Poisson representations of measure-valued processes

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Poisson random measures

S a Polish space and ν be a σ-finite measure on $\mathcal{B}(S)$.

ξ is a *Poisson random measure* with mean measure ν if

a) ξ is a random counting measure on S.

b) For each $A \in \mathcal{S}$ with $\nu(A) < \infty$, $\xi(A)$ is Poisson distributed with parameter $\nu(A)$.

c) For $A_1, A_2, \ldots \in \mathcal{S}$ disjoint, $\xi(A_1), \xi(A_2), \ldots$ are independent.

Lemma 1 *If $H : S \to S_0$, Borel measurable, and $\widehat{\xi}(A) = \xi(H^{-1}(A))$, then $\widehat{\xi}$ is a Poisson random measure on S_0 with mean measure $\widehat{\nu}$ given by $\widehat{\nu}(A) = \nu(H^{-1}(A))$*

Moment identities

If ξ is a Poisson random measure with mean measure ν

$$E[e^{\int f(z)\xi(dz)}] = e^{\int (e^f - 1)d\nu},$$

or letting $\xi = \sum_i \delta_{Z_i}$,

$$E[\prod_i g(Z_i)] = e^{\int (g-1)d\nu}.$$

Similarly,

$$E[\sum_j h(Z_j) \prod_i g(Z_i)] = \int hgd\nu e^{\int (g-1)d\nu},$$

and

$$E[\sum_{i\neq j} h(Z_i)h(Z_j) \prod_k g(Z_k)] = (\int hgd\nu)^2 e^{\int (g-1)d\nu},$$

Conditionally Poisson systems

Let ξ be a random counting measure on S and Ξ be a locally finite random measure on S.

ξ is *conditionally Poisson* with Cox measure Ξ if, conditioned on Ξ, ξ is a Poisson point process with mean measure Ξ.

$$E[e^{-\int_S f d\xi}] = E[e^{-\int_S (1-e^{-f})d\Xi}]$$

for all nonnegative $f \in M(S)$.

If ξ is conditionally Poisson system on $S \times [0,\infty)$ with Cox measure $\Xi \times m$ where m is Lebesgue measure, then for $f \in M(S)$

$$E[e^{-\int_{S\times[0,K]} f d\xi}] = E[e^{-K\int_S (1-e^{-f})d\Xi}]$$

and for $f \geq 0$,

$$\Xi(f) = \lim_{K\to\infty} \frac{1}{K} \int_{S\times[0,K]} f d\xi = \lim_{\epsilon\to 0} \epsilon \int_{S\times[0,\infty)} e^{-\epsilon u} f(x)\xi(dx \times du) \quad a.s.$$

Relationship to exchangeability

Lemma 2 *Suppose* ξ *is a conditionally Poisson random measure on* $S \times [0,\infty)$ *with Cox measure* $\Xi \times m$*. If* $\Xi < \infty$ *a.s., then we can write* $\xi = \sum_{i=1}^{\infty} \delta_{(X_i,U_i)}$ *with* $U_1 < U_2 < \cdots$ *a.s. and* $\{X_i\}$ *is exchangeable.*

Conditioned on Ξ*,* $\{U_i\}$ *is Poisson with parameter* $\Xi(S)$ *and* $\{X_i\}$ *is iid with distribution* $\Xi(S)^{-1}\Xi$*.*

Convergence

Lemma 3 *If $\{\xi_n\}$ is a sequence of conditionally Poisson random measures on $S \times [0,\infty)$ with Cox measures $\{\Xi_n \times m\}$. Then $\xi_n \Rightarrow \xi$ if and only if $\Xi_n \Rightarrow \Xi$, Ξ the Cox measure for ξ.*

If $\xi_n \to \xi$ in probability, then $\Xi_n \to \Xi$ in probability

A population model

Consider a process with state space $E = \cup_n [0,r]^n$.

$0 \leq g \leq 1$ and $f(u,n) = \prod_{i=1}^n g(u_i)$

For $a > 0$, and $-\infty < b \leq ra$, define

$$Af(u,n) = f(u,n) \sum_{i=1}^n 2a \int_{u_i}^r (g(v)-1)dv + f(u,n) \sum_{i=1}^n (au_i^2 - bu_i) \frac{g'(u_i)}{g(u_i)}.$$

In other words, particle levels satisfy

$$\dot{U}_i(t) = aU_i^2(t) - bU_i(t),$$

and a particle with level z gives birth at rate $2a(r-z)$ to a particle whose initial level is uniformly distributed between z and r.

$N(t) = \#\{i : U_i(t) < r\}$

$\alpha(n, du)$ the joint distribution of n iid uniform $[0,r]$ random variables.

A calculation

As before, $\widehat{f}(n) = \int f(u,n)\alpha(n,du) = e^{-\lambda_g n}$, $e^{-\lambda_g} = \frac{1}{r}\int_0^r g(u)du$

To calculate $\int Afu,n)\alpha(n,du)$, observe that

$$r^{-1}2a\int_0^r g(z)\int_z^r (g(v)-1)dv = are^{-2\lambda_g} - 2ar^{-1}\int_0^r g(z)(r-z)dz$$

and

$$\begin{aligned} r^{-1}\int_0^r (az^2-bz)g'(z)dz &= -r^{-1}\int_0^r (2az-b)(g(z)-1)dz \\ &= -2ar^{-1}\int_0^r zg(z)dz + ar + b(e^{-\lambda_g}-1). \end{aligned}$$

Then

$$\begin{aligned} \int Af(u,n)\alpha(n,du) &= ne^{-\lambda_g(n-1)}\left(are^{-2\lambda_g} - 2are^{-\lambda_g} + ar + b(e^{-\lambda_g}-1)\right) \\ &= C\widehat{f}(n), \end{aligned}$$

where

$$C\widehat{f}(n) = arn(\widehat{f}(n+1) - \widehat{f}(n)) + (ar-b)n(\widehat{f}(n-1) - \widehat{f}(n)).$$

Conclusion

Let $\widetilde{N}$ be a solution of the martingale problem for

$$C\widehat{f}(n) = arn(\widehat{f}(n+1) - \widehat{f}(n)) + (ar-b)n(\widehat{f}(n-1) - \widehat{f}(n)),$$

that is, $\widetilde{N}$ is a branching process with birth rate ar and death rate $(ar-b)$.

Then there exists a solution $(U_1(t), \ldots, U_{N(t)}(t), N(t))$ of the martingale problem for A such that N has the same distribution as $\widetilde{N}$.

The limit as $r \to \infty$

If $n = O(r)$, then the scaling is correct for the Feller diffusion.

A converges for every g such that $0 \le g \le 1$, $g(z) = 1$, $z \ge u_g$.

$f(u) = \prod_i g(u_i)$

$$Af(u) = f(u)\sum_i 2a\int_{u_i}^{u_g}(g(v)-1)dv + f(u)\sum_i(au_i^2 - bu_i)\frac{g'(u_i)}{g(u_i)}.$$

$\alpha(y, du)$ is the distribution of a Poisson process on $[0,\infty)$ with intensity y.

$$\widehat{f}(y) = \alpha f(y) = \int f(u)\alpha(y,du) = e^{-y\int_0^\infty(1-g(z))dz} = e^{-y\beta_g}$$

and

$$\begin{aligned}\alpha Af(y) &= e^{-y\beta_g}\left(2ay\int_0^\infty g(z)\int_z^\infty(g(v)-1)dvdz + y\int_0^\infty(az^2-bz)g'(z)dz\right)\\ &= e^{-y\beta_g}(ay\beta_g^2 - by\beta_g)\\ &= ay\widehat{f}''(y) + by\widehat{f}'(y)\end{aligned}$$

Particle representation of Feller diffusion

Let $\{U_i(0)\}$ be a conditionally Poisson process on $[0,\infty)$ with (conditional) intensity $Y(0)$. Then, $\{U_i(t)\}$ is conditionally Poisson with intensity $Y(t)$,

$$Y(t) = \lim_{r\to\infty} \frac{1}{r}\#\{i : U_i(t) \le r\},$$

and Y is a Feller diffusion with generator $Cf(y) = ayf''(y) + byf'(y)$

$\gamma : \mathcal{N}(\mathbb{R}) \to [0,\infty)$

$$\gamma(u) = \lim_{r\to\infty} \frac{1}{r}\#\{i : u_i \le r\}$$

$\alpha(y, du)$ Poisson process distribution on $[0,\infty)$ with intensity y. $\alpha(y, \gamma^{-1}(y)) = 1$.

Extinction

Assume $U_1(0) < U_2(0) < \cdots$. Then for all t, all levels are above

$$U_1(t) = \frac{U_1(0)e^{-bt}}{1 - \frac{a}{b}U_1(0)(1 - e^{-bt})}$$

Let $\tau = \inf\{t : Y(t) = 0\}$

$P\{\tau > t\} = P\{U_1(0) < \left[(1 - e^{-bt})a/b\right]^{-1}\} = 1 - e^{-yb/[(1-e^{-bt})a]}$

If $b \leq 0$, conditioning on nonextinction for all t is equivalent to setting $U_1(0) = 0$. The generator becomes

$$Af(u) = f(u)\sum_i 2a\int_{u_i}^{u_g}(g(v) - 1)dv + f(u)\sum_i(au_i^2 - bu_i)\frac{g'(u_i)}{g(u_i)} + f(u)2a\int_0^{u_g}(g(v) - 1)dv$$

and

$$\alpha Af(y) = ay\widehat{f}''(y) + (2a + by)\widehat{f}'(y)$$

Branching Markov processes

$f(x,u,n)=\prod_{i=1}^n g(x_i,u_i)$, where $g:E\times[0,\infty)\to(0,1]$

As a function of x, g is in the domain $\mathcal{D}(B)$ of the generator of a Markov process in E, g is continuously differentiable in u, and $g(x,u)=1$ for $u\geq r$.

$$Af(x,u,n)=f(x,u,n)\sum_{i=1}^n\frac{Bg(x_i,u_i)}{g(x_i,u_i)}+f(x,u,n)\sum_{i=1}^n 2a(x_i)\int_{u_i}^r(g(x_i,v)-1)dv$$
$$+f(x,u,n)\sum_{i=1}^n(a(x_i)u_i^2-b(x_i)u_i)\frac{\partial_{u_i}g(x_i,u_i)}{g(x_i,u_i)}$$

Each particle has a location $X_i(t)$ in E and a level $U_i(t)$ in $[0,r]$.

The locations evolve independently as Markov processes with generator B, the levels satisfy

$$\dot{U}_i(t)=a(X_i(t))U_i^2(t)-b(X_i(t))U_i(t)$$

and particles that reach level r die.

Particles give birth at rates $2a(X_i(t))(r-U_i(t))$; the initial location of a new particle is the location of the parent at the time of birth; and the initial level is uniformly distributed on $[U_i(t),r]$.

Generator for $X(t) = (X_1(t), \ldots, X_{N(t)})$

Setting $e^{-\lambda_g(x_i)} = r^{-1}\int_0^r g(x_i, z)dz$ and $\widehat{f}(x,n) = e^{-\sum_{i=1}^n \lambda_g(x_i)}$, and calculating as in the previous example, we have

$$\begin{aligned} C\widehat{f}(x,n) = & \sum_{i=1}^n B_{x_i}\widehat{f}(x,n) + \sum_{i=1}^n a(x_i)r(\widehat{f}((x,x_i),n+1) - \widehat{f}(x,n)) \\ & + \sum_{i=1}^n (a(x_i)r - b(x_i))(\widehat{f}(d(x|x_i),n-1) - \widehat{f}(x,n)), \end{aligned}$$

where B_{x_i} is the generator B applied to $\widehat{f}(x,n)$ as a function of x_i.

Infinite population limit

Letting $r \to \infty$, Af becomes

$$Af(x,u) = f(x,u)\sum_i \frac{Bg(x_i,u_i)}{g(x_i,u_i)} + f(x,u)\sum_i 2a(x_i)\int_{u_i}^{u_g}(g(x_i,v)-1)dv$$
$$+f(x,u)\sum_i (a(x_i)u_i^2 - b(x_i)u_i)\frac{\partial_{u_i} g(x_i,u_i)}{g(x_i,u_i)}$$

Particle locations evolve as independent Markov processes with generator B.

Levels satisfy

$$\dot{U}_i(t) = a(X_i(t))U_i^2(t) - b(X_i(t))U_i(t)$$

A particle with level $U_i(t)$ gives birth to new particles at its location $X_i(t)$ and initial level in the interval $[U_i(t)+c, U_i(t)+d]$ at rate $2a(X_i(t))(d-c)$.

A particle dies when its level hits ∞.

The measure-valued limit

For $\mu \in \mathcal{M}_f(E)$, let $\alpha(\mu, dx \times du)$ be the distribution of a Poisson random measure on $E \times [0,\infty)$ with mean measure $\mu \times m$. Then setting $h(y) = \int_0^\infty (1 - g(y,v))dv$

$$\alpha f(\mu) = \int f(x,u)\alpha(\mu, dx \times du) = \exp\{-\int_E h(y)\mu(dy)\},$$

and

$$\begin{aligned}
\alpha A f(\mu) &= \exp\{-\int_E h(y)\mu(dy)\}\Big[\int_E\int_0^\infty Bg(y,v)dv\mu(dy) \\
&\qquad + \int_E\int_0^\infty 2a(y)g(y,z)\int_z^\infty (g(y,v)-1)dvdz\mu(dy) \\
&\qquad + \int_E\int_0^\infty (a(y)v^2 - b(y)v)\partial_v g(y,v)dv\mu(dy)\Big] \\
&= \exp\{-\int_E h(y)\mu(dy)\}\int_E \left(-Bh(y) + a(y)h(y)^2 - b(y)h(y)\right)\mu(dy)
\end{aligned}$$

It follows that the Cox measure (or more precisely, the E marginal of the Cox measure) corresponding to the particle process at time t, call it $Z(t)$, is a solution of the martingale problem for $\mathbb{A} = \{(\alpha f, \alpha A f) : f \in \mathcal{D}\}$.

Branching processes in random environments

a and b functions of another stochastic process ξ, say an irreducible finite Markov chain with generator Q.

$f(l,u,n) = f_0(l)f_1(u) = f_0(l)\prod_{i=1}^n g(u_i)$,

$$A_r f(l,u,n) = rf_1(u)Qf_0(l) + f(l,u,n)\sum_{i=1}^n 2a(l)\int_{u_i}^r (g(v)-1)dv$$
$$+f(l,u,n)\sum_{i=1}^n (a(l)u_i^2 - \sqrt{r}b(l)u_i)\frac{g'(u_i)}{g(u_i)},$$

which projects to

$$\begin{aligned} C_r\widehat{f}(l,n) &= rQ\widehat{f}(l,n) + a(l)rn(\widehat{f}(l,n+1)-\widehat{f}(n)) \\ &\quad +(ra(l)-\sqrt{r}b(l))n(\widehat{f}(l,n-1)-\widehat{f}(l,n)), \end{aligned}$$

where $\widehat{f}(l,n) = f_0(l)e^{-\lambda_g n}$. The process corresponding to C_r is a branching process in a random environment determined by ξ.

Scaling limit

Writing the process corresponding to A_r as

$$(\xi(rt), X_1(t), \ldots, X_{N_r(t)}, U_1(t), \ldots, U_{N_r(t)})$$

the process corresponding to C_r is $(\xi(rt), N_r(t))$.

Note that the levels satisfy

$$\dot{U}_i(t) = a(\xi(rt))U_i^2(t) - \sqrt{r}b(\xi(rt))U_i(t).$$

Let π be the stationary distribution for Q and assume that $\sum_l \pi(l)b(l) = 0$. Then

$$Z^{(r)}(t) = \sqrt{r}\int_0^t b(\xi(rs))ds$$

converges to a Brownian motion Z with variance parameter

$$\sum_k \sum_l \pi(k)q_{kl}(h_0(l) - h_0(k))^2 = -2\sum_l \pi(l)h_0(l)b(l) \equiv 2\overline{c},$$

where $h_0(l)$ is a solution of $Qh_0(l) = b(l)$. In the limit, the levels will satisfy

$$dU_i(t) = (\overline{a}U_i(t)^2 + \overline{c}U_i(t))dt + \sqrt{2\overline{c}}U_i(t)dW(t), \tag{1}$$

where $\overline{a} = \sum \pi(l)a(l)$.

Note that the limiting levels are all driven by the same Brownian motion.

Limiting generator

$$h_1(l,u,n) = h_0(l) f_1(u,n) \sum_{i=1}^{n} u_i \frac{g'(u_i)}{g(u_i)},$$

Passing to the limit as $r \to \infty$, $A_r(f_1 + \frac{1}{\sqrt{r}h})$ converges to

$$\begin{aligned}
&\widetilde{A} f_1(u,l) \\
&\quad = f_1(u) \sum_i 2a(l) \int_{u_i}^{\infty} (g(v)-1)dv + f_1(u) \sum_i a(l) u_i^2 \frac{g'(u_i)}{g(u_i)} \\
&\qquad - h_0(l) b(l) f_1(u) \sum_{j=1}^{n} \left(\sum_{i \neq j} u_j u_i \frac{g'(u_i) g'(u_j)}{g(u_i) g(u_j)} + \frac{u_j g'(u_j) + u_j^2 g''(u_j)}{g(u_j)} \right).
\end{aligned}$$

An additional perturbation h_2 gives $A_r(f_1 + \frac{1}{\sqrt{r}} h_1 + \frac{1}{r} h_2)$ converging to

$$\begin{aligned}
&A f_1(u) \\
&\quad = f_1(u) \sum_i 2\overline{a} \int_{u_i}^{\infty} (g(v)-1)dv + f_1(u) \sum_i \overline{a} u_i^2 \frac{g'(u_i)}{g(u_i)} \\
&\qquad + \overline{c} f_1(u) \sum_{j=1}^{n} \left(\sum_{i \neq j} u_j u_i \frac{g'(u_i) g'(u_j)}{g(u_i) g(u_j)} + \frac{u_j g'(u_j) + u_j^2 g''(u_j)}{g(u_j)} \right).
\end{aligned}$$

Limiting diffusion

Let

$$\beta_g = \int_0^\infty (1-g(z))dz = \int_0^\infty zg'(z)dz = -\frac{1}{2}\int_0^\infty z^2 g''(z)dz.$$

We have

$$\begin{aligned}
\alpha Af(y) &= e^{-y\beta_g}\Big(2\overline{a}y\int_0^\infty g(z)\int_z^\infty (g(v)-1)dvdz + y\int_0^\infty (\overline{a}z^2+\overline{c}z)g'(z)dz \\
&\qquad + \overline{c}y^2(\int_0^\infty zg'(z)dz)^2 + \overline{c}y\int_0^\infty z^2 g''(z)dz\Big) \\
&= e^{-y\beta_g}((\overline{a}y+\overline{c}y^2)\beta_g^2 - \overline{c}y\beta_g) \\
&= C\widehat{f}(y),
\end{aligned}$$

where

$$C\widehat{f}(y) = (\overline{a}y+\overline{c}y^2)\widehat{f}''(y) + \overline{c}y\widehat{f}'(y),$$

which identifies the diffusion limit for $r^{-1}N_r$.

Exit measures

Let B be the generator of a diffusion and D a bounded domain with all points in ∂D regular.

$$
\begin{aligned}
Af(x,u) &= f(x,u)\sum_i 1_D(x_i)\frac{Bg(x_i,u_i)}{g(x_i,u_i)} \\
&\quad + f(x,u)\sum_i 1_D(x_i)2a(x_i)\int_{u_i}^{u_g}(g(x_i,v)-1)dv \\
&\quad + f(x,u)\sum_i 1_D(x_i)(a(x_i)u_i^2 - b(x_i)u_i)\frac{\partial_{u_i} g(x_i,u_i)}{g(x_i,u_i)}
\end{aligned}
$$

For simplicity assume $\inf_x a(x) > 0$, $b \le 0$, then $\tau = \inf\{t : \sum_i \mathbf{1}_D(X_i(t) > 0\} < \infty$.

Define $\xi = \sum_i \delta_{(X_i(\tau),U_i(\tau))}$ and

$$
Z_D(\Gamma) = \lim_{r\to\infty}\frac{1}{r}\sum_{i:U_i(\tau)\le r}\mathbf{1}_\Gamma(X_i(\tau)),
$$

the *exit measure.*

Nonlinear PDE

Assume $\{(X_i(0), U_i(0))\}$ is a Poisson random measure with mean measure $\delta_x \times m$. Then for $h \geq 0$, bounded and continuous,

$$u(x) = -\log E[e^{-\langle h, Z_D^x \rangle}]$$

solves the differential equation

$$Bu = au^2 - bu$$

$$u = h \quad \text{on } \partial D.$$

$$e^{-\langle h, Z_D^x \rangle} = E[e^{\int \log(1-h/r) 1_{[0,r]} d\xi} | Z_D] = E[e^{\sum_{U_i(\tau) \leq r} \log(1-h(X_i(\tau))/r)} | Z_D].$$

Taking expectations,

$$E[e^{-\langle h, Z_D^x \rangle}] = E\Big[\prod_{U_i(\tau) \leq r} \Big(1 - \frac{h(X_i(\tau))}{r}\Big)\Big]$$

Homogenization

$$
\begin{aligned}
Bu(x) &= a(x, x/\epsilon)u(x)^2 - b(x, x/\epsilon)u(x) \\
u &= h \text{ on } \partial D
\end{aligned}
$$

$a(x, y)$, $b(x, y)$ periodic in y, and

$$
Bg(x) = \frac{1}{2}\sum_{j,k=1}^{d} c_{jk}(x)\frac{\partial^2 g}{\partial x_j \partial x_k}(x) + \sum_{j=1}^{d} d_j(x)\frac{\partial g}{\partial x_j}(x).
$$

Particle generator:

$$
\begin{aligned}
A^\epsilon f(x,u) &= f(x,u)\sum_i \frac{Bg(x_i,u_i)}{g(x_i,u_i)} 1_D(x_i) \\
&\quad + f(x,u)\sum_i 2a(x_i, x_i/\epsilon)\int_{u_i}^{u_g} (g(x_i,v)-1)dv 1_D(x_i) \\
&\quad + f(x,u)\sum_i (a(x_i,x_i/\epsilon)u_i^2 - b(x_i,x_i/\epsilon)u_i)\frac{\partial_{u_i} g(x_i,u_i)}{g(x_i,u_i)} 1_D(x_i)
\end{aligned}
$$

Effective coefficients

Let $\overline{a}(x) = \int a(x,y)\pi_x(dy)$, $\overline{b}(x) = \int b(x,y)\pi_x(dy)$, where π_x is determined by

$$\int \widehat{B}g(x,y)\pi_x(dy) = 0, \quad g \in \mathbb{R}^d,$$

where

$$\widehat{B}g(x,y) = \frac{1}{2}\sum_{j,k=1}^{d} c_{jk}(x)\frac{\partial^2 g}{\partial y_j \partial y_k}(y).$$

$$\lim_{\epsilon\to 0} E[e^{-\langle Z_D^{\epsilon,x},h\rangle}] = \lim_{\epsilon\to 0} E\Big[\prod_{U_i^\epsilon(\tau)\le r}\Big(1-\frac{h(X_i(\tau))}{r}\Big)\Big] = E[e^{-\langle \overline{Z}_D^x,h\rangle}]$$

Conditioning on non-extinction

Let a be constant and $b \equiv 0$. and let $U_*(0)$ be the minimum of the initial levels.

$$U_*(t) = \frac{U_*(0)}{1 - aU_*(0)t}.$$

Let τ be the time of extinction, Then $\{\tau > T\} = \{U_*(0) < \frac{1}{aT}\}$. Conditioning on $\{\tau > T\}$ and letting $T \to \infty$ is equivalent to conditioning on the initial Poisson process having a level at zero. The resulting generator becomes

$$\begin{aligned} Af(x,u) = f(x,u)\sum_i \frac{Bg(x_i,u_i)}{g(x_i,u_i)} + f(x,u)\sum_{i>0} 2a\int_{u_i}^{r_g}(g(x_i,v)-1)dv \\ +f(x,u)2a\int_0^{r_g}(g(x_0,v)-1)dv \\ +f(x,u)\sum_{i>0} au_i^2\frac{\partial_{u_i}g(x_i,u_i)}{g(x_i,u_i)}, \end{aligned} \tag{2}$$

Generator for measure-valued process

The generator for the measure-valued process is given by setting

$$\alpha_0 f(\mu) = \int f(x,u)\alpha_0(\mu, dx \times du) = \frac{1}{|\mu|}\int_E g(z,0)\mu(dz)\exp\{-\langle h,\mu\rangle\},$$

and

$$\begin{aligned}\alpha_0 Af(\mu) &= \langle -Bh(y) + ah(y)^2, \mu\rangle \frac{1}{|\mu|}\int_E g(z,0)\mu(dz)\exp\{-\langle h,\mu\rangle\} \\ &\quad + \frac{1}{|\mu|}\int_E (Bg(z,0) - 2ag(z,0)h(z))\mu(dz)\exp\{-\langle h,\mu\rangle\}\end{aligned}$$

Markov mappings

Theorem 4 *$A \subset \overline{C}(E) \times \overline{C}(E)$ a pre-generator with bp-separable graph.*

$\mathcal{D}(A)$ closed under multiplication and separating.

$\gamma : E \to E_0$, Borel measurable.

α a transition function from E_0 into E satisfying

$$\alpha(y, \gamma^{-1}(y)) = 1$$

Define

$$C = \{(\int_E f(z)\alpha(\cdot, dz), \int_E Af(z)\alpha(\cdot, dz)) : f \in \mathcal{D}(A)\} .$$

Let $\mu_0 \in \mathcal{P}(E_0)$, $\nu_0 = \int \alpha(y, \cdot)\mu_0(dy)$.

If $\widetilde{Y}$ is a solution of the MGP for (C, μ_0), then there exists a solution Z of the MGP for (A, ν_0) such that $Y = \gamma \circ Z$ and $\widetilde{Y}$ have the same distribution on $M_{E_0}[0, \infty)$.

$$E[f(Z(t))|\mathcal{F}_t^Y] = \int f(z)\alpha(Y(t), dz)$$

(at least for almost every t).

Uniqueness

Corollary 5 *If uniqueness holds for the MGP for* (A, ν_0)*, then uniqueness holds for the* $M_{E_0}[0,\infty)$*-MGP for* (C, μ_0)*. If* $\widetilde{Y}$ *has sample paths in* $D_{E_0}[0,\infty)$*, then uniqueness holds for the* $D_{E_0}[0,\infty)$*-martingale problem for* (C, μ_0)*.*

Abstract

Poisson representations of measure-valued processes

Measure-valued diffusions and measure-valued solutions of stochastic partial differential equations can be represented in terms of the Cox measures of particle systems that are conditionally Poisson at each time t. The representations are useful for characterizing the processes, establishing limit theorems, and analyzing the behavior of the measure-valued processes. Examples will be given and some of the useful methodology will be described.