Diffusions and diffusion approximations revisited

Kolmogorov's equations

The *transition function* of a Markov process X with state space E is defined by $P(s,x,t,dy) = P\{X(t) \in dy | X(s) = x\}$.

A transition function satisfies the *Chapman-Kolmogorov equation*

$$P(s,x,r,\Gamma) = \int_E P(s,x,t,dy)P(t,y,r,\Gamma), \quad s < t < r.$$

Kolmogoroff (1931) considers transition functions for $E = \mathbb{R}^d$ with the following properties:

$$\begin{aligned}
\lim_{t\to s} \frac{1}{t-s} \int_{|y-x|\le 1} (y-x)P(s,x,t,dy) &= b(s,x) \\
\lim_{t\to s} \frac{1}{t-s} \int_{|y-x|\le 1} (y-x)(y-x)^T P(s,x,t,dy) &= a(s,x) \\
\lim_{t\to s} \frac{1}{t-s} \int_{|y-x|\le 1} |y-x|^3 P(s,x,t,dy) &= 0
\end{aligned}$$

Note that $a(s,x)$ must be nonnegative definite.

Kolmogorov backward equation

Let $u(s,x,t) = E[f(X(t))|X(s)=x] = \int_{\mathbb{R}^d} f(y)P(s,x,t,dy)$.

$$\begin{aligned} u(s-h,x,t) - u(s,x,t) \\ &= E[\int_{\mathbb{R}^d}(u(s,y,t)-u(s,x,t))P(s-h,x,s,dy) \\ &= E[\int_{\mathbb{R}^d}(\partial_x u(s,x,t)\cdot(y-x) + \frac{1}{2}(y-x)^T\partial_x^2 u(s,x,t)(y-x) \\ &\qquad\qquad +O(|y-x|^3))P(s-h,x,s,dy) \\ &\approx h\partial_x u(s,x,t)\cdot b(s-h,x) + \frac{1}{2}\sum a_{ij}(s-h,x)\partial_{x_i}\partial_{x_j}u(s,x,t) \end{aligned}$$

strongly suggesting $-\partial_s u(s,x,t) = L_s u(s,x,t)$, where

$$L_s f(x) = \partial_x f(x)\cdot b(s,x) + \frac{1}{2}\sum a_{ij}(s,x)\partial_{x_i}\partial_{x_j}f(x)$$

Kolmogorov's forward equation

Again setting $u(s,x,t) = E[f(X(t))|X(s)=x] = \int_{\mathbb{R}^d} f(y)P(s,x,t,dy)$.

$$\begin{aligned} u(s,x,t+h) - u(s,x,t) \\ &= E[\int_{\mathbb{R}^d}(\int_{\mathbb{R}^d} f(z)P(t,y,t+h,dz) - f(y))P(s,x,t,dy) \\ &= E[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(\partial_y f(y)\cdot(z-y) + \frac{1}{2}(z-y)^T\partial_y^2 f(y)(z-y) \\ &\qquad + O(|z-y|^3))P(t,y,t+h,dz)P(s,x,t,dy) \end{aligned}$$

strongly suggesting $\partial_t \int_{\mathbb{R}^d} f(y)P(s,x,t,dy) = \int_{\mathbb{R}^d} L_t f(y)P(s,x,t,dy)$ where

$$L_t f(y) = \partial_y f(y)\cdot b(t,y) + \frac{1}{2}\sum a_{ij}(t,y)\partial_{y_i}\partial_{y_j} f(y).$$

Assuming a smooth density, $p(s,x,t,y)dy = P(s,x,t,dy)$,

$$\partial_t p(s,x,t,y) = L_t^* p(s,x,t,y)$$

Doeblin's approach Doeblin (2000)

Let $d = 1$. For $0 = t_0 < \cdots < t_m = t$

$$X(t) - X(0) - \sum_{i=0}^{m-1} E[X(t_{i+1}) - X(t_i)|\mathcal{F}_{t_i}]$$

$$= X(t) - X(0) - \sum_{i=0}^{m-1} \int_{\mathbb{R}} (y - X(t_i))P(t_i, X(t_i), t_{i+1}, dy)$$

$$\approx X(t) - X(0) - \sum_{i=0}^{m-1} b(t_i, X(t_i))(t_{i+1} - t_i)$$

$$\approx X(t) - X(0) - \int_0^t b(s, X(s))ds$$

In modern terminology,

$$Z(t) = X(t) - X(0) - \int_0^t b(s, X(s))ds$$

is a *martingale.*

Time-change to a Brownian motion

Let θ satisfy

$$\int_0^{\theta(t)} a(s, X(s))ds = t.$$

Then

$$W(t) = Z(\theta(t))$$

is a Brownian motion and hence (as Yor (2000) notes)

$$Z(t) = W(\int_0^t a(s, X(s))ds)$$

so

$$X(t) = X(0) + W(\int_0^t a(s, X(s))ds) + \int_0^t b(s, X(s))ds \qquad (1)$$

Itô's approach

Doeblin and others recognized that $X(t+\Delta t) - X(t)$ was, in a strong sense, approximately Gaussian with (conditional) mean $b(t, X(t))\Delta t$ and covariance $a(t, X(t))\Delta t$.

Write $a(s,x) = \sigma^2(s,x)$ (or $a(s,x) = \sigma(s,x)\sigma(s,x)^T$ if $d > 1$). Itô (1946, 1951) exploits the Gaussian observation writing

$$\begin{aligned} X(t) - X(0) &= \sum (X(t_{i+1}) - X(t_i)) \\ &\approx \sum \sigma(t_i, X(t_i))(W(t_{i+1}) - W(t_i)) + b(t_i, X(t_i)(t_{i+1} - t_i) \end{aligned}$$

and formulating the equation

$$X(t) = X(0) + \int_0^t \sigma(s, X(s))dW(s) + \int_0^t b(s, X(s))ds \qquad (2)$$

based on the stochastic integral introduced in Itô (1944).

An aside on quadratic variations and continuous martingales

If M is a continuous (local) martingale, then

$$[M]_t = \lim \sum (M(t_{i+1}) - M(t_i))^2$$

exists and defining θ by $\theta(t) = \inf\{s : [M]_s \geq t\}$, $\widetilde{W}(t) = M(\theta(t))$ is a standard Brownian motion and

$$M(t) = \widetilde{W}([M]_t).$$

An Itô integral $M(t) = \int_0^t Y(s)dW(s)$ is a local martingale with quadratic variation $\int_0^t Y(s)^2 ds$, so

$$\int_0^t Y(s)dW(s) = \widetilde{W}(\int_0^t Y(s)^2 ds).$$

In modern terminology, a solution of the time-change equation is a *weak solution* of the corresponding Itô equation.

Doeblin's change of variable

If X is a Markov process in $\mathbb{R}$ and $\varphi(x,t)$ is (strictly) increasing in x, then $Y(t)=\varphi(X(t),t)$ is Markov.

Specifically, Doeblin shows

$$\lim_{t\to s}\frac{1}{t-s}\int_{|z-Y(s)|\leq 1}(z-Y(s))Q(s,Y(s),t,dz)$$
$$=\varphi'(X(s),s)b(s,X(s))+\frac{1}{2}\varphi''(X(s),s)a(s,X(s))+\partial_s\varphi(X(s),s)$$
$$\lim_{t\to s}\frac{1}{t-s}\int_{|z-Y(s)|\leq 1}(z-Y(s))^2Q(s,Y(s),t,dz)$$
$$=\varphi'(X(s),s)^2a(s,X(s))$$

Calculation by Itô's formula

Of course, the above conclusion follows immediately from Itô's formula

$$\varphi(X(t),t) = \varphi(X(s),s) + \int_s^t \varphi'(X(r),r)\sigma(X(r),r)dW(r) \\ + \int_s^t L_r\varphi(X(r),r) + \partial_r\varphi(X(r),r))dr$$

Doeblin's analysis gives

$$\varphi(X(t),t) = \varphi(X(s),s) + \widetilde{W}_\varphi(\int_s^t \varphi'(X(r),r)^2 a(X(r),r)) \\ + \int_s^t L_r\varphi(X(r),r) + \partial_r\varphi(X(r),r))dr$$

where $\widetilde{W}_\varphi$ is a standard Brownian motion depending on φ.

Martingale properties of diffusions

In effect, Doeblin's argument implies

$$M_\varphi(t) = \varphi(X(t),t) - \varphi(X(s),s) - \int_s^t (L_r\varphi(X(r),r) + \partial_r\varphi(X(r),r))dr$$

is a martingale and that M_φ can be represented as

$$M_\varphi(t) = \widetilde{W}(\int_s^t \varphi'(X(r),r)^2 a(r,X(r))dr).$$

This martingale property plays a central role in the general semigroup approach to Markov processes (see Dynkin (1965)) and ultimately led to the development by Stroock and Varadhan of the *martingale problem* (see Stroock and Varadhan (1979), Ethier and Kurtz (1986)).

The time-change equation for Markov chain

Consider a continuous-time Markov chain in $\mathbb{Z}^d$ satsifying

$$P\{X(t+\Delta t) = X(t) + l | X(t) = k\} \approx \beta_k(l)\Delta t.$$

Then $X(t) = X(0) + \sum_l l N_l(t)$, where $N_l(t)$ is the number of jumps of l at or before time t. N_l is a counting process with intensity $\beta_l(X(t))$, that is,

$$N_l(t) - \int_0^t \beta_l(X(s))ds$$

is a martingale. Consequently, we can write

$$N_l(t) = Y_l(\int_0^t \beta_l(X(s))ds),$$

where the Y_l are independent, unit Poisson processes, and

$$X(t) = X(0) + \sum_l l Y_l(\int_0^t \beta_l(X(s))ds).$$

Diffusion approximations

Suppose X_n is a Markov chain in $\frac{1}{\sqrt{n}}\mathbb{Z}^d$ with intensities of the form $n\beta_l^n(s, X_n(s))$. Then

$$\begin{aligned} X_n(t) &= X_n(0) + \frac{1}{\sqrt{n}}\sum_l l Y_l(n\int_0^t \beta_l^n(s, X_n(s))ds) \\ &= X_n(0) + \frac{1}{\sqrt{n}}\sum_l l \widetilde{Y}_l(n\int_0^t \beta_l^n(s, X_n(s))ds) + \int_0^t b_n(s, X_n(s))ds \\ &= X_n(0) + \sum_l l W_l^n(\int_0^t \beta_l^n(s, X_n(s))ds) + \int_0^t b_n(s, X_n(s))ds, \end{aligned}$$

where

$$W_l^n(u) = \frac{Y_l(nu) - nu}{\sqrt{n}}, \quad b_n(s,x) = \sum_l \frac{1}{\sqrt{n}} l\beta_l^n(s,x).$$

Limiting equation

Assume $\beta_l^n \to \beta_l$ and $F_n \to F$. Then (probably), $X_n \Rightarrow X$ satisfying

$$X(t) = X(0) + \sum_l lW_l(\int_0^t \beta_l(s, X(s))ds) + \int_0^t b(s, X(s))ds,$$

and (probably) X is a diffusion with $a(s,x) = \sum \beta_l(s,x)ll^T$.

See Kurtz (1977/78) and Ethier and Kurtz (1986), Chapter 11.

Note that

$$\tau_0(t) = \int_0^t b(s, X(s))ds, \quad \tau_l(t) = \int_0^t \beta_l(s, X(s))ds$$

satisfies a system of random differential equations

$$\begin{aligned} \dot{\tau}_0(t) &= b(t, \Gamma(\tau(t))) \\ \dot{\tau}_l(t) &= \beta_l(t, \Gamma(\tau(t))), \end{aligned}$$

where $\Gamma(\tau) = X(0) + \sum_l lW_l(\tau_l) + \tau_0$.

Convergence of stochastic differential equations

$$\begin{aligned} X_n(t) &= X_n(0) + \frac{1}{\sqrt{n}} \sum_l l Y_l(n \int_0^t \beta_l^n(s, X_n(s))ds) \\ &= X_n(0) + \frac{1}{\sqrt{n}} \sum_l l \widetilde{Y}_l(n \int_0^t \beta_l^n(s, X_n(s))ds) + \int_0^t b_n(s, X_n(s))ds \\ &= X_n(0) + \sum_l l \int_0^t \sqrt{\beta_l^n(s-, X_n(s-))} dM_l^n(s) + \int_0^t b_n(s, X_n(s))ds, \end{aligned}$$

where

$$M_l^n(t) = \int_0^t \frac{1}{\sqrt{\beta_l^n(r-, X_n(r-))}} \frac{1}{\sqrt{n}} d\widetilde{Y}(n \int_0^r \beta_l^n(s, X_n(s))ds).$$

Limiting equation

M_l^n is a local martingale with

$$[M_l^n]_t = \int_0^t \frac{1}{n\beta_l^n(r, X_n(r))} dY(n\int_0^r \beta_l^n(s, X_n(s))ds) \to t$$

(probably) and

$$[M_l^n, M_{l'}^n]_t = 0$$

which implies the M_l^n converge to independent standard Brownian motions. Then (probably) $X_n \Rightarrow X$ satisfying

$$X(t) = X(0) + \sum_l \int_0^t \sqrt{\beta_l(s, X(s))} dW_l(s) + \int_0^t b(s, X(s))ds.$$

See Słomiński (1989) and Kurtz and Protter (1991).

Convergence based on the martingale problem

Let

$$\begin{aligned} L_s^n f(x) &= \sum_l n\beta_l^n(s,x)(f(x+\frac{1}{\sqrt{n}}l)-f(x)) \\ &= \sum_l n\beta_l^n(s,x)(f(x+\frac{1}{\sqrt{n}}l)-f(x)-\frac{1}{\sqrt{n}}l\cdot\partial f(x))+b_n(s,x)\cdot\partial f(x)\,. \end{aligned}$$

Then $f(X_n(t))-f(X_n(0))-\int_0^t L_s^n f(s,X_n(s))ds$ is a martingale.

$$L_s^n f(x) \rightarrow L_s f(x) = \frac{1}{2}\sum_l \beta_l(s,x) l^T \partial^2 f(x) l + b(s,x)\cdot\partial f(x)$$

and (probably) $X_n \Rightarrow X$ where

$$f(X(t))-f(X(0))-\int_0^t L_s f(X(s))ds$$

is a martingale.

Characterizing diffusions by time-change equations

How general is

$$X(t) = X(0) + \sum_l l W_l(\int_0^t \beta_l(s, X(s))ds) + \int_0^t b(s, X(s))ds \,?$$

The drift is completely general, but the diffusion matrix must be representable as

$$a(s, x) = \sum_l \beta_l(s, x) l l^T.$$

We want the sum to be finite or at least countable.

Uniformly positive definite matrices

The collection of nonnegative definite matrices is convex and the extreme points are matrices of the form zz^T.

By a result of Motzkin and Wasow (1953) (see Kurtz (1980), Lemma 4.8), for $\epsilon > 0$ there exists a finite set $\{z_k\} \subset \mathbb{R}^d$ such that for all A satisfying

$$\epsilon|x|^2 \leq x^T Ax \leq \epsilon^{-1}|x|^2$$

there exist $\beta_k(A)$ such that

$$A = \sum_{k=1}^{m} \beta_k(A) z_k z_k^T.$$

Multiple time-change equations Kurtz (1980)

Given independent Markov processes Y_k, require

$$X_k(t) = Y_k\left(\int_0^t \beta_k(s, X(s))ds\right). \qquad (3)$$

Set $\tau_k(t) = \int_0^t \beta_k(s, X(s))ds$, and for $\alpha \in [0, \infty)^\infty$, define

$$\mathcal{F}_\alpha^Y = \sigma(Y_k(s_k) : s_k \leq \alpha_k, k = 1, 2, \ldots)$$

and

$$\mathcal{F}_\alpha^X = \sigma(\{\tau_1(t) \leq s_1, \tau_2(t) \leq s_2, \ldots\} : s_i \leq \alpha_i, i = 1, 2, \ldots, t \geq 0).$$

Compatibility conditions

Assume Y_k is the unique solution of the martingale problem for (A_k, ν_k^0).

Define $\mathcal{D}(H_k) = \{f_k \in \mathcal{D}(A_k) : \inf_y f_k > 0\}$ and $H_k f_k = A_k f_k / f_k$. Then

$$M_{f_1,\ldots,f_k}(\alpha) = \prod_{i=1}^{k} f_i(Y_i(\alpha)) \exp\{-\int_0^{\alpha_i} H_i f_i(Y_i(s))ds\}$$

is a martingale with respect to the filtration $\{\mathcal{F}_\alpha^Y\}$.

Compatibility (see Kurtz (2007), Example 3.20) is equivalent to the requirement that $M_{f_1,\ldots,f_k}$ be a martingale with respect to $\{\mathcal{F}_\alpha^X \vee \mathcal{F}_\alpha^Y\}$ for all k and all $f_i \in \mathcal{D}(H_i)$.

$\tau(t) = (\tau_1(t), \tau_2(t), \ldots)$ is a stopping time with respect to $\{\mathcal{F}_\alpha^X \vee \mathcal{F}_\alpha^Y\}$ in the sense that

$$\{\tau(t) \leq \alpha\} = \{\tau_1(t) \leq \alpha_1, \tau_2(t) \leq \alpha_2, \ldots\} \in \mathcal{F}_\alpha^X \vee \mathcal{F}_\alpha^Y, \quad \alpha \in [0, \infty)^\infty.$$

Notes

1. Doeblin (2000) contains the material from the sealed envelope (the *Pli cacheté)* submitted to the French Academy of Sciences by Doeblin. Bru and Yor (2002) contains extensive material about Doeblin and his work on diffusions as well as his earlier published work and includes an English translation of a portion of the Pli cacheté.

2. The picture of Wolfgang Doeblin is taken from the documentary *Wolfgang Doeblin: A Mathematician Rediscovered.*

   \http://www.wolfgang-doeblin-video.org/

Abstract

Diffusions and diffusion approximations revisited

Abstract: Wolfgang Doeblin was already an established mathematician when, while serving in the French army at the age of 25, he committed suicide after his company was surrounded by German troops during the initial invasion of France. 60 years later probabilists were amazed to learn that shortly prior to his death, he had mailed a sealed envelope to the French Academy that contained a manuscript that in many ways anticipated the work of Ito that has been the foundation of the explosive development of the theory of diffusion processes for the last several decades. Doeblin's approach and its relationship to Ito's will be described and connections of his approach to diffusion approximations for continuous time Markov chains will be illustrated through examples.

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