Particle representations for continuum models

- de Finetti theorem
- Limit theorems for de Finetti measures
- McKean Vlasov limit
- Fluid models for internet protocols
- Stock price set by infinitely many competing traders
- Hydrodynamic limit for symmetric simple exclusion process
- Consistency of numerical schemes for filtering equations
- Sampling from a large population

de Finetti's theorem

$\xi_1, \xi_2, \ldots$ is *exchangeable* if

$$P\{\xi_1 \in \Gamma_1, \ldots, \xi_m \in \Gamma_m\} = P\{\xi_{s_1} \in \Gamma_1, \ldots, \xi_{s_m} \in \Gamma_m\}$$

$(s_1, \ldots, s_m)$ any permutation of $(1, \ldots, m)$.

Theorem 1 (de Finetti) *Let $\xi_1, \xi_2, \ldots$ be exchangeable. Then there exists a random probability measure Φ such that for every bounded, measurable g,*

$$\lim_{N\to\infty} \frac{g(\xi_1) + \cdots + g(\xi_N)}{N} = \int g(x)\Phi(dx) \quad a.s.$$

so $\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N \delta_{\xi_i} = \Phi$.

In addition

$$E[\prod_{i=1}^m g_i(\xi_i)|\Phi] = \prod_{i=1}^m \langle \Phi, g_i \rangle = \prod_{i=1}^m \int g_i d\Phi.$$

Basic convergence lemma

Lemma 2 *For $n = 1, 2, \ldots$, let $\{\xi_1^n, \ldots, \xi_{N_n}^n\}$ be exchangeable in S (allowing) $N_n = \infty$.) Let Ξ^n be the empirical measure,*

$$\Xi^n = \frac{1}{N_n} \sum_{i=1}^{N_n} \delta_{\xi_i^n}.$$

Assume $N_n \to \infty$, and for each $m = 1, 2, \ldots$, $\{\xi_1^n, \ldots, \xi_m^n\} \Rightarrow \{\xi_1, \ldots, \xi_m\}$ in S^m.

Then $\{\xi_i\}$ is exchangeable and setting $\xi_i^n = s_0 \in S$ for $i > N_n$, $\{\Xi^n, \xi_1^n, \xi_2^n \ldots\} \Rightarrow \{\Xi, \xi_1, \xi_2, \ldots\}$ in $\mathcal{P}(S) \times S^\infty$, where Ξ is the deFinetti measure for $\{\xi_i\}$.

If for each m, $\{\xi_1^n, \ldots, \xi_m^n\} \to \{\xi_1, \ldots, \xi_m\}$ in probability in S^m, then $\Xi^n \to \Xi$ in probability in $\mathcal{P}(S)$.

Convergence lemma for processes

Lemma 3 *Let* $X^n = (X^n_1, \ldots, X^n_{N_n})$ *be exchangeable families of* $D_E[0,\infty)$*-valued random variables such that* $N_n \Rightarrow \infty$ *and* $X^n \Rightarrow X$ *in* $D_E[0,\infty)^\infty$*. Define*

$$\Xi_n = \frac{1}{N_n}\sum_{i=1}^{N_n} \delta_{X_i^n} \in \mathcal{P}(D_E[0,\infty))$$

$$\Xi = \lim_{m\to\infty} \frac{1}{m}\sum_{i=}^{m} \delta_{X_i}$$

$$V_n(t) = \frac{1}{N_n}\sum_{i=1}^{N_n} \delta_{X_i^n(t)} \in \mathcal{P}(E)$$

$$V(t) = \lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^{m} \delta_{X_i(t)}$$

Then

a) For $t_1, \ldots, t_l \notin \{t : E[\Xi\{x : x(t) \neq x(t-)\}] > 0\}$

$$(\Xi_n, V_n(t_1), \ldots, V_n(t_l)) \Rightarrow (\Xi, V(t_1), \ldots, V(t_l)).$$

b) If $X^n \Rightarrow X$ *in* $D_{E^\infty}[0,\infty)$*, then* $V_n \Rightarrow V$ *in* $D_{\mathcal{P}(E)}[0,\infty)$*.*

Remarks

a) The set $D_\Xi = \{t : E[\Xi\{x : x(t) \neq x(t-)\}] > 0\}$ is at most countable.

b) If for $i \neq j$, with probability one, X_i and X_j have no simultaneous discontinuities, then $D_\Xi = \emptyset$ and convergence of X^n to X in $D_E[0,\infty)^\infty$ implies convergence in $D_{E^\infty}[0,\infty)$.

Classical McKean-Vlasov limit

For $i = 1, \ldots, n$

$$X_i^n(t) = X_i^n(0) + \int_0^t \sigma(X_i^n(s), V^n(s))dW_i(s) + \int_0^t b(X_i^n(s), V^n(s))ds$$

where $V^n(t) = \frac{1}{n}\sum_{i=1}^n \delta_{X_i^n(t)}$

Example: $X_i^n(t) = X_i^n(0) + W_i(t) + \dfrac{1}{n}\displaystyle\sum_{j=1}^n \int_0^t b(X_i^n(s) - X_j^n(s))ds$

Any limit point must satisfy

$$X_i(t) = X_i(0) + \int_0^t \sigma(X_i(s), V(s))dW_i(s) + \int_0^t b(X_i(s), V(s))ds$$

which has a unique solution under Lipschitz conditions.

Conditions for uniqueness

Theorem 4 *Assume that* $\{X_i(0)\}$ *and* $\{W_i\}$ *are independent. If*

$$|\sigma(x,\nu) - \sigma(y,\mu)| + |b(x,\nu) - b(y,\mu)| \leq C(|x-y| + \rho_1(\mu,\nu))$$

for the Wasserstein metric

$$\rho_1(\mu,\nu) = \sup_{f:|f(x)-f(y)|\leq|x-y|} |\int f d\nu - \int f d\mu|,$$

then the system

$$X_i(t) = X_i(0) + \int_0^t \sigma(X_i(s), V(s))dW_i(s) + \int_0^t b(X_i(s), V(s))ds$$

with $V(t) = \lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^m \delta_{X_i(t)}$ *has a unique solution.*

Corollary 5 (Propagation of chaos) V *is measurable with respect to the tail* σ*-algebra for* $\{(X_i(0), W_i)\}$ *and hence must be deterministic. Consequently, the* X_i *are independent.*

Limiting PDE

$$\varphi(X_i(t)) = \varphi(X_i(0)) + \int_0^t \nabla\varphi(X_i(s))^T \sigma(X_i(s), V(s))dW_i(s) + \int_0^t L(V(s))f(X_i(s))ds$$

where

$$L(v)\varphi(x) = \frac{1}{2}\sum a_{ij}(x,v)\frac{\partial^2}{\partial x_i \partial x_j}\varphi(x) + \sum b_i(x,v)\frac{\partial}{\partial x_i}\varphi(x)$$

$$a(x,v) = \sigma(x,v)\sigma^T(x,v)$$

Since $\langle V(t), \varphi\rangle = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \varphi(X_i(t))$, V satisfies

$$\langle V(t), \varphi\rangle = \langle V(0), \varphi\rangle + \int_0^t \langle V(s), L(V(s))\varphi\rangle ds$$

Under the Lipschitz conditions, $V^n \Rightarrow V$.

A stochastic partial differential equation

Suppose

$$X_i(t) = X_i(0) + \int_0^t \sigma(X_i(s), V(s))dW_i(s) + \int_0^t b(X_i(s), V(s))ds + \int_0^t \alpha(X_i(s), V(s))dW(s)$$

where α is Lipschitz. Then uniqueness holds and the solution is adapted to the filtration generated by the common Brownian motion W. V is a solution of the SPDE

$$\langle V(t), \varphi\rangle = \langle V(0), \varphi\rangle + \int_0^t \langle V(s), L(V(s))\varphi\rangle ds + \int_0^t \langle V(s), \nabla\varphi(\cdot)^T \alpha(\cdot, V(s))\rangle dW(s)$$

A Markov mapping theorem

By a Markov mapping argument, uniqueness for the particle system implies uniqueness for the SPDE.

- Use Itô's formula to derive the generator for a martingale problem for the infinite system.
- Observe that uniqueness for the infinite system implies uniqueness for the corresponding martingale problem.
- Apply the fact that $E[f(X_1(t),\ldots,X_m(t))|\mathcal{F}_t^V] = \langle V^m(t), f\rangle$ (μ^m denotes the m-fold product of μ) to derive a martingale problem for V.
- Show that every solution of the SPDE is a solution of the martingale problem for V.
- Apply the Markov mapping theorem to conclude that every solution of the SPDE has a representation as the deFinetti measure of a solution of the infinite system.

A brief introduction to transmission protocols

Files sent over the internet are typically broken into smaller *packets*. The packets must be reassembled in the correct order by the receiving computer. The sending computer must be assured that all packets have been correctly received. Consequently, the receiving computer sends an *acknowledgement* of each packet received.

Packet losses are typically due to *congestion* in the network. A *transmission control protocol* (TCP) controls the rate at which packets are sent based on the losses it perceives. The rate is determined by a *window size*, that is, the maximum number of unacknowledged packets the sending computer can have in "flight." If congestion is low and no packet losses are experienced, the window size will increase. If congestion is high and packet losses occur, the window size will decrease.

Modeling congestion control mechanisms (cf. McDonald, Reynier, Bacelli)

Idealize model: N sources (computers) feeding a single queue (router).

$Q_N(t)$ normalized queue length (the number of packets in the queue divided by N) at time t

$F_N(Q_N(t))$ probability a packet sent to the queue at time t is rejected/dropped; $F_N(q) = 1$ for $q > q_0$, q_0 the normalized buffer size.

$W_i(t)$ window size for source i at time t

$D_i(t)$ number of "dropped" packets generated by source i

$A_i(t)$ number of packets generated by source i accepted by the queue

T_i delay after a packet is processed before the source receives an acknowledgement

d time until a source concludes that an unacknowledged packet has been dropped

Number of packets "in flight"

$\gamma_N(t)$ the arrival time in the queue of the most recently served packet.

$A_i(\gamma_N(t - T_i))$ number of acknowledgements received by source i by time t

$$X_i(t) = A_i(t) - A_i(\gamma_N(t - T_i)) + D_i(t) - D_i(t - d)$$

Rate at which new packets are sent

$$\lambda(W_i(t) - X_i(t))^+$$

Stochastic equations for the model

Y_i^a, Y_i^d, $i = 1, 2, \ldots$ independent unit Poisson processes, $s_N(t) = \frac{[Nt]}{N}$

$$A_i(t) = Y_i^a\Big(\int_0^t \lambda(W_i(s) - X_i(s))^+(1 - F_N(Q_N(s)))ds\Big)$$

$$D_i(t) = Y_i^d\Big(\int_0^t \lambda(W_i(s) - X_i(s))^+ F_N(Q_N(s))ds\Big)$$

$$W_i(t) = W_i(0) + \int_0^t \frac{1}{W_i(s-)} dA_i(\gamma_N(s - T_i)) - \int_0^t \frac{1}{2} W_i(s-)dD_i(s - d)$$

$$\begin{aligned} Q_N(t) &= Q_N(0) + \frac{1}{N}\sum_{i=1}^N A_i(t) - \int_0^t \mathbf{1}_{\{Q_N(s-)>0\}} ds_N(s) \\ &= Q_N(0) + \frac{1}{N}\sum_{i=1}^N A_i(t) - s_N(t) + I_N(t) \end{aligned}$$

Many source limit

$F_N = F + H_N$, where F is continuous, H_N is nondecreasing, and there exists $\epsilon_N \to 0$ such that $H_N(q) = 0$ for $q < q_0 - \epsilon_N$ and $H_N(q) = 1 - F(q)$ for $q \geq q_0$.

Let

$$\begin{aligned} K_N(t) &= \int_0^t H_N(Q_N(s))ds \\ \Lambda_N(t) &= \int_0^t \frac{1}{N}\sum_{i=1}^N \lambda(W_i(s) - X_i(s))^+ dK_N(s), \end{aligned}$$

$$\widetilde{A}_i(t) = \widetilde{Y}_i^a\left(\int_0^t \lambda(W_i(s) - X_i(s))^+(1 - F_N(Q_N(s)))ds\right),$$

where $\widetilde{Y}_i^a(u) = Y_i^a(u) - u$.

Limit of the queue length

$$\begin{aligned} Q_N(t) &= Q_N(0) + \frac{1}{N}\sum_{i=1}^N \widetilde{A}_i(t) + \int_0^t \frac{1}{N}\sum_{i=1}^N \lambda(W_i(s) - X_i(s))^+(1 - F(Q_N(s)))ds \\ &\quad - \int_0^t \frac{1}{N}\sum_{i=1}^N \lambda(W_i(s) - X_i(s))^+ dK_N(s) - s_N(t) + \int_0^t \mathbf{1}_{\{Q_N(s-)=0\}} ds_N(s). \end{aligned}$$

γ_N is determined by

$$Q_N(t) = \frac{1}{N}\sum_{i=1}^N A_i(t) - \frac{1}{N}\sum_{i=1}^N A_i(\gamma_N(t)).$$

Q_N is asymptotically continuous and $|\gamma_N - \widetilde{\gamma}_N| \to 0$, where $\widetilde{\gamma}_N$ is the solution of

$$Q_N(t) = \int_{\widetilde{\gamma}_N(t)}^t \frac{1}{N}\sum_{i=1}^N \lambda(W_i(s) - X_i(s))^+(1 - F(Q_N(s)))ds - \Lambda_N(t) + \Lambda_N(\gamma_N(t)).$$

In particular, γ_N is asymptotically continuous.

Existence of limit

Relative compactness of the individual counting processes follows from the fact that they cannot have asymptotically coalescing jumps which in turn follows from the boundedness of λ and the asymptotic countinuity of γ_N.

Joint relative compactness in the Skorohod (J_1) topology follows from the fact that with probability one no two of the limiting processes will have simultaneous jumps.

$V_t^{W,X}$ the de Finetti measure for $\{(W_i(t), X_i(t))\}$

$$r(t) = \int (w-x)^+ V_t^{W,X}(dw, dx),$$

Limiting model

$$A_i(t) = Y_i^a\Big(\int_0^t \lambda(W_i(s) - X_i(s))^+(1 - F(Q(s)) - \dot{K}(s))ds\Big)$$

$$D_i(t) = Y_i^d\Big(\int_0^t \lambda(W_i(s) - X_i(s))^+(F(Q(s)) + \dot{K}(s))ds\Big)$$

$$W_i(t) = W_i(0) + \int_0^t \frac{1}{W_i(s-)} dA_i(\gamma(s - T_i)) - \int_0^t \frac{1}{2} W_i(s-)dD_i(s - d)$$

$$Q(t) = Q(0) + \int_0^t \lambda r(s)(1 - F(Q(s)))ds - t - \Lambda(t) + I(t)$$

I increases only when $Q = 0$, Λ increases only when $Q = q_0$.

$$K(t) = \int_0^t \frac{1}{\lambda r(s)} d\Lambda(s),$$

γ is determined by

$$Q(t) = \int_{\gamma(t)}^t \lambda r(s)(1 - F(Q(s)))ds - \Lambda(t) + \Lambda(\gamma(t)),$$

Competing traders (cf. Yoonjung Lee)

N traders (N even) and $N/2$ shares of stock.

$S(t)$ price of a share of stock at time t

Individual trader's (log) valuation:

$$X_i(t) = X_i(0) + \sigma_1 W_i(t) + \sigma_2 W(t) + \mu t + \int_0^t \nu(S(s) - X_i(s))ds$$

$\theta_i(t)$ is 1 if the ith trader owns the stock and -1 if the ith trader does not own the stock.

$$\begin{aligned} \theta_i(t) &= \theta_i(0)(-1)^{K_i(t)} \\ K_i(t) &= Y_i(\int_0^t \lambda\{\theta_i(s)(S(s) - X_i(s))\}^+ ds). \end{aligned}$$

Price setting by market maker

Q_N number of shares owned by the market maker

$$Q_N(t) = N - \sum_i \theta_i^+(t) = -\frac{1}{2}\sum_{i=1}^N \theta_i(t)$$

$$S(t) = S(0) - \int_0^t \beta \frac{1}{N} Q_N(s)ds$$

Equation for Q_N

Let $V_N^+(t) = \frac{1}{N}\sum_{i=1}^N \mathbf{1}_{\{\theta_i=1\}}\delta_{X_i(t)}$ and $V_N^-(t) = \frac{1}{N}\sum_{i=1}^N \mathbf{1}_{\{\theta_i=-1\}}\delta_{X_i(t)}$.

$$\frac{1}{N}Q_N(t) = \frac{1}{N}Q_N(0) + \frac{1}{N}\sum_{i=1}^N \int_0^t \theta_i(s)\lambda\{\theta_i(s)(S(s)-X_i(s))\}^+ ds$$

$$+\frac{1}{N}\sum_{i=1}^N \int_0^t \theta_i(r-)d\widetilde{Y}_i(\int_0^r \lambda\{\theta_i(s)(S(s)-X_i(s))\}^+ ds)$$

$$= \frac{1}{N}Q_N(0) + \frac{1}{N}\sum_{i=1}^N \int_0^t \theta_i(r-)d\widetilde{Y}_i(\int_0^r \lambda\{\theta_i(s)(S(s)-X_i(s))\}^+ ds)$$

$$+\int_0^t \lambda \int (S(s)-x)^+ V_N^+(s,dx)ds - \int_0^t \lambda \int (x-S(s))^+ V_N^-(s,dx)ds$$

Limiting model

$$
\begin{aligned}
X_i(t) &= X_i(0) + \sigma_1 W_i(t) + \sigma_2 W(t) + \mu t + \int_0^t \nu(S(s) - X_i(s))ds \\
\theta_i(t) &= \theta_i(0)(-1)^{K_i(t)} \\
K_i(t) &= Y_i(\int_0^t \lambda\{\theta_i(s)(S(s) - X_i(s))\}^+ ds)
\end{aligned}
$$

and $V^+(t) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \mathbf{1}_{\{\theta_i(t)=1\}}\delta_{X_i(t)}$ and similarly for V^-.

$$
\begin{aligned}
Q(t) &= Q(0) + \int_0^t \lambda \int (S(s) - x)^+ V^+(s, dx)ds \\
&\quad - \int_0^t \lambda \int (x - S(s))^+ V^-(s, dx)ds \\
S(t) &= S(0) - \int_0^t \beta Q(s)ds
\end{aligned}
$$

Equations for V^+ and V^-

$$\begin{aligned}\varphi(X_i(t),\theta_i(t)) &= \varphi(X_i(0),\theta_i(0)) + \sigma_1\int_0^t \varphi'(X_i(s),\theta_i(s))dW_i(s)\\ &\quad +\sigma_2\int_0^t \varphi'(X_i(s),\theta_i(s))dW(s)\\ &\quad +\int_0^t \frac{1}{2}(\sigma_1^2+\sigma_2^2)\varphi''(X_i(s),\theta_i(s))ds\\ &\quad +\int_0^t (\mu+\nu(S(s)-X_i(s)))\varphi'(X_i(s),\theta_i(s))ds\\ &\quad +\int_0^t (\varphi(X_i(s),-\theta_i(s-)) - \varphi(X_i(s),\theta_i(s-))dK_i(s)\end{aligned}$$

Suppose $\varphi(x,-1)=0$, $\varphi(x,1)=\varphi(x)$. Then

$$\begin{aligned}\langle V^+(t),\varphi\rangle &= \langle V(0)^+,\varphi\rangle + \sigma_2\int_0^t \langle V^+(s),\varphi'\rangle dW(s) + \int_0^t \langle V^+(s), L(S(s))\varphi\rangle ds\\ &\quad + \int_0^t \lambda(\langle V^-(s),(\cdot - S(s))^+\varphi(\cdot)\rangle - \langle V^+(s),(S(s)-\cdot)^+\varphi(\cdot)\rangle ds\end{aligned}$$

A "simplified" SPDE

Let $\lambda, \beta \to \infty$. Then

$$X_i(t) = X_i(0) + \sigma_1 W_i(t) + \sigma_2 W(t) + \mu t + \int_0^t \nu(S(s) - X_i(s))ds$$

$$\langle V(t), \varphi \rangle = \langle V(0), \varphi \rangle + \sigma_2 \int_0^t \langle V(s), \varphi' \rangle dW(s) + \int_0^t \langle V(s), L(S(s))\varphi \rangle ds$$

with

$$L(s)\varphi(x) = \frac{1}{2}(\sigma_1^2 + \sigma_2^2)\varphi''(x) + (\mu + \nu(S(s) - x))\varphi'(x)$$

and $S(t)$ is the median of $V(t)$.

Symmetric simple exclusion model

X_i^n particle location on $E_n = \frac{1}{n}\mathbb{Z}$ mod 1 (at most one particle per location)

$X_i^n \to X_i^n + \frac{k}{n}$ at rate $n^2\lambda(|k|)$ unless new location is already occupied (note symmetry)

Equivalent formulation

Swap contents of $\frac{l}{n}$ and $\frac{k}{n}$ at rate $n^2\lambda(|k-l|)$

$$\begin{aligned} X_i^n(t) &= X_i^n(0) + \frac{1}{n}\sum_{l\neq k}\int_0^t (l-k)\mathbf{1}_{\{X_i^n(s-)=k\}} dY_{kl}(n^2\lambda(|k-l|)s) \\ &= X_i^n(0) + \frac{1}{n}\sum_{l\neq k}\int_0^t (l-k)\mathbf{1}_{\{X_i^n(s-)=k\}} d\widetilde{Y}_{kl}(n^2\lambda(|k-l|)s) \end{aligned}$$

$Y_{kl} = Y_{lk}$ independent unit Poisson processes and $\widetilde{Y}_{kl}(u) = Y_{kl}(u) - u$.

Convergence

$$
\begin{aligned}
[X_i^n]_t &= \frac{1}{n^2}\sum_{l\neq k}\int_0^t (l-k)^2 \mathbf{1}_{\{X_i^n(s-)=k\}} dY_{kl}(n^2\lambda(|k-l|)s) \\
&= \frac{1}{n^2}\sum_{l\neq k}\int_0^t (l-k)^2 \mathbf{1}_{\{X_i^n(s-)=k\}} d\widetilde{Y}_{kl}(n^2\lambda(|k-l|)s) + \sigma^2 t
\end{aligned}
$$

Then $[X_i^n]_t \to \sigma^2 t$, $\sigma^2 = \sum_k k^2\lambda(|k|)$, and by the martingale central limit theorem, $X_i^n \Rightarrow X_i$ where $X_i(t) = X_i(0) + \sigma W_i(t)$.

$$
[X_i^n, X_j^n]_t = -\frac{1}{n^2}\sum_{l\neq k}\int_0^t (l-k)^2 \mathbf{1}_{\{X_i^n(s-)=k, X_j^n(s-)=l\}} dY_{kl}(n^2\lambda(|k-l|)s)
$$

converges to zero which implies the W_i are independent.

Therefore $V_n \Rightarrow V$, where $V(t,\Gamma) = \int_\Gamma v(t,x)dx$ and

$$
v_t = \frac{1}{2}\sigma^2 \Delta v
$$

Filtering (cf. Fujisaki, Kallianpur, Kunita)

$$\begin{aligned} X(t) &= X(0) + \int_0^t \sigma(X(s))dB(s) + \int_0^t b(X(s))ds + \int_0^t \alpha(X(s))dW(s) \\ &= X(0) + \int_0^t \sigma(X(s))dB(s) + \int_0^t (b(X(s)) - \alpha(X(s))h(X(s)))ds \\ &\quad + \int_0^t \alpha(X(s))dY(s) \\ Y(t) &= \int_0^t h(X(s))ds + W(t). \end{aligned}$$

Kallianpur Striebel formula

Let

$$L(t) = \exp\{\int_0^t h(X(s))dY(s) - \frac{1}{2}\int_0^t h^2(X(s))ds\}$$

so

$$L(t) = 1 + \int_0^t L(s)h(X(s))dY(s)$$

Assume that under Q, B and Y are independent $\{\mathcal{F}_t\}$-Brownian motions. If $dP = L(t)dQ$ on $\mathcal{F}_t$, then under P, B and W are independent Brownian motions.

$$E^P[f(X(t))|\mathcal{F}_t^Y] = \frac{E^Q[f(X(t))L(t)|\mathcal{F}_t^Y]}{E^Q[L(t)|\mathcal{F}_t^Y]} = \frac{\phi(f,t)}{\phi(1,t)}$$

Convergence lemma

Lemma 6 *Suppose* $X_1, X_2, \ldots$ *are iid and* Y *is independent of* $\{X_i\}$. *Let* $Z_i = H(X_i, Y)$. *Then* $\{Z_i\}$ *is exchangeable and*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} H(X_i, Y) = E[Z_1 | Y]$$

Particle representation

$$\phi(f,t) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(X_i(t))L_i(t)$$

where

$$\begin{aligned} X_i(t) &= X(0) + \int_0^t \sigma(X_i(s))dB_i(s) + \int_0^t (b(X_i(s)) - \alpha(X_i(s))h(X_i(s)))ds \\ &\quad + \int_0^t \alpha(X_i(s))dY(s) \\ L_i(t) &= 1 + \int_0^t L_i(s)h(X_i(s))dY(s). \end{aligned}$$

Filtering equation

$$
\begin{aligned}
f(X_i(t))L_i(t) &= f(X_i(0)) + \int_0^t L_i(s)\sigma(X_i(s))f'(X_i(s))dB_i(s) \\
&\quad + \int_0^t Af(X_i(s))L_i(s)ds \\
&\quad + \int_0^t (f'(X_i(s)) + f(X_i(s))h(X_i(s)))L_i(s)dY(s)
\end{aligned}
$$

where

$$
Af(x) = \frac{1}{2}(\sigma^2(x) + \alpha^2(x))f''(x) + b(x)f'(x).
$$

Then

$$
\phi(f,t) = \phi(f,0) + \int_0^t \phi(Af,s)ds + \int \phi(f' + fh, s)dY(s)
$$

Finite state, discrete time approximation

$$\begin{aligned} X_i^\epsilon(t+\epsilon) &= X_i^\epsilon(t) + H_1^\epsilon(X_i^\epsilon(t), \xi^1_{i,[t/\epsilon]}) + H_2^\epsilon(X_i^\epsilon(t), \xi^2_{i,[t/\epsilon]}, Y(t+\epsilon) - Y(t)) \\ &= H_0^\epsilon(X_i^\epsilon(t), \xi^1_{i,[t/\epsilon]}, \xi^2_{i,[t/\epsilon]}, Y(t+\epsilon) - Y(t)) \end{aligned}$$

where

$$\int_0^1 H_1^\epsilon(x,u)du = (b(x) - \alpha(x))\epsilon + o(\epsilon)$$

$$\int_0^t H_1^\epsilon(x,u)^2 du = \sigma^2(x)\epsilon + o(\epsilon) \qquad \int_0^t |H_1^\epsilon(x,u)|^3 du = o(\epsilon),$$

$$\int_0^1 H_2^\epsilon(x,u,\Delta y)du = \alpha(x)\Delta y + o(\epsilon)$$

$$\int_0^1 |H_2^\epsilon(x,u,\Delta y) - \alpha(x)\Delta y|^2 du = o(\epsilon).$$

Particle representation for approximation

Let L_i^ϵ satisfy $L_i^\epsilon(0)=1$ and

$$L_i^\epsilon(t+\epsilon)=L_i^\epsilon(t)(1+h(X_i^\epsilon(t))(Y(t+\epsilon)-Y(t))),$$

so

$$\begin{aligned} f(X_i^\epsilon(t+\epsilon))L_i^\epsilon(t+\epsilon) &= f(H_0^\epsilon(X_i^\epsilon(t),\xi^1_{i,[t/\epsilon]},\xi^2_{i,[t/\epsilon]},Y(t+\epsilon)-Y(t))) \\ &\qquad (1+h(X_i^\epsilon(t))(Y(t+\epsilon)-Y(t)))L_i^\epsilon(t) \end{aligned}$$

The algorithm

Define

$$T^\epsilon f(x,\Delta y) = \int_0^1 \int_0^1 f(x + H_1^\epsilon(x,u^1) + H_2^\epsilon(x,u^2,\Delta y))du^1 du^2$$

so

$$\begin{aligned}\phi^\epsilon(f,t+\epsilon) &= \phi^\epsilon(T^\epsilon f(\cdot, Y(t+\epsilon)-Y(t)),t) \\ &\quad +\phi^\epsilon(T^\epsilon f(\cdot, Y(t+\epsilon)-Y(t))h,t)(Y(t+\epsilon)-Y(t))\end{aligned}$$

Taking $f = \mathbf{1}_{\{z\}}$, define $p^\epsilon_{x,z}(\Delta y) = T^\epsilon f(x,\Delta y)$. Then

$$\begin{aligned}\phi^\epsilon(x,t+\epsilon) &= \sum_{x'} \phi^\epsilon(x',t)p_{x'x}(Y(t+\epsilon)-Y(t)) \\ &\quad + \sum_{x'} \phi^\epsilon(x',t)p_{x'x}(Y(t+\epsilon)-Y(t))h(x')(Y(t+\epsilon)-Y(t))\end{aligned}$$

Sampling from a population model

Model I

$(X_1(t), \ldots, X_N(t))$ "types" of N individuals in a population at time t

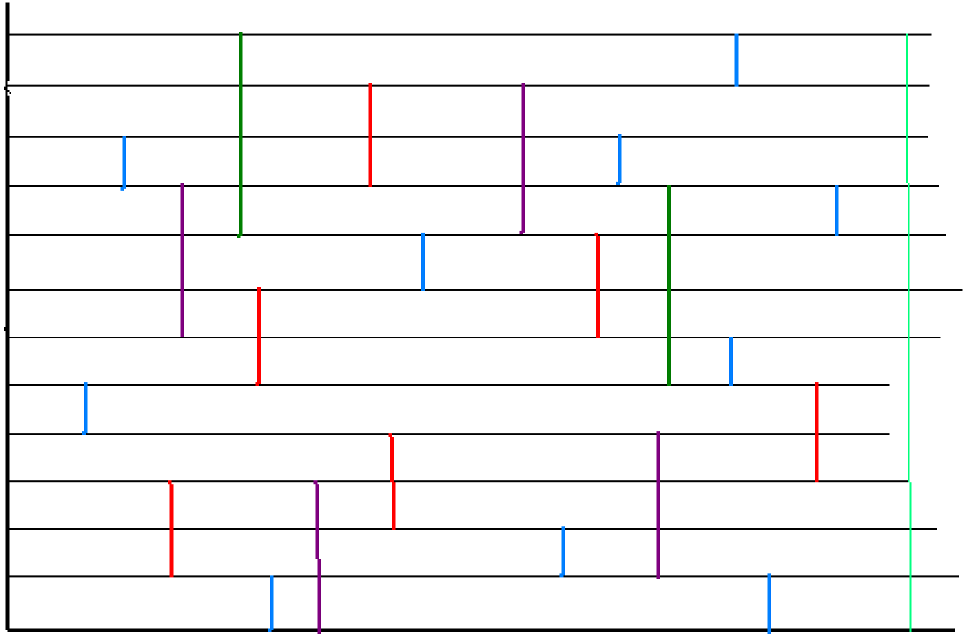
At rate λ, a pair of individuals is selected at random, one is killed and replaced by a copy of the other

In between birth/death events, individuals may change type (mutate) independently of the other individuals in the population

Model II

Same as Model I except that when the pair is selected the higher numbered individual is killed and replaced by a copy of the lower numbered individual

Lookdown process



Equivalence of models

Theorem 7 *Let* X^I *be a realizataion of model I and* X^{II} *be a realization of model II with*

$$V^I(0) = \frac{1}{N}\sum_{i=1}^N \delta_{X_i^I(0)} = \frac{1}{N}\sum_{i=1}^N \delta_{X_i^{II}(0)} = V^{II}(0)$$

If $\{X_i^{II}(0)\}$ *is exchangeable, then for each* $t > 0$, $\{X_i^{II}(t)\}$ *and* V^I *and* V^{II} *have the same distribution.*

Stochastic equations

$$X_i(t) = X_i(0) + \int_{[0,1]\times[0,t]} (H(X_i(s-), z) - X_i(s-))\xi_i(dz \times ds)$$
$$+ \sum_{1\le j<i} \int_0^t (X_j(s-) - X_i(s-))dL_{ij}(s)$$

ξ_i independent Poisson random measures with mean measure $\theta m \times m$ on $[0,1] \times [0,\infty)$

L_{ij} independent Poisson processes with intensity λ

Generator

$Bf(r) = \theta\int_0^1 (f(H(r,z)) - f(r))dz$

$$Af(x) = \sum_{i=1}^{\infty} B_i f(x) + \sum_{j<i} \lambda(f(\eta_i(x|x_j)) - f(x))$$

$$f(X_1(t),\ldots,X_m(t)) - \int_0^t \sum_{i=1}^{m} B_i f(X_1(s),\ldots,X_m(s))ds$$
$$+ \sum_{1\le j<i\le m} \int_0^t \lambda(f(\eta_i(X(s)|X_j(s)) - f(X_1(s),\ldots,X_m(s))ds$$

Fleming-Viot process

Theorem 8 *If* $\{X_i(0)\}$ *is exchangeable, then for each* $t > 0$, $\{X_i(t)\}$ *is exchangeable, and*

$$E[f(X_1(t),\dots X_m(t))|\mathcal{F}_t^V] = \langle f, V(t)^m\rangle.$$

It follows that

$$\langle f, V(t)^m\rangle - \int_0^t \sum_{i=1}^m \langle B_i f, V(s)^m\rangle ds$$
$$+\int_0^t \sum_{1\le j<i\le m} (\langle \Phi_{ij} f, V(s)^{m-1}\rangle - \langle f, V(s)^m\rangle) ds$$

is a $\{\mathcal{F}_t^V\}$*-martingale, where* $\Phi_{ij} f$ *is the function of* $m-1$ *variables obtained by setting* $x_i = x_j$.

Abstract

Many stochastic and deterministic models are derived as continuum limits of discrete stochastic systems as the size of the systems tends to infinity. Discrete "particles" are each assigned a small mass and the limiting "mass distribution," typically characterized as a solution of a deterministic or stochastic partial differential equation, gives the desired model. A number of examples will be described in which keeping the discrete particles in the limit provides a useful tool for justifying the limit and analyzing the limiting model. Examples include derivation of fluid models for internet protocols, many-server queueing approximations, models of stock prices set by infinitely many competing traders, and consistency of numerical schemes for filtering equations.