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Particle Representations for Measure-Valued Population Processes with Spatially Varying Birth Rates

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ABSTRACT. Representations of measure-valued processes in terms of countable systems of particles are constructed for models with spatially varying birth and death rates. In previous constructions for models with birth and death rates not depending on location or type, the particles were assigned integer-valued "levels", the joint distributions of the particle types were exchangeable, and the measure-valued process K was given by $K(t) = P(t)\bar{Z}(t)$, where P was the "total mass" process and $\bar{Z}(t)$ was the de Finetti measure for the exchangeable particle types at time t. In the present construction, particles are assigned real-valued levels and for each time t the joint distribution of locations and levels is conditionally Poisson distributed with mean measure $K(t) \times m$. The representation gives an explicit construction of the boundary measure in Dynkin's probabilistic solution of the nonlinear partial differential equation $\lambda(x)v(x)^\gamma - Bv(x) = \rho(x)$, $x \in D$, $v(x) = f(x)$, $x \in \partial D$. The representation also provides a way of generalizing Perkins's models for measure-valued processes in which the individual particle motion depends on the distribution of the population. Questions of uniqueness, however, remain open for most of the models in this larger class.

1. Exchangeable population models

We begin by considering a class of finite population models. Let $N(t)$ denote the total population size at time t, and let $X(t) = (X_1(t), \dots, X_{N(t)}(t))$ denote the locations of population members in a complete, separable metric space E. The state space for the models is then $\hat{E} = \cup_{k=0}^\infty E^k$, where E^0 denotes the single state in which the population size is zero. If the state x is in E^k, we will sometimes write (x, k) to emphasize the length of the vector. We will refer to individual population members as *particles*. The behavior of each particle will depend on the others only through the empirical measure

$$Z(t) = \sum_{i=1}^{N(t)} \delta_{X_i(t)},$$

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that is, the order of the particles is not significant.

The models we consider will be Markov, specified by their generators, that is, the operators that characterize the processes as solutions of martingale problems. To specify a model, we must define the operator Af for f in an appropriate domain $\mathcal{D}(A)$.

For $k = 1, 2, \ldots$, let Γ_k be the collection of all permutations of $\{1, \ldots, k\}$. For $x \in E^k$ and $\sigma \in \Gamma_k$, let $x_\sigma = (x_{\sigma_1}, \ldots, x_{\sigma_k})$. Let B be the generator of a Markov process on E. B will determine the individual particle motion. For $1 \le i \le k$, $B_i f(x,k)$ will denote the operator B applied to f as a function of x_i. For example, for $g_i \in \mathcal{D}(B)$, $i = 1, 2, \ldots$ with $g_i \equiv 1$ for i sufficiently large, define

$$f(x,k) = \prod_{j=1}^{k} g_j(x_j). \tag{1.1}$$

Then

$$B_i f(x,k) = Bg_i(x_i) \prod_{1 \le j \le k, j \ne i} g_j(x_j) = \frac{Bg_i(x_i)}{g_i(x_i)} f(x,k).$$

Let $\lambda_{-1}(x_i, x)$ denote the "death rate" for the ith particle, and for $m \ge 1$, let $\lambda_m(x_i, x)$ be the intensity for a birth event in which the ith particle gives birth to m "offspring". We assume that offspring are initially placed at the location of the "parent". For $m = -1$ and $m \ge 1$, we assume that for $x \in E^k$ and $\sigma \in \Gamma_k$, $\lambda_m(x_i, x) = \lambda_m(x_{\sigma_j}, x_\sigma)$, whenever $x_i = x_{\sigma_j}$. In particular, the birth and death rates for a particle depend only on its location and the empirical measure $\sum_{j=1}^k \delta_{x_j}$.

In terms of these parameters, we have the generator

$$\begin{aligned} A_0 f(x,k) &= \sum_{i=1}^{k} B_i f(x,k) \\ &\quad + \sum_{i=1}^{k} \sum_{m=1}^{\infty} \lambda_m(x_i, x) \frac{1}{\binom{k+m}{m}} \sum_{1 \le j_1 < \cdots < j_m \le k+m} \\ &\qquad (f(\theta_{j_1,\ldots,j_m}(x|x_i), k+m) - f(x,k)) \\ &\quad + \sum_{i=1}^{k} \lambda_{-1}(x_i, x)(f(d_i(x), k-1) - f(x,k)). \end{aligned} \tag{1.2}$$

For $x \in E^k$, $\theta_{j_1,\ldots,j_m}(x|z)$ is the element $x' \in E^{k+m}$ obtained from x by setting $x'_{j_l} = z$, $l = 1, \ldots, m$, and defining the remaining k components of x' to be the components of x, preserving the order, and $d_i(x) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k) \in E^{k-1}$. We take $\mathcal{D}(A_0)$ to be the linear space generated by functions of the form (1.1).

For simplicity, we assume that

$$\sup_k \sup_{x \in E^k} \sum_m m \lambda_m(x_i, x) < \infty. \tag{1.3}$$

This condition states that the per particle birth rate is uniformly bounded and, in particular, implies that the population size cannot blow up in finite time. If uniqueness holds for the martingale problem of B, then this fact implies uniqueness will hold for A_0.

The following theorem (Corollary 3.5 from Kurtz (1998)) plays an essential role in our construction. Let (S,d) and (S_0,d_0) be complete, separable metric spaces. An operator $A\subset B(S)\times B(S)$ is *dissipative* if $\|f_1-f_2-\epsilon(g_1-g_2)\|\geq\|f_1-f_2\|$ for all $(f_1,g_1),(f_2,g_2)\in A$ and $\epsilon>0$; A is a *pre-generator* if A is dissipative and there are sequences of functions $\mu_n:S\to\mathcal{P}(S)$ and $\lambda_n:S\to[0,\infty)$ such that for each $(f,g)\in A$

$$g(x)=\lim_{n\to\infty}\lambda_n(x)\int_S(f(y)-f(x))\mu_n(x,dy) \tag{1.4}$$

for each $x\in S$. A is *graph separable* if there exists a countable subset $\{g_k\}\subset\mathcal{D}(A)\cap\bar{C}(S)$ such that the graph of A is contained in the bounded, pointwise closure of the linear span of $\{(g_k,Ag_k)\}$. (More precisely, we should say that there exists $\{(g_k,h_k)\}\subset A\cap\bar{C}(S)\times B(S)$ such that A is contained in the bounded pointwise closure of $\{(g_k,h_k)\}$, but typically A is single-valued, so we use the more intuitive notation Ag_k.) These two conditions are satisfied by essentially all operators A that might reasonably be thought to be generators of Markov processes.

For an S_0-valued, measurable process Y, $\hat{\mathcal{F}}_t^Y$ will denote the completion of the σ-algebra $\sigma(\int_0^r h(Y(s))ds, r\leq t, h\in B(S_0))$. For almost every t, $Y(t)$ will be $\hat{\mathcal{F}}_t^Y$-measurable, but in general, $\hat{\mathcal{F}}_t^Y$ does not contain $\mathcal{F}_t^Y=\sigma(Y(s):s\leq t)$. Let $\mathbf{T}^Y=\{t:Y(t)\text{ is }\hat{\mathcal{F}}_t^Y\text{ measurable}\}$. If Y is cadlag and has no fixed points of discontinuity (that is, for every t, $Y(t)=Y(t-)$ a.s.), then $\mathbf{T}^Y=[0,\infty)$. $D_S[0,\infty)$ denotes the space of cadlag S-valued functions with the Skorohod topology and $M_S[0,\infty)$ denotes the space of Borel measurable functions, $x:[0,\infty)\to S$, topologized by convergence in Lebesgue measure.

THEOREM 1.1. *Let (S,d) and (S_0,d_0) be complete, separable metric spaces. Let $A\subset\bar{C}(S)\times\bar{C}(S)$ be a graph separable, pre-generator, and suppose that $\mathcal{D}(A)$ is closed under multiplication and is separating. Let $\gamma:S\to S_0$ be Borel measurable, and let α be a transition function from S_0 into S ($y\in S_0\to\alpha(y,\cdot)\in\mathcal{P}(S)$ is Borel measurable) satisfying $\int h\circ\gamma(z)\alpha(y,dz)=h(y)$, $y\in S_0$, $h\in B(S_0)$, that is, $\alpha(y,\gamma^{-1}(y))=1$. Define*

$$C=\{(\int_S f(z)\alpha(\cdot,dz),\int_S Af(z)\alpha(\cdot,dz)):f\in\mathcal{D}(A)\}\,.$$

Let $\mu_0\in\mathcal{P}(S_0)$, and define $\nu_0=\int\alpha(y,\cdot)\mu_0(dy)$.

a) *If $\tilde{Y}$ is a solution of the martingale problem for (C,μ_0), then there exists a solution X of the martingale problem for (A,ν_0) such that $\tilde{Y}$ has the same distribution on $M_{S_0}[0,\infty)$ as $Y=\gamma\circ X$. If Y and $\tilde{Y}$ are cadlag, then Y and $\tilde{Y}$ have the same distribution on $D_{S_0}[0,\infty)$.*

b) *For $t\in\mathbf{T}^Y$,*

$$P\{X(t)\in\Gamma|\hat{\mathcal{F}}_t^Y\}=\alpha(Y(t),\Gamma),\quad\Gamma\in\mathcal{B}(S). \tag{1.5}$$

c) *If, in addition, uniqueness holds for the martingale problem for (A,ν_0), then uniqueness holds for the $M_{S_0}[0,\infty)$-martingale problem for (C,μ_0). If $\tilde{Y}$ has sample paths in $D_{S_0}[0,\infty)$, then uniqueness holds for the $D_{S_0}[0,\infty)$-martingale problem for (C,μ_0).*

d) *If uniqueness holds for the martingale problem for (A,ν_0), then Y restricted to $\mathbf{T}^Y$ is a Markov process.*

REMARK 1.2. Theorem 1.1 can be extended to cover a large class of generators whose range contains discontinuous functions. (See Kurtz (1998), Corollary 3.5 and Theorem 2.7.) In particular, suppose $A_1, \dots, A_m$ satisfy the conditions of Theorem 1.1 for a common domain $\mathcal{D} = \mathcal{D}(A_1) = \cdots = \mathcal{D}(A_m)$ and $\beta_1, \dots, \beta_m$ are nonnegative functions in $B(S)$. Then the conclusions of Theorem 1.1 hold for

$$Af = \beta_1 A_1 f + \cdots + \beta_m A_m f.$$

1.1. Example: Empirical measure process. Let $A = A_0$ defined in (1.2), let $S = \hat{E}$, and let $S_0 = M_c^f(E)$, the space of finite counting measures on E. Define $\gamma : S \to S_0$ by

$$\gamma(x,k) = \sum_{i=1}^{k} \delta_{x_i}.$$

Note that each $\mu \in M_c^f(E)$ is of the form $\mu = \sum_{i=1}^k \delta_{x_i}$ for some k and $x \in E^k$, and for μ of this form, define

$$\alpha_0(\mu, \cdot) = \frac{1}{k!} \sum_{\sigma \in E^k} \delta_{x_\sigma}.$$

Define

$$C_0 = \{(\alpha_0 f, \alpha_0 A_0 f) : f \in \mathcal{D}(A_0)\}.$$

We can interpret $\alpha_0 f$ as a function on $M_c^f(E)$ or as a function on $\hat{E}$, $h_f(x,k) = \alpha_0 f(\sum_{i=1}^k \delta_{x_i})$, that is symmetric in the sense that $h_f(x,k) = h_f(x_\sigma, k)$ for all $\sigma \in \Gamma_k$. Note that if f is symmetric, then $h_f(x,k) = f(x,k)$ and $\alpha_0 A_0 f(\sum_{i=1}^k \delta_{x_i}) = A_0 f(x,k)$. It follows that if X is a solution of the martingale problem for A_0, then $\sum_{i=1}^{N(t)} \delta_{X_i(t)}$ is a solution of the martingale problem for C_0.

Conversely, if $B \subset \bar{C}(E) \times \bar{C}(E)$ is a graph separable pregenerator and $\mathcal{D}(B)$ is closed under multiplication and separates points and the λ_m satisfy (1.3) and are continuous, then A_0 satisfies the conditions of Theorem 1.1 and hence any solution of the martingale problem for C_0 corresponds to a solution of the martingale problem for A_0. Consequently, the two martingale problems are essentially equivalent. (Note that there are variations of Theorem 1.1 that apply under less restrictive conditions. See Kurtz (1998).)

2. Marked population models

Next, we introduce a family of "marked" population processes. F will denote the space of marks, so the new state space S will be a subset of $\mathcal{M}_c^f(E \times F)$. In all of the examples $F \subset [0, \infty)$, and with order in mind, we will sometimes refer to the marks as "levels".

With reference to Theorem 1.1, let $S_0 = \mathcal{M}_c^f(E)$, and let γ be defined by $\gamma(\xi) = \xi(\cdot \times F)$, $\xi \in \mathcal{M}_c^f(E \times F)$. For each $\mu \in \mathcal{M}_c^f(E)$, let $\hat{\alpha}(\mu, \cdot)$ be an exchangeable distribution on $F^{\mu(E)}$. Let $\mu = \sum_{i=1}^k \delta_{x_i}$, and define $\alpha_1(\mu, \cdot) \in \mathcal{P}(E \times F)$ by

$$\alpha_1(\mu, G) = \hat{\alpha}(\mu, \{u \in F^k : \sum_{i=1}^{k} \delta_{(x_i, u_i)} \in G\}), \quad G \in \mathcal{B}(\mathcal{M}_c^f(E \times F)).$$

Let $A_1 \subset \bar{C}(S) \times \bar{C}(S)$ and define

$$C_1 = \{(\alpha_1 f, \alpha_1 A_1 f) : f \in \mathcal{D}(A_1)\}.$$

Assuming that $A_1 \subset \bar{C}(S) \times \bar{C}(S)$ is a graph separable, pre-generator, and that $\mathcal{D}(A_1)$ is closed under multiplication and is separating, then Theorem 1.1 applies.

2.1. Example: Neutral model. Let $F = \{1, 2, \dots\}$, and for $\mu = \sum_{i=1}^k \delta_{x_i}$, let $\hat{\alpha}(\mu, \cdot)$ satisfy $\hat{\alpha}(\mu, \Gamma_k) = 1$, that is, $\hat{\alpha}(\mu, \cdot)$ is just the distribution of a random permutation. To simplify notation, we identify μ with x as above and define the generator in terms of functions of (x, u). If we ensure that all functions involved have the property that $h(x, u) = h(x_\sigma, u_\sigma)$ for all permutations σ, the functions will depend only on the measures $\sum_{i=1}^k \delta_{(x_i, u_i)}$.

If $u = (u_1, \dots, u_k)$ is a permutation of $(1, \dots, k)$ and $1 \le l < m \le k+1$, define

$$r_i^{lm}(u) = \begin{cases} u_i & u_i < m \\ u_i + 1 & u_i \ge m. \end{cases}$$

$\eta_{lm}(u) = j$ if $u_j = l$, and $\eta_0(u) = j$ where j is the unique index such that $u_j = k$.

Assume that $B \subset \bar{C}(B) \times \bar{C}(E)$, that $\mathcal{D}(B)$ is an algebra (that is, a linear subspace that is closed under multiplication) that separates points and contains the constant functions, and that the martingale problem for B is well posed. Let $\mathcal{D}(A_2)$ be the collection of functions of the form

$$f(x, u, k) = \prod_{i=1}^k g(x_i, u_i),$$

where $g(\cdot, j) \in \mathcal{D}(B)$ and for some $j_g \ge 0$, $g(\cdot, j) \equiv 1$ for $j > j_g$. Assume that $\lambda_m : \cup_k E^k \to [0, \infty)$, $m = -1, 1$, satisfy $\lambda_m(x) = \lambda_m(x_\sigma)$ and $\sup_x \lambda_m(x) < \infty$. Define

$$\begin{aligned} (2.1) \qquad & A_2 f(x, u, k) \\ &= \sum_{i=1}^k f(x, u, k) \frac{Bg(x_i, u_i)}{g(x_i, u_i)} \\ &+ \frac{2\lambda_1(x)}{k+1} \sum_{1 \le l < m \le k+1} (g(x_{\eta_{lm}(u)}, m) \prod_{i=1}^k g(x_i, r_i^{lm}(u)) - f(x, u, k)) \\ &+ \lambda_{-1}(x) k f(x, u, k)) (\frac{1}{g(x_{\eta_0(u)}, k)} - 1). \end{aligned}$$

Note that when there is a "death", it is the particle with the highest level that is eliminated. When there is a "birth", particles with lower levels are more likely to become parents.

If $\alpha(x, k, du) = \frac{1}{k!} \sum_{\sigma \in \Gamma_k} \delta_\sigma(du)$, then defining

$$\begin{aligned} A_0 f(x, k) &= \sum_{i=1}^k B_i f(x, k) \\ & \qquad + \lambda_1(x) \sum_{i=1}^k \frac{1}{k+1} \sum_{j=1}^{k+1} (f(\theta_j(x|x_i), k+1) - f(x, k)) \\ & \qquad + \lambda_{-1}(x) \sum_{i=1}^k (f(d_i(x), k-1) - f(x, k)), \end{aligned}$$

we have $\alpha A_2 f = A_0 \alpha f$. Note that A_0 is a special case of A_0 defined in (1.2) in Section 1. Here A_0 is the generator for a model that is *neutral* in the sense that the birth and death rates are the same for all particles regardless of location.

Let (X, U) be a solution of the martingale problem for A_2. Suppose that $U(0)$ is independent of $X(0)$ and is uniformly distributed over all permutations of $(1, \dots, N(0))$. Let μ_0 denote the distribution of $X(0)$. Then, by Theorem 1.1, X is a solution of the martingale problem for (A_0, μ_0).

Assume that λ_1 and λ_{-1} are bounded and continuous on S. It follows that the martingale problem for A_2 is well-posed. For $f \in \mathcal{D}(A_2)$, define $\hat{f}(x, k) = f(x_1, \dots, x_k, 1, \dots, k, k)$ and set

$$
\begin{aligned}
A_3 \hat{f}(x, k) = & \sum_{i=1}^{k} B_i \hat{f}(x, k) \\
& + \lambda_1(x) \sum_{1 \le l < m \le k+1} (\hat{f}(\theta_m(x|x_l), k+1) - \hat{f}(x, k)) \\
& + \lambda_{-1}(x)(\hat{f}(d_k(x), k-1)) - \hat{f}(x, u)),
\end{aligned}
$$

where for $1 \le l < m \le k+1$ and $x \in E^k$, $x' = \theta_m(x|x_l) \in E^{k+1}$ is given by

$$
x_i' = \begin{cases} x_i & i < m \\ x_{i-1} & m < i \le k+1 \\ x_l & i = m. \end{cases}
$$

Let (X, U) be as above. Let $V_i(t) = j$ if $U_j(t) = i$, that is, $U_{V_i(t)}(t) = i$, and define $(Y_1(t), \dots, Y_{N(t)}(t)) = (X_{V_1(t)}(t), \dots, X_{V_{N(t)}(t)}(t))$. Then Y is a solution of the martingale problem for A_3.

Define $\gamma : E^k \to \mathcal{M}(E)$, by $\gamma(x) = \sum_{i=1}^k \delta_{x_i}$, and for $\mu = \sum_{i=1}^k \delta_{x_i}$, define $\alpha_0(\mu, dx) = \frac{1}{k!} \sum_{\sigma \in \Gamma_k} \delta_{x_\sigma}$. Then for $f \in \mathcal{D}(A_0) = \mathcal{D}(A_3)$,

$$
\alpha_0 A_3 f = \alpha_0 A_0 f.
$$

Let $C = \{(\alpha_0 f, \alpha_0 A_0 f) : f \in \mathcal{D}(A_0)\} = \{(\alpha_0 f, \alpha_0 A_3 f) : f \in \mathcal{D}(A_3)\}$. By the discussion in Section 1, if X^0 is a solution of the martingale problem for A_0, then $\gamma(X^0)$ is a solution of the martingale problem for C. But by Theorem 1.1, any solution of the martingale problem for C corresponds to a solution of the martingale problem for A_3. Consequently, for each solution X^0 of the martingale problem for A_0, there exists a solution X of the martingale problem for A_3 such that $\gamma(X^0)$ and $\gamma(X)$ have the same distribution. The process corresponding to A_3 is a special case of Model II of [**2**].

3. Models with location/type dependent birth and death rates

3.1. Critical models. Let $F = [0, n]$. Define

$$(3.1) \quad \begin{aligned} & A^n f(x,u,k) \\ &= \sum_{i=1}^{k} B_i f(x,u,k) \\ &+ \sum_{i=1}^{k} 2\lambda(x_i,x)\frac{1}{k+1}\sum_{j=1}^{k+1}\int_{u_i}^{n}(f(\theta_j(x,u|x_i,v),k+1) - f(x,u,k))dv \\ &+ \sum_{i=1}^{k} 2\lambda(x_i,x)u_i(f(d_i(x,u),k-1) - f(x,u,k)) \end{aligned}$$

for f in an appropriate domain. Assume that for $x \in E^k$, and $\sigma \in \Gamma_k$, $k = 1, 2, \ldots$ and i and j such that $x_i = x_{\sigma_j}$, $\lambda(x_i, x) = \lambda(x_{\sigma_j}, x_\sigma)$.

If $\alpha^n(x, k, du) = n^{-k}du_1 \cdots du_k$, then $\alpha^n A^n f = A_0^n \alpha^n f$, where

$$\begin{aligned} & A_0^n f(x,k) \\ &= \sum_{i=1}^{k} B_i f(x,k) \\ &+ \sum_{i=1}^{k} n\lambda(x_i,x)\frac{1}{k+1}\sum_{j=1}^{k+1}(f(\theta_j(x|x_i),k+1) - f(x,k)) \\ &+ \sum_{i=1}^{k} n\lambda(x_i,x)(f(d_i(x),k-1) - f(x,k)). \end{aligned}$$

Here, A_0^n is again a special case of (1.2). In particular, if $\lambda(x_i, x) \equiv \lambda(x_i)$, then A_0^n is the generator of a critical branching Markov process and the scaling in n is such that a sequence of solutions X^n should satisfy

$$(3.2) \qquad Z_n = \frac{1}{n}\sum_{i=1}^{N_n(t)} \delta_{X_i^n} \Rightarrow Z,$$

where Z is a Dawson-Watanabe process.

In the remainder of the paper, we concentrate on the Dawson-Watanabe setting, that is, we assume that $\lambda(x_i, x) = \lambda(x_i)$. We will see that this assumption makes existence and uniqueness for the limiting model easy. The relationship between A^n and A_0^n, however, insures that the particle representation will be valid for more general models.

With (3.2) in mind, consider (3.1) as $n \to \infty$. To be specific, let $\mathcal{D}(A)$ be the collection of functions of the form

$$f(x,u,k) = \prod_{i=1}^{k} g(x_i,u_i),$$

where $0 < g \leq 1$ is bounded away from zero and there exists u_g such that $g(x_i, u_i) = 1$ if $u_i > u_g$.

If $n > u_g$, then A^n becomes

$$(3.3)\qquad \begin{aligned} Af(x,u) &= \sum_{u_i<u_g} f(x,u)\frac{Bg(x_i,u_i)}{g(x_i,u_i)} \\ &\quad + \sum_{u_i<u_g} 2\lambda(x_i)\int_{u_i}^{u_g} f(x,u)(g(x_i,v)-1)dv \\ &\quad + \sum_{u_i<u_g} 2\lambda(x_i)u_i f(x,u)(\frac{1}{g(x_i,u_i)}-1), \end{aligned}$$

and the convergence of A^n is immediate. Assuming that the martingale problem for B is well-posed and (for simplicity) λ is bounded and continuous, the martingale problem for A is well-posed. We identify the process with the counting measure

$$\Psi(t) = \sum_i \delta_{(X_i(t),U_i(t))}.$$

Define $\gamma : \mathcal{M}_c(E\times[0,\infty)) \to \mathcal{M}^f(E)$ by

$$\gamma(x,u) = \lim_{r\to\infty}\frac{1}{r}\sum_i I_{[0,r]}(u_i)\delta_{x_i}$$

if the limit exists. Set

$$K(t) = \lim_{r\to\infty}\frac{1}{r}\sum_i I_{[0,r]}(U_i(t))\delta_{X_i(t)} = \gamma(X(t),U(t)).$$

If $\Psi(0)$ is a Poisson random measure with mean measure $K(0)\times m$, then we claim that, conditioned on $\mathcal{F}_t^K \equiv \sigma(K(s), s\le t)$, $\Psi(t)$ is a Poisson random measure with mean measure $K(t)\times m$.

Let $(S,\mathcal{S})$ be a measurable space, and let ν be a σ-finite measure on $\mathcal{S}$. We need the following facts about a *Poisson random measure*, ξ, with *mean measure* ν:

a) ξ is a random counting measure on S.
b) For each $A\in\mathcal{S}$ with $\nu(A)<\infty$, $\xi(A)$ is Poisson distributed with parameter $\nu(A)$.
c) For $A_1, A_2, \ldots \in \mathcal{S}$ disjoint, $\xi(A_1), \xi(A_2), \ldots$ are independent.

If ξ is a Poisson random measure with mean measure ν

$$E[e^{\int f(z)\xi(dz)}] = e^{\int (e^f-1)\delta\nu},$$

or letting $\xi = \sum_i \delta_{V_i}$,

$$E[\prod_i g(V_i)] = e^{\int (g-1)d\nu}.$$

Similarly,

$$E[\sum_j h(V_j)\prod_i g(V_i)] = \int hgd\nu e^{\int (g-1)d\nu}.$$

We define α so that if $\mu\in\mathcal{M}^f(E)$, then $\alpha(\mu,\cdot)\in\mathcal{P}(\mathcal{M}_c(E\times[0,\infty))$ is the distribution of a Poisson random measure on $E\times[0,\infty)$ with mean measure $\mu\times m$, that is

$$\int_{\mathcal{M}_c(E\times[0,\infty))} e^{\langle f,z\rangle}\alpha(\mu,dz) = e^{\int (e^{f(x,u)}-1)\mu(dx)du}.$$

Therefore, if $f(x,u) = \prod g(x_i,u_i)$, then

$$\alpha f(\mu) = e^{\int_E \int_0^\infty (g(x,u)-1)du\mu(dx)} \tag{3.4}$$

and

$$\begin{aligned}
&\alpha Af(\mu)\\
&= \alpha f(\mu)\int_E B\int_0^\infty (g(x,u)-1)du\mu(dx)\\
&\quad +\alpha f(\mu)\int_E 2\lambda(x)\int_0^\infty\int_u^\infty g(x,u)(g(x,v)-1)dvdu\mu(dx)\\
&\quad +\alpha f(\mu)\int_E 2\lambda(x)\int_0^\infty u(1-g(x,u))du\mu(dx)\\
&= \alpha f(\mu)\int_E B\int_0^\infty (g(x,u)-1)du\mu(dx)\\
&\quad +\alpha f(\mu)\int_E 2\lambda(x)\int_0^\infty\int_u^\infty (g(x,u)-1)(g(x,v)-1)dvdu\mu(dx)\\
&= \alpha f(\mu)\int_E B\int_0^\infty (g(x,u)-1)du\mu(dx)\\
&\quad +\alpha f(\mu)\int_E \lambda(x)\left(\int_0^\infty (g(x,u)-1)du\right)^2\mu(dx).
\end{aligned}$$

Consequently, for $f(\mu) = e^{-\langle h,\mu\rangle}$ and $C = \{(\alpha f, \alpha Af) : f \in \mathcal{D}(A)\}$,

$$Cf(\mu) = f(\mu)\langle -Bh + \lambda h^2, \mu\rangle. \tag{3.5}$$

But C of this form is the generator for a Dawson-Watanabe process. (See, for example, [**5**], Section 9.4.3.) Since, as defined, Af need not be a bounded function, Theorem 1.1 does not immediately apply; however, Theorem 1.1 can be extended to cover certain operators whose range includes unbounded functions and this extension would apply in the current setting. Alternatively, we could take F to be the space consisting of copies of the closed intervals $[k, k+1]$, $k = 0, 1, \dots$, that is, F includes two copies of each integer. Writing $F = \uplus_{k=0}^\infty [k, k+1]$, assume that the domain of A consists of functions of the form

$$f(x,u) = \prod_i g(x_i,u_i),$$

where $0 \le g(x_i,u_i) \le \rho_g < 1$ for $u_i \in \uplus_{k=0}^{k_g}[k,k+1]$ and $g(x_i,u_i) = 1$ for $u_i \in \uplus_{k=k_g+1}^\infty [k,k+1]$. Then, under the assumption that λ is bounded, Af is bounded. In any case, we have the following theorem.

THEOREM 3.1. *Suppose that $\tilde{K}$ is a solution of the martingale problem for C given by (3.5), and let $\nu_0 = E[\alpha(Z(0),\cdot)]$. Then there exists a solution*

$$\Psi(t) = \sum_i \delta_{(X_i(t),U_i(t))}$$

of the martingale problem for (A,ν_0) such that K defined by

$$K(t) = \lim_{r\to\infty} \frac{1}{r}\sum_i I_{[0,r]}(U_i(t))\delta_{X_i(t)}$$

has the same distribution as $\tilde{K}$.

3.2. Subcritical models. The particle construction above can be extended easily to subcritical models of the form

$$\begin{aligned}
&A_0^n f(x,k)\\
&=\sum_{i=1}^k B_i f(x,k)\\
&\quad+\sum_{i=1}^k n\lambda(x_i)(f((x,x_i),k+1)-f(x,k))\\
&\quad+\sum_{i=1}^k n(\lambda(x_i)+\frac{1}{n}\lambda_0(x_i))(f(\gamma_i(x),k-1)-f(x,k)).
\end{aligned}$$

For $f(x,u)=\prod g(x_i,u_i)$, the limit for the corresponding marked model is given by

$$\begin{aligned}
&Af(x,u)\\
&=\sum_{u_i<u_g} f(x,u)\frac{Bg(x_i,u_i)}{g(x_i,u_i)}\\
&\quad+\sum_{u_i<u_g} 2\lambda(x_i)\int_{u_i}^{u_g} f(x,u)(g(x_i,v)-1)dv\\
&\quad+\sum_{u_i<u_g}(2\lambda(x_i)u_i+\lambda_0(x_i))f(x,u)(\frac{1}{g(x_i,u_i)}-1),
\end{aligned}$$

and we have $\alpha f(\mu)=e^{\langle\int_0^\infty(g(\cdot,u)-1)du,\mu\rangle}$ and

$$\begin{aligned}
\alpha Af(\mu) &= \alpha f(\mu)\Big[\int_E B\int_0^\infty(g(x,u)-1)du\mu(dx)\\
&\qquad+\int_E\lambda(x)\left(\int_0^\infty(g(x,u)-1)du\right)^2\mu(dx)\\
&\qquad+\int_E\int_0^\infty\lambda_0(x)(1-g(x,u))du\mu(dx)\Big].
\end{aligned}$$

Taking $h=\int_0^\infty(1-g(\cdot,u))du$ in this formula, for $f(\mu)=e^{-\langle h,\mu\rangle}$, we have

$$Cf(\mu)=f(\mu)\langle -Bh+\lambda h^2+\lambda_0 h,\mu\rangle.$$

3.3. Ordered model. The indexing of the above particle models has no significance. If we order the particles according to increasing level, the generator becomes

$$\begin{aligned}
&Af(x,u)\\
&=\sum_i B_i f(x,u)\\
&\quad+\sum_i 2\lambda(x_i)\sum_{j=i}^\infty\int_{u_j}^{u_{j+1}}(f(\theta_j(x,u|x_i,v))-f(x,v))dv\\
&\quad+\sum_i(2\lambda(x_i)u_i+\lambda_0(x_i))(f(d_i(x,u))-f(x,u)).
\end{aligned}$$

For this ordering, $P(t) = |K(t)|$ is given by

$$P(t) = \lim_{m\to\infty} \frac{m}{u_m},$$

and

$$K(t) = \lim_{m\to\infty} \frac{1}{u_m} \sum_{i=1}^{m} \delta_{X_i(t)}.$$

Assume that B is the generator for a diffusion process satisfying an Itô equation

$$X(t) = X(0) + \int_0^t \sigma(X(s))ds + \int_0^t b(X(s))ds.$$

Then we can write a system of equations for the particle model.

$$\begin{aligned} X_k(t) = X_k(0) &+ \int_0^t \sigma(X_k(s))dW_k(s) + \int_0^t b(X_k(s))ds \\ &+ \sum_{1\le i<j<k} (X_{k-1}(s-) - X_k(s-))dL_{ij}^b(s) \\ &+ \sum_{i<k} (X_i(s-) - X_k(s-))dL_{ik}^b(s) \\ &+ \sum_{j<k} (X_{k+1}(s-) - X_k(s-))dL_j^d(s), \end{aligned}$$

$$\begin{aligned} U_k(t) &= U_k(0) + \sum_{1\le i<j<k} (U_{k-1}(s-) - U_k(s-))dL_{ij}^b(s) \\ &\quad + \sum_{i<k} \int_{[0,\infty)\times[0,\infty)\times[0,t]} (u - U_k(s-)) I_{[U_{k-1}(s-),U_k(s-))}(u) \\ &\qquad\qquad I_{[0,2\lambda(X_i(s-)))}(v) N_i(du \times dv \times ds) \\ &\quad + \sum_{j\le k} (U_{k+1}(s-) - U_k(s-))dL_j^d(s), \end{aligned}$$

and

$$\begin{aligned} L_{ij}^b(t) &= \int_{[0,\infty)\times[0,\infty)\times[0,t]} I_{[U_{j-1}(s-),U_j(s-))}(u) \\ &\qquad I_{[0,2\lambda(X_i(s-)))}(v) N_i(du \times dv \times ds) \\ L_i^d(t) &= \int_{[0,\infty)\times[0,\infty)\times[0,t]} I_{[0,U_i(s-))}(u) \\ &\qquad I_{[0,2\lambda(X_i(s-)))}(v) N_i(du \times dv \times ds) \\ &\quad + \int_{[0,\infty)\times[0,t]} I_{[0,\lambda_0(X_i(s))]}(v) N_i^0(dv \times ds). \end{aligned}$$

3.4. Model with population dependent motion and birth and death rates. In the system of equations above it is simple to introduce dependence on the total mass distribution K in the motion and birth and death rates.

$$\begin{aligned}
X_k(t) &= X_k(0) + \int_0^t \sigma(X_k(s), K(s))dW_k(s) \\
&\quad + \int_0^t b(X_k(s), K(s))ds \\
&\quad + \sum_{1\le i<j<k} (X_{k-1}(s-) - X_k(s-))dL_{ij}^b(s) \\
&\quad + \sum_{i<k} (X_i(s-) - X_k(s-))dL_{ik}^b(s) \\
&\quad + \sum_{j<k} (X_{k+1}(s-) - X_k(s-))dL_j^d(s),
\end{aligned}$$

$$\begin{aligned}
U_k(t) &= U_k(0) + \sum_{1\le i<j<k} (U_{k-1}(s-) - U_k(s-))dL_{ij}^b(s) \\
&\quad + \sum_{i<k} \int_{[0,\infty)\times[0,\infty)\times[0,t]} (u - U_k(s-)) I_{[U_{k-1}(s-),U_k(s-))}(u) \\
&\qquad\qquad I_{[0,2\lambda(X_i(s-),K(s-)))}(v) N_i(du \times dv \times ds) \\
&\quad + \sum_{j\le k} (U_{k+1}(s-) - U_k(s-))dL_j^d(s),
\end{aligned}$$

and

$$\begin{aligned}
L_{ij}^b(t) &= \int_{[0,\infty)\times[0,\infty)\times[0,t]} I_{[U_{j-1}(s-),U_j(s-))}(u) \\
&\qquad\qquad I_{[0,2\lambda(X_i(s-),K(s-)))}(v) N_i(du \times dv \times ds) \\
L_i^d(t) &= \int_{[0,\infty)\times[0,\infty)\times[0,t]} I_{[0,U_i(s-))}(u) \\
&\qquad\qquad I_{[0,2\lambda(X_i(s-),K(s-)))}(v) N_i(du \times dv \times ds) \\
&\quad + \int_{[0,\infty)\times[0,t]} I_{[0,\lambda_0(X_i(s),K(s-))]}(v) N_i^0(dv \times ds).
\end{aligned}$$

The generator for K becomes

$$Cf(\mu) = f(\mu)\langle -B(K)h + \lambda(\cdot, K)h^2 + \lambda_0(\cdot, K)h, \mu\rangle, \tag{3.6}$$

for $f(\mu) = e^{-\langle h,\mu\rangle}$, where

$$B(\mu)h(z) = \frac{1}{2}\sum_{ij} a_{ij}(z,\mu)\frac{\partial^2}{\partial z_i \partial z_j}h(z) + \sum_i b_i(z,\mu)\frac{\partial}{\partial z_i}h(z)$$

for $a(z,\mu) = \sigma(z,\mu)\sigma(z,\mu)^T$.

In the case λ and λ_0 constant, models with generators of the form (3.6) were introduced by Perkins [**7**]. In this setting, the analog of the system was given in Donnelly and Kurtz [**2**]. For λ and λ_0 constant, uniqueness of the above system can be proved under Lipshitz assumptions on σ and b. If λ and λ_0 depend on z and/or μ, uniqueness is open.

4. Models with simultaneous births

The models above can be extended to allow for multiple simultaneous births. In particular, for f of the form

$$f(x,u,k) = \prod_{i=1}^{k} g(x_i,u_i),$$

let

$$\begin{aligned} A^n f(x,u) &= \sum_i f(x,u)\frac{Bg(x_i,u_i)}{g(x_i,u_i)} \\ &\quad +f(x,u)\sum_i\sum_{k=1}^{\infty}(k+1)\lambda_k^n(x_i)n^{-k} \\ &\qquad\qquad \int_{[u_i,n]^k}(\prod_{l=1}^{k} g(x_i,v_l)-1)dv_1\cdots dv_k \\ &\quad +f(x,u)\sum_i\sum_{k=1}^{\infty}(k+1)\lambda_k^n(x_i)(1-(1-\frac{u_i}{n})^k)(\frac{1}{g(x_i,u_i)}-1). \end{aligned}$$

Note that if $\lambda_1^n(x) = n\lambda_1(x)$ and $\lambda_k^n \equiv 0$ for $k > 1$, then A^n conincides with (3.3). If $\alpha^n(x,k,du) = n^{-k}du_1\cdots du_k$, then $\alpha^n A^n f = A_0^n\alpha^n f$, where for $f(x,k) = \prod_{i=1}^{k} g(x_i)$,

$$\begin{aligned} &A_0^n f(x,k) \\ &\quad = \sum_{i=1}^{k} B_i f(x,k) \\ &\qquad +f(x,u)\sum_i\sum_{k=1}^{\infty}\lambda_k^n(x_i)(g(x_i)^k-1) \\ &\qquad +f(x,u)\sum_i\sum_{k=1}^{\infty}\lambda_k^n(x_i)k(\frac{1}{g(x_i)}-1). \end{aligned}$$

We see that $\lambda_k^n(x_i)$ is the intensity for the birth of k offspring for a particle located at x_i, and setting

$$\lambda_{-1}^n(x_i) = \sum_{k=1}^{\infty} k\lambda_k^n(x_i),$$

$\lambda_{-1}^n(x_i)$ is the death rate that makes the process critical.

Assume that for $u_i > u_g$, $g(x_i,u_i) = 1$, and define $h(x_i,u_i) = \int_{u_i}^{u_g}(1-g(x_i,v))dv$. Then

$$\int_{[u_i,n]^k}(\prod_{l=1}^{k} g(x_i,v_l)-1)dv_1\cdots dv_k = (n-u_i-h(x_i,u_i))^k-(n-u_i)^k$$

and

$$\begin{aligned}
&A^n f(x,u) \\
&= \sum_i f(x,u)\frac{Bg(x_i,u_i)}{g(x_i,u_i)} \\
&+f(x,u)\sum_i\sum_{k=1}^{\infty}(k+1)\lambda_k^n(x_i)\left((1-\frac{u_i+h(x_i,u_i)}{n})^k-(1-\frac{u_i}{n})^k\right) \\
&+f(x,u)\sum_i\sum_{k=1}^{\infty}(k+1)\lambda_k^n(x_i)(1-(1-\frac{u_i}{n})^k)(\frac{1}{g(x_i,u_i)}-1).
\end{aligned}$$

Consequently, if

$$\lim_{n\to\infty}\sum_{k=1}^{\infty}(k+1)\lambda_k^n(x_i)(1-(1-\frac{u_i}{n})^k)=\Lambda(x_i,u_i), \tag{4.1}$$

$A^n f \to Af$ given by

$$\begin{aligned}
Af(x,u) &= \sum_i f(x,u)\frac{Bg(x_i,u_i)}{g(x_i,u_i)} \\
&\quad +f(x,u)\sum_i(\Lambda(x_i,u_i)-\Lambda(x_i,u_i+h(x_i,u_i))) \\
&\quad +f(x,u)\sum_i\Lambda(x_i,u_i)(\frac{1}{g(x_i,u_i)}-1).
\end{aligned}$$

We assume that the convergence in (4.1) is uniform in $x_i \in E$ and in u_i on bounded intervals. Assumption (4.1) is essentially equivalent to (9.4.36) of Ethier and Kurtz (1986).

Let $\Lambda^n(x_i,u_i)$, $n=1,2,\dots$ denote the sequence on the left of (4.1), and observe that

$$\frac{\partial^m}{\partial u_i^m}\Lambda^n(x_i,u_i)=(-1)^{m+1}\sum_{k=m}^{\infty}\lambda_k^n(x_i)\frac{(k+1)k\cdots(k-m+1)}{n^m}(1-\frac{u_i}{n})^{k-m}.$$

The fact that the derivatives alternate in sign and decrease in absolute value implies that

$$\lim_{n\to\infty}\frac{\partial^m}{\partial u_i^m}\Lambda^n(x_i,u_i)=\frac{\partial^m}{\partial u_i^m}\Lambda(x_i,u_i)\equiv\partial^m\Lambda(x_i,u_i)$$

for each m, where the convergence is uniform in u_i on bounded intervals that are bounded away from $u_i=0$. It also follows that $\partial^1\Lambda(x_i,\cdot)$ is completely monotone and hence must be of the form

$$\partial^1\Lambda(x_i,v)=\int_0^{\infty}e^{-vz}\hat{\nu}(x_i,dz).$$

Writing $\hat{\nu}(x_i,\cdot)=\lambda(x_i)\delta_0+\nu(x_i,\cdot)$ where the support of $\nu(x_i,\cdot)$ is in $(0,\infty)$, we have

$$\Lambda(x_i,v)=\lambda(x_i)v+\int_0^{\infty}z^{-1}(1-e^{-vz})\nu(x_i,dz). \tag{4.2}$$

Since $\Lambda(x_i, v) < \infty$, we must have

$$\int_0^\infty \frac{1}{1 \vee z} \nu(x_i, dz) < \infty.$$

In terms of ν,

$$\begin{aligned} Af(x,u) &= \sum_i f(x,u) \frac{Bg(x_i,u_i)}{g(x_i,u_i)} \\ &\quad + f(x,u) \sum_i \lambda(x_i) \int_{u_i}^\infty (g(x_i,v)-1)dv \\ &\quad + f(x,u) \sum_i \int_0^\infty \left(e^{z \int_{u_i}^\infty (g(x_i,v)-1)dv} - 1 \right) z^{-1} e^{-u_i z} \nu(x_i, dz) \\ &\quad + f(x,u) \sum_i \Lambda(x_i,u_i) (\frac{1}{g(x_i,u_i)} - 1). \end{aligned}$$

The fourth term on the right indicates that at rate $\Lambda(x_i, u_i)$, a particle at location x_i and level u_i dies. The second term on the right corresponds to single births. For $u_i \le a < b$, at rate $\lambda(x_i)(b-a)$, a particle at location x_i and level u_i gives birth to a single particle with level in the interval $(a, b]$. The third term corresponds to multiple births. When such a birth occurs to the particle at level u_i, a positive random variable ζ is generated, and the levels of the offspring form a Poisson process on $[u_i, \infty)$ with intensity ζ. To be precise, suppose that a particle with level u_i lives from time τ_i^b until time τ_i^d and that $X_i(t)$ gives the location of the particle for $\tau_i^b \le t < \tau_i^d$. Then ν and X_i determine a point process ξ_i on $[\tau_i^b, \tau_i^d) \times [0, \infty)$ through the requirement that

$$\xi_i((\tau_i^b, t] \times G) - \int_{\tau_i^b}^t \int_G z^{-1} e^{-u_i z} \nu(X_i(s), dz) ds, \quad \tau_i^b \le t < \tau_i^d,$$

is a martingale for each $G \in \mathcal{B}(E)$. Writing

$$\xi_i = \sum_k \delta_{(S_k, \zeta_k)},$$

at time S_k, there is a birth event in which new particles are created whose levels form a Poisson process with intensity ζ_k on $[u_i, \infty)$. Note that

$$\int_0^\infty z^{-1} e^{-u_i z} \nu(x_i, dz)$$

may be infinite, so that a particle may have infinitely many such birth events in a finite amount of time; however, during a finite time interval, only finitely many births will have levels in a bounded interval. In particular, let $u_i \le a < b$. Noting that for a Poisson process with intensity z, $z(b-a)$ is the expected number of points in the interval $(a, b]$,

$$\lambda(x_i)(b-a) + \int_0^\infty z(b-a) z^{-1} e^{-u_i z} \nu(x_i, dz) = (b-a)\partial^1 \Lambda(x_i, u_i) < \infty,$$

is the expected number of births with levels in the interval $(a, b]$ per unit time occuring to a parent at level u_i and location x_i.

4.1. Example: Offspring distribution with finite variance. Suppose $\lambda_k^n(x) = n\lambda_k(x)$ and

$$\lambda(x) = \sum_{k=1}^{\infty}(k+1)k\lambda_k(x) < \infty.$$

Then

$$\Lambda(x_i, u_i) = \sum_{k=1}^{\infty}(k+1)k\lambda_k(x_i)u_i,$$

and

$$\begin{aligned} Af(x,u) &= \sum_i f(x,u)\frac{Bg(x_i,u_i)}{g(x_i,u_i)} \\ &\quad + f(x,u)\sum_{u_i<u_g}\lambda(x_i)\int_{[u_i,u_g]}(g(x_i,v)-1)dv \\ &\quad + f(x,u)\lambda(x_i)u_i(\frac{1}{g(x_i,u_i)}-1), \end{aligned}$$

which is essentially the same as (3.3).

4.2. Example: Offspring distribution in domain of attraction of stable law. For $1 < \beta < 2$, let

$$\lambda_k^n(x_i) = \frac{n^{\beta-1}\lambda(x)}{(k+1)^{\beta+1}}.$$

Then

$$\Lambda^n(x_i,u_i) = n^{\beta-1}\sum_{k=1}^{\infty}\frac{\lambda(x_i)}{(k+1)^{\beta}}(1-(1-\frac{u_i}{n})^k) \to \lambda(x_i)\int_0^{\infty} z^{-\beta}(1-e^{-u_i z})dz$$

which gives

$$\Lambda(x_i,u_i) = \lambda(x_i)u_i^{\beta-1}\frac{\Gamma(2-\beta)}{\beta-1},$$

$\nu(x_i, dz) = \lambda(x_i)z^{-(\beta-1)}dz$, and

$$\begin{aligned} Af(x,u) &= \sum_i f(x,u)\frac{Bg(x_i,u_i)}{g(x_i,u_i)} \\ &\quad + f(x,u)\frac{\Gamma(2-\beta)}{\beta-1}\sum_i \lambda(x_i)(u_i^{\beta-1}-(u_i+h(x_i,u_i))^{\beta-1}) \\ &\quad + f(x,u)\lambda(x_i)u_i^{\beta-1}\frac{\Gamma(2-\beta)}{\beta-1}(\frac{1}{g(x_i,u_i)}-1). \end{aligned}$$

4.3. Generator for measure-valued process. Let $h_0(x_i) = h(x_i,0) = \int_0^{\infty}(1-g(x_i,v))dv$. With α as in (3.4) and $f(x,u) = \prod_i g(x_i,u_i)$, we have

$$\alpha f(\mu) = e^{-\langle h_0,\mu\rangle}$$

and

$$
\begin{aligned}
&\alpha Af(\mu)\\
&= \alpha f(\mu)\int_E B\int_0^\infty (g(x,v)-1)dv\mu(dx)\\
&\quad +\alpha f(\mu)\int_E\int_0^\infty g(x,v)(\Lambda(x,v)-\Lambda(x,v+h(x,v)))dv\mu(dx)\\
&\quad +\alpha f(\mu)\int_E\int_0^\infty \Lambda(x,v)(1-g(x,v))dv\mu(dx)\\
&= \alpha f(\mu)\left(-\langle Bh_0,\mu\rangle+\int_E\int_0^\infty(\Lambda(x,v)-g(x,v)\Lambda(x,v+h(x,v)))dv\mu(dx)\right)\\
&= \alpha f(\mu)\left(-\langle Bh_0,\mu\rangle+\int_E\int_0^{h_0(x)}\Lambda(x,v)dv\mu(dx)\right),
\end{aligned}
$$

where the last equality follows from the fact that

$$\frac{\partial}{\partial v}(v+h(x,v)) = g(x,v).$$

For the example of Section 4.2, we have

$$\alpha Af(\mu) = \alpha f(\mu)\left(-\langle Bh_0,\mu\rangle+\frac{\Gamma(2-\beta)}{\beta(\beta-1)}\langle\lambda h_0^\beta,\mu\rangle\right).$$

5. Dynkin's boundary value problem

We now consider a particle model in which the motion process is absorbing on the boundary of an open set $D \subset E$. Let $B_0 \subset \bar{C}(E)\times\bar{C}(E)$ be a graph separable, pre-generator, and suppose that $\mathcal{D}(B_0)$ is closed under multiplication and is separating. (In particular, B_0 satisfies the conditions of Theorem 1.1.) Define

$$Bf = I_D B_0 f.$$

(Then B satisfies the conditions of Remark 1.2.) If X is a solution of the martingale problem for B_0 and $\tau = \inf\{t : X(t) \notin D\}$, then $X(\cdot\wedge\tau)$ is a solution of the martingale problem for B. We assume that $\tau < \infty$ a.s. and write $X(\infty)$ for $X(\tau)$.

For $f(x,u) = \prod_i g(x_i,u_i)$, let

$$
\begin{aligned}
Af(x,u) \;=\; &\sum_i f(x,u)\frac{Bg(x_i,u_i)}{g(x_i,u_i)}\\
&+f(x,u)\sum_i(\Lambda(x_i,u_i)-\Lambda(x_i,u_i+h(x_i,u_i)))\\
&+f(x,u)\sum_i\Lambda(x_i,u_i)(\frac{1}{g(x_i,u_i)}-1),
\end{aligned}
$$

where Λ is as in Section 4, that is, Λ is of the form (4.2). We assume that Λ is bounded on $D\times[0,a]$ for each $a>0$ and that $\Lambda(x_i,u_i)=0$ for $x_i\notin D$. We do not require Λ to be continuous; however, A still satisfies the conditions of Remark 1.2. Consequently, each solution of the martingale problem for $C = \{(\alpha f,\alpha Af) : f \in \mathcal{D}(A)\}$ has a particle representation given by a solution of the martingale problem for A. Define

$$\psi(x,r) = \int_0^r \Lambda(x,v)dv,$$

so that

$$\alpha Af(\mu) = \alpha f(\mu)\langle -Bh + \psi(\cdot, h), \mu\rangle.$$

Following Dynkin [4], suppose that V satisfies

$$-BV(x) + \psi(x, V(x)) = \rho(x), \quad x \in D \tag{5.1}$$

$$V(x) = \varphi(x), \quad x \in \partial D, \tag{5.2}$$

where $\rho \geq 0$ and V is nonnegative and bounded. We define $\rho(x) = 0$ for $x \notin D$, so (5.1) holds for all x. Let $g(x_i, u_i) = 1 - V(x_i)g_0(u_i)$, where $\int_0^\infty g_0(v)dv = 1$ and $0 \leq Vg_0 \leq 1$. Set $f(x, u) = \prod_i g(x_i, u_i)$ and $g_1(u_i) = \int_{u_i}^\infty g_0(v)dv$. Then

$$\begin{aligned} Af(x,u) &= f(x,u)\sum_i \Big(\frac{-g_0(u_i)BV(x_i)}{1 - g_0(u_i)V(x_i)} + (\Lambda(x_i, u_i) - \Lambda(x_i, u_i + V(x_i)g_1(u_i))) \\ &\qquad + \Lambda(x_i, u_i)\frac{V(x_i)g_0(u_i)}{1 - V(x_i)g_0(u_i)}\Big), \end{aligned}$$

$\alpha f(\mu) = e^{-\langle V,\mu\rangle}$ and $\alpha Af(\mu) = \langle \rho, \mu\rangle e^{-\langle V,\mu\rangle}$.

Assume that $X_i(0) = x$ for all i and that $\{U_i(0)\}$ is a Poisson random measure with mean measure m. Then

$$\begin{aligned} e^{-V(x)} &= E[e^{-\langle V, K(0)\rangle}] \\ &= E[e^{-\langle V,K(t)\rangle - \int_0^t \langle \rho, K(s)\rangle ds}] \\ &= E[\prod_i (1 - V(X_i(t))g_0(U_i(t))e^{-\int_0^t \langle \rho, K(s)\rangle ds}] \\ &= E[\prod_i (1 - \varphi(X_i(\infty))g_0(U_i(\infty)))e^{-\int_0^\infty \langle \rho, K(s)\rangle ds}] \\ &= E[e^{-\langle \varphi, K(\infty)\rangle - \int_0^\infty \langle \rho, K(s)\rangle ds}], \end{aligned}$$

where $\{(X_i(\infty), U_i(\infty))\}$ are the boundary absorption points and the levels for all particles that exit D before dying and

$$K(\infty) = \lim_{r\to\infty} \frac{1}{r}\sum_i I_{[0,r]}(U_i(\infty))\delta_{X_i(\infty)}.$$

The second equality follows from the fact that $e^{-\langle V,K(t)\rangle - \int_0^t \langle \rho, K(s)\rangle ds}$ is a martingale. The third equality follows from (1.5), that is,

$$P\{\Psi(t) \in G|\mathcal{F}_t^K\} = \alpha(K(t), G),$$

where $\alpha(\mu, \cdot)$ is the distribution of a Poisson random measure with mean measure $\mu \times m$. The fourth equality follows by the bounded convergence theorem.

Taking logs, we have

$$\begin{aligned} V(x) &= -\log E[e^{-\langle \varphi, K(\infty)\rangle - \int_0^\infty \langle \rho, K(s)\rangle ds}] \\ &= -\log E[\prod_i (1 - \varphi(X_i(\infty))g_0(U_i(\infty)))e^{-\int_0^\infty \langle \rho, K(s)\rangle ds}]. \end{aligned}$$

The first equality is just (1.11) of Dynkin [4].

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