conclusions of the lemma hold. Then R is closed under bounded, pointwise limits of increasing functions and under uniform convergence.

For $0 \leq t_1 < \cdots < t_m$, $f_i \in B([0,\infty) \times E)$, $f_0 \in B(\mathbb{W})$, let

$$f(t,x,w) = f_0(t,w) \prod f_i(t, x(t_i \wedge t-)) \in R. \tag{A.2}$$

Then letting $\{T^*(t)\}$ denote the semigroup for the time-reversed process, g can be expressed in terms of $\{T^*(t)\}$ and the f_i. For example, if $m = 2$,

$$g(t,x(t-),w) = f_0(t,w)T^*(t - t_2 \wedge t)[f_2 T^*(t_2 \wedge t - t_1 \wedge t) f_1](x(t-)).$$

Let H_0 be the collection of functions of the form (A.2). Then by the appropriate version of the monotone class theorem (for example, Corollary 4.4 in the Appendix of [6]), R contains all bounded functions that are $\sigma(H_0)$ measurable, that is all bounded measurable functions such that $f(t,x,w) = f(t,x(\cdot \wedge t-),w)$. □

Corollary A.2. *For each $T > 0$,*

$$\mathbb{E}[\sup_{0 \leq s \leq T} f(s,X,W)|\sigma(W) \vee \mathcal{G}_t^X] \geq g(t,X(t-),W) \quad \forall t \in [0,T] \quad a.s. \tag{A.3}$$

Proof. The uniqueness of the optional projection up to indistinguishability ensures that if $Z_1(t) \leq Z_2(t)$ for all t with probability one, then the optional projection of Z_1 is less than or equal to the optional projection of Z_2 for all t with probability one. □

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