Particle representations for some SPDEs and measure-valued processes

- Law of large numbers
- ODEs and associated first-order PDEs
- Diffusion models and parabolic PDEs
- Systems with common randomness and SPDEs
- Limit theorems
- McKean-Vlasov equations
- Stock price set by infinitely many competing traders
- Hydrodynamic limit for symmetric simple exclusion model

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Law of large numbers

X *random variable* = numerical measurement in an experiment = measurable function on $(\Omega, \mathcal{F}, P)$

Distribution of X: $\mu_X(\Gamma) = P\{X \in \Gamma\}$

Expectation of X: $E[X] = \int_{\mathbb{R}} x\mu_X(dx)$

$X_1, X_2, \ldots$ *independent* with distribution μ_X = measurements from repeated trials of the experiment

$$P\{X_1 \in \Gamma_1, \ldots, X_m \in \Gamma_m\} = \prod_{i=1}^{m} \mu_X(\Gamma_i)$$

Theorem 1 *(Law of large numbers) If* $X_1, X_2, \ldots$ *are independent with distribution* μ_X*, then*

$$\lim_{n\to\infty} \frac{X_1 + \cdots + X_n}{n} = E[X]$$

with probability one.

Transformations of random variables

$X_1, X_2, \ldots$ independent and identically distributed (iid)

Then for Borel measurable g, $g(X_1), g(X_2), \ldots$ iid and

$$\lim_{n\to\infty} \frac{g(X_1) + \cdots + g(X_n)}{n} = E[g(X)] = \int g(x)\mu_X(dx) = \langle \mu_X, g \rangle$$

almost surely.

ODEs and associated first-order PDEs

For $b : \mathbb{R}^d \to \mathbb{R}^d$, consider $\dot{X}_i = b(X_i)$ or in integrated form

$$X_i(t) = X_i(0) + \int_0^t b(X_i(s))ds,$$

where $X_1(0), X_2(0), \ldots$ are iid. Then

$$\varphi(X_i(t)) = \varphi(X_i(0)) + \int_0^t b(X_i(s)) \cdot \nabla\varphi(X_i(s))ds \ .$$

Define

$$V(t, \Gamma) = \mu_{X_i(t)}(\Gamma) = P\{X_i(t) \in \Gamma\}.$$

Then

$$\langle V(t), \varphi\rangle = \langle V(0), \varphi\rangle + \int_0^t \langle V(s), b \cdot \nabla\varphi\rangle ds,$$

so V is a weak solution of

$$v_t(t, x) = -\sum_{j=1}^d \frac{\partial}{\partial x_j}(b_j(x)v(t, x))$$

Mapping from E^∞ to $\mathcal{P}(E)$

E a Polish space

$\gamma : x = (x_1, x_2, \ldots) \in E^\infty \to \mu \in \mathcal{P}(E)$ if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} g(x_i) = \int_E g(x)\mu(dx), \quad g \in C_b(E)$$

otherwise $\gamma(x) = \mu_0$

e.g. $V(t) = \gamma(\{X_i(t)\})$ a.s.

$\{X_i\}$ is a representation of V

Brownian motion

Central limit theorem: $\xi_1, \xi_2, \ldots$ independent, identically distributed with $E[\xi_i] = 0$ and $Var(\xi_i) = 1$. Then

$$P\{\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i \leq z\} \to \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

Functional version:

$$W_n(t) \equiv \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \xi_i \text{ then } W_n \Rightarrow W$$

Properties

W has *independent increments*, that is, for $0 = t_0 < t_1 < \cdots < t_k$

$$W(t_i) - W(t_{i-1}), \quad i = 1, \ldots, k$$

are independent

$$[W]_t \equiv \lim_{\max t_i - t_{i-1} \to 0} \sum (W(t_i) - W(t_{i-1}))^2 = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{[nt]} \xi_i^2 = t$$

Stochastic integration

If $Y(t)$ is independent of $\{W(t+s)-W(t): s \geq 0\}$, $t \geq 0$, then

$$Z(t) = \int_0^t Y(s)dW(s) = \lim \sum Y(t_i)(W(t_{i+1}) - W(t_i))$$

and (typically)

$$E[Z(t)] = 0 \quad \text{and} \quad [Z]_t = \int_0^t Y^2(s)ds$$

$$Var(Z(t)) = E[Z^2(t)] = E[[Z]_t] = \int_0^t E[Y^2(s)]ds$$

We also have

$$\int_0^t U(s)dZ(s) = \int_0^t U(s)Y(s)dW(s)$$

Diffusion models and parabolic PDEs

$$X_i(t) = X_i(0) + \int_0^t b(X_i(s))ds + \int_0^t \sigma(X_i(s))dW_i(s)$$

Think

$$X_i(t+\Delta t) \approx X_i(t) + b(X_i(t))\Delta t + \sigma(X_i(t))(W_i(t+\Delta t) - W_i(t))$$

Then

$$\begin{aligned}\varphi(X_i(t)) = \varphi(X_i(0)) + \int_0^t b(X_i(s))\varphi'(X_i(s))ds + \int_0^t \frac{1}{2}\sigma^2(X_i(s))\varphi''(X_i(s))ds \\ + \int_0^t \sigma(X_i(s))\varphi'(X_i(s))dW_i(s)\end{aligned}$$

$$L\varphi(x) = \frac{1}{2}\sigma^2(x)\varphi''(x) + b(x)\varphi'(x) \qquad \langle V(t), \varphi\rangle = \int \varphi(x)V(t,dx)$$

$$\langle V(t), \varphi\rangle = \langle V(0), \varphi\rangle + \int_0^t \langle V(s), L\varphi\rangle ds$$

Weak solution of PDE

V is a weak solution of

$$v_t = L^* v = \frac{1}{2}\frac{\partial^2}{\partial x^2}(\sigma^2(x)v(t,x)) - \sum_{j=1}^{d}\frac{\partial}{\partial x_j}(b_j(x)v(t,x))$$

Gaussian white noise integral

ν a Borel measure on (U, r_U)

$M(A\times[0,t])$ normal with $E[M(A\times[0,t])]=0$ and $Var(M(A\times[0,t]))=\nu(A)t$ and

$$E[M(A\times[0,t])M(B\times[0,s])]=\nu(A\cap B)t\wedge s$$

Then $M(\cup A_i\times[0,t])=\sum M(A_i\times[0,t])$

$$Z(t)=\int_{U\times[0,t]}Y(u,s)M(du\times ds)$$

satisfies

$$E[Z(t)]=0 \qquad [Z]_t=\int_0^t\int_U Y^2(u,s)\nu(du)ds$$

$$E[Z^2(t)]=E[[Z]_t]=\int_0^t\int_U E[Y^2(u,s)]\nu(du)ds$$

Systems with common randomness and SPDEs

$$X_i(t) = X_i(0) + \int_0^t b(X_i(s))ds + \int_0^t \sigma(X_i(s))dW_i(s) + \int_{U\times[0,t]} \alpha(X_i(s),u)M(du\times ds)$$

$$\begin{aligned}\varphi(X_i(t)) &= \varphi(X_i(0)) + \int_0^t b(X_i(s))\varphi'(X_i(s))ds \\ &+ \int_0^t \frac{1}{2}a(X_i(s))\varphi''(X_i(s))ds \\ &+ \int_0^t \sigma(X_i(s))\varphi'(X_i(s))dW_i(s) \\ &+ \int_{U\times[0,t]} \alpha(X_i(s),u)\varphi'(X_i(s))M(du\times ds)\end{aligned}$$

where $a(x) = \sigma^2(x) + \int_U \alpha^2(x,u)\nu(du)$.

de Finetti's theorem

$X_1, X_2, \ldots$ is *exchangeable* if

$$P\{X_1 \in \Gamma_1, \ldots, X_m \in \Gamma_m\} = P\{X_{s_1} \in \Gamma_1, \ldots, X_{s_m} \in \Gamma_m\}$$

$(s_1, \ldots, s_m)$ any permutation of $(1, \ldots, m)$.

Theorem 2 *(de Finetti) Let $X_1, X_2, \ldots$ be exchangeable. Then there exists a random probability measure μ such that for every bounded, measurable g,*

$$\lim_{n \to \infty} \frac{g(X_1) + \cdots + g(X_n)}{n} = \int g(x)\mu(dx)$$

almost surely.

The SPDE

$$L\varphi(x) = \frac{1}{2}a(x)\varphi''(x) + b(x)\varphi'(x) \qquad \langle V(t), \varphi\rangle = \int \varphi(x)V(t,dx)$$

$$\langle V(t), \varphi\rangle = \langle V(0), \varphi\rangle + \int_0^t \langle V(s), L\varphi\rangle ds + \int_{U\times[0,t]} \langle V(s), \alpha(\cdot,u)\varphi'\rangle M(du\times ds)$$

V is a weak solution of

$$dv(x,t) = L^*v(x,t)dt - \int_U \partial_x(\alpha(x,u)v(x,t))M(du\times dt)$$

Nonlinear system

$$X_i(t) = X_i(0) + \int_0^t b(X_i(s), V(s))ds + \int_0^t \sigma(X_i(s), V(s))dW_i(s)$$
$$+ \int_{U\times[0,t]} \alpha(X_i(s), V(s), u)M(du \times ds)$$

$$V(t, \Gamma) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_\Gamma(X_i(t))$$

that is,

$$V(t) = \gamma(\{X_i(t)\})$$

should exist by exchangeability.

SPDE

$a(x,v) = \sigma^2(x,v) + \int_U \alpha^2(x,v,u)\nu(du).$

$$L(v)\varphi(x) = \frac{1}{2}a(x,v)\varphi''(x) + b(x,v)\varphi'(x) \qquad \langle V(t),\varphi\rangle = \int \varphi(x)V(t,dx)$$

$$\begin{aligned}\langle V(t),\varphi\rangle &= \langle V(0),\varphi\rangle + \int_0^t \langle V(s), L(V(s))\varphi\rangle ds \\ &\quad + \int_{U\times[0,t]} \langle V(s), \alpha(\cdot, V(s), u)\varphi'\rangle M(du\times ds)\end{aligned}$$

Uniqueness

Let ρ_W be the Wasserstein metric on $\mathcal{P}(\mathbb{R}^d)$, that is,

$$\rho_W(\mu,\nu) = \sup_{f\in B_1} |\int f d\mu - \int f d\nu|$$

$$B_1 = \{f : |f(x)| \leq 1, |f(x)-f(y)| \leq |x-y|, x, y \in \mathbb{R}^d\}$$

Theorem 3 *Suppose*

$$\begin{aligned} |\sigma(x,v)-\sigma(x',v')| + |b(x,v)-b(x',v')| \\ +\sqrt{\int_{\mathbb{R}^d} |\alpha(x,v,u)-\alpha(x',v',u)|^2\nu(du)} \\ \leq K(|x-x'| + \rho_W(v,v')). \end{aligned}$$

Then there exists a unique solution of the system.

Uniqueness for the SPDE

$$P\{X_1(t) \in \Gamma | \mathcal{F}_t^V\} = P\{X_1(t) \in \Gamma | \mathcal{F}_t^M\} = V(t, \Gamma)$$

Uniqueness for the system implies uniqueness for the SPDE.

Convergence lemma for processes

Lemma 4 *Let* $X^n = (X_1^n, \ldots, X_{N_n}^n)$ *be exchangeable families of* $D_E[0,\infty)$*-valued random variables such that* $N_n \Rightarrow \infty$ *and* $X^n \Rightarrow X$ *in* $D_E[0,\infty)^\infty$*. Define*

$$\Xi_n = \frac{1}{N_n}\sum_{i=1}^{N_n} \delta_{X_i^n} \in \mathcal{P}(D_E[0,\infty))$$

$$\Xi = \lim_{m\to\infty} \frac{1}{m}\sum_{i=}^{m} \delta_{X_i}$$

$$V_n(t) = \frac{1}{N_n}\sum_{i=1}^{N_n} \delta_{X_i^n(t)} \in \mathcal{P}(E)$$

$$V(t) = \lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^{m} \delta_{X_i(t)}$$

Then

a) For $t_1, \ldots, t_l \notin \{t : E[\Xi\{x : x(t) \neq x(t-)\}] > 0\}$

$$(\Xi_n, V_n(t_1), \ldots, V_n(t_l)) \Rightarrow (\Xi, V(t_1), \ldots, V(t_l)).$$

b) If $X^n \Rightarrow X$ *in* $D_{E^\infty}[0,\infty)$*, then* $V_n \Rightarrow V$ *in* $D_{\mathcal{P}(E)}[0,\infty)$*.*

Classical McKean-Vlasov limit For $i = 1, \ldots, n$

$$X_i^n(t) = X_i^n(0) + \int_0^t \sigma(X_i^n(s), V^n(s))dW_i(s) + \int_0^t b(X_i^n(s), V^n(s))ds$$

where $V^n(t) = \frac{1}{n}\sum_{i=1}^n \delta_{X_i^n(t)}$

Example: $X_i^n(t) = X_i^n(0) + W_i(t) + \frac{1}{n}\sum_{j=1}^{n}\int_0^t b(X_i^n(s) - X_j^n(s))ds$

Under the Lipschitz conditions, $V^n \Rightarrow V$ satisfying

$$\langle V(t), \varphi\rangle = \langle V(0), \varphi\rangle + \int_0^t \langle V(s), L(V(s))\varphi\rangle ds$$

$$L(v)\varphi(x) = \frac{1}{2}\sum a_{ij}(x,v)\frac{\partial^2}{\partial x_i \partial x_j}\varphi(x) + \sum b_i(x,v)\frac{\partial}{\partial x_i}\varphi(x)$$

$$a(x,v) = \sigma(x,v)\sigma^T(x,v)$$

Competing traders

N traders (N even) and $N/2$ shares of stock.

Individual trader's (log) valuation:

$$X_i(t) = X_i(0) + \sigma_1 W_i(t) + \sigma_2 W(t) + \mu t + \int_0^t \nu(S(s) - X_i(s))ds$$

$\theta_i(t)$ is 1 if the ith trader owns the stock and -1 if the ith trader does not own the stock.

$$\begin{aligned}\theta_i(t) &= \theta_i(0)(-1)^{N_i(t)} \\ N_i(t) &= Y_i(\int_0^t \lambda\{\theta_i(s)(S(s) - X_i(s))\}^+ ds).\end{aligned}$$

where the Y_i are independent Poisson processes and $a^+ = a \vee 0$.

Price setting by market maker

Q_N number of shares owned by the market maker

$$Q_N(t) = -\frac{1}{2}\sum_{i=1}^{N}\theta_i(t)$$

$$S(t) = S(0) - \int_0^t \beta \frac{1}{N} Q_N(s) ds$$

Equation for Q_N

Let $V_N^+(t) = \frac{1}{N}\sum_{i=1}^N \mathbf{1}_{\{\theta_i=1\}}\delta_{X_i(t)}$ and $V_N^-(t) = \frac{1}{N}\sum_{i=1}^N \mathbf{1}_{\{\theta_i=-1\}}\delta_{X_i(t)}$.

$$\frac{1}{N}Q_N(t) = \frac{1}{N}Q_N(0) + \frac{1}{N}\sum_{i=1}^N \int_0^t \theta_i(s)\lambda\{\theta_i(s)(S(s)-X_i(s))\}^+ ds$$

$$+\frac{1}{N}\sum_{i=1}^N \int_0^t \theta_i(r-)d\tilde{Y}_i(\int_0^r \lambda\{\theta_i(s)(S(s)-X_i(s))\}^+ ds)$$

$$= \frac{1}{N}Q_N(0) + \frac{1}{N}\sum_{i=1}^N \int_0^t \theta_i(r-)d\tilde{Y}_i(\int_0^r \lambda\{\theta_i(s)(S(s)-X_i(s))\}^+ ds)$$

$$+\int_0^t \lambda \int (S(s)-x)^+ V_N^+(s,dx)ds - \int_0^t \lambda \int (x-S(s))^+ V_N^-(s,dx)ds$$

Limiting model

$$Q(t) = Q(0) + \int_0^t \lambda \int (S(s) - x)^+ V^+(s, dx) ds - \int_0^t \lambda \int (x - S(s))^+ V^-(s, dx) ds$$

$$S(t) = S(0) - \int_0^t \beta Q(s) ds$$

$$\int g(x) V^+(t, dx) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n g(X_i(t)) \mathbf{1}_{\{\theta_i(t)=1\}}$$

$$\int g(x) U(t, dx) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n g(X_i(t)) \theta_i(t)$$

$$U(t) = V^+(t) - V^-(t)$$

SPDE

$$\begin{aligned}\varphi(X_i(t)) &= \varphi(X_i(0)) + \sigma_1 \int_0^t \varphi'(X_i(s))dW_i(s) + \sigma_2 \int_0^t \varphi'(X_i(s))dW(s) \\ &\quad + \int_0^t \frac{1}{2}(\sigma_1^2 + \sigma_2^2)\varphi''(X_i(s))ds \\ &\quad + \int_0^t (\mu + \nu(S(s) - X_i(s)))\varphi'(X_i(s))ds\end{aligned}$$

$$\langle V(t), \varphi\rangle = \langle V(0), \varphi\rangle + \sigma_2 \int_0^t \langle V(s), \varphi'\rangle dW(s) + \int_0^t \langle V(s), L(S(s))\varphi\rangle ds$$

Equation for U

$$
\begin{aligned}
\varphi(X_i(t))\theta_i(t) &= \varphi(X_i(0))\theta_i(0) + \sigma_1 \int_0^t \varphi'(X_i(s))\theta_i(s)dW_i(s) \\
&\quad + \sigma_2 \int_0^t \varphi'(X_i(s))\theta_i(s)dW(s) \\
&\quad + \int_0^t \frac{1}{2}(\sigma_1^2 + \sigma_2^2)\varphi''(X_i(s))\theta_i(s)ds \\
&\quad + \int_0^t (\mu + \nu(S(s) - X_i(s)))\varphi'(X_i(s))\theta_i(s)ds \\
&\quad - \int_0^t 2\varphi(X_i(s))\theta_i(s-)dN_i(s)
\end{aligned}
$$

$$
\begin{aligned}
\langle U(t), \varphi \rangle &= \langle U(0), \varphi \rangle + \sigma_2 \int_0^t \langle U(s), \varphi' \rangle dW(s) \\
&\quad + \int_0^t (\langle U(s), L(S(s))\varphi \rangle + \langle V^+(s), B^+(S(s))\varphi \rangle \\
&\qquad + \langle V^-(s), B^-(S(s))\varphi \rangle) ds
\end{aligned}
$$

where

$$
\begin{aligned}
B^+(s)\varphi(x) &= \lambda(s-x)^+\varphi(x) \\
B^-(s)\varphi(x) &= \lambda(x-s)^+\varphi(x)
\end{aligned}
$$

A "simplified" SPDE

Let $\lambda, \beta \to \infty$. Then

$$X_i(t) = X_i(0) + \sigma_1 W_i(t) + \sigma_2 W(t) + \mu t + \int_0^t \nu(S(s) - X_i(s))ds$$

$$\langle V(t), \varphi \rangle = \langle V(0), \varphi \rangle + \sigma_2 \int_0^t \langle V(s), \varphi' \rangle dW(s) + \int_0^t \langle V(s), L(S(s))\varphi \rangle ds$$

with

$$L(s)\varphi(x) = \frac{1}{2}(\sigma_1^2 + \sigma_2^2)\varphi''(x) + (\mu + \nu(S(s) - x))\varphi'(x)$$

and $S(t)$ is the median of $V(t)$.

Hydrodynamic limit for symmetric simple exclusion model

X_i^n particle location on $E_n = \frac{1}{n}\mathbb{Z}$ mod 1 (at most one particle per location)

$X_i^n \to X_i^n + \frac{k}{n}$ at rate $n^2\lambda(|k|)$ unless new location is already occupied (note symmetry)

Equivalent formulation

Swap contents of $\frac{l}{n}$ and $\frac{k}{n}$ at rate $n^2\lambda(|k-l|)$

$$\begin{aligned} X_i^n(t) &= X_i^n(0) + \frac{1}{n}\sum_{l\neq k}\int_0^t (l-k)\mathbf{1}_{\{X_i^n(s-)=k\}}dY_{kl}(n^2\lambda(|k-l|)s) \\ &= X_i^n(0) + \frac{1}{n}\sum_{l\neq k}\int_0^t (l-k)\mathbf{1}_{\{X_i^n(s-)=k\}}d\tilde{Y}_{kl}(n^2\lambda(|k-l|)s) \end{aligned}$$

$Y_{kl} = Y_{lk}$ independent unit Poisson processes and $\tilde{Y}_{kl}(u) = Y_{kl}(u) - u$.

Convergence

$$[X_i^n]_t = \frac{1}{n^2}\sum_{l\neq k}\int_0^t (l-k)^2 \mathbf{1}_{\{X_i^n(s-)=k\}} dY_{kl}(n^2\lambda(|k-l|)s)$$
$$= \frac{1}{n^2}\sum_{l\neq k}\int_0^t (l-k)^2 \mathbf{1}_{\{X_i^n(s-)=k\}} d\tilde{Y}_{kl}(n^2\lambda(|k-l|)s) + \sigma^2 t$$

Then $[X_i^n]_t \to \sigma^2 t$, $\sigma^2 = \sum_k k^2\lambda(|k|)$, and by the martingale central limit theorem, $X_i^n \Rightarrow X_i$ where $X_i(t) = X_i(0) + \sigma W_i(t)$.

$$[X_i^n, X_j^n]_t = -\frac{1}{n^2}\sum_{l\neq k}\int_0^t (l-k)^2 \mathbf{1}_{\{X_i^n(s-)=k, X_j^n(s-)=l\}} dY_{kl}(n^2\lambda(|k-l|)s)$$

converges to zero which implies the W_i are independent.

Therefore $V_n \Rightarrow V$, where $V(t,\Gamma) = \int_\Gamma v(t,x)dx$ and

$$v_t = \frac{1}{2}\sigma^2\Delta v$$

Basic convergence lemma

Lemma 5 *For $n = 1, 2, \ldots$, let $\{\xi_1^n, \ldots, \xi_{N_n}^n\}$ be exchangeable in S (allowing) $N_n = \infty$.) Let Ξ^n be the empirical measure,*

$$\Xi^n = \frac{1}{N_n} \sum_{i=1}^{N_n} \delta_{\xi_i^n}.$$

Assume $N_n \to \infty$, and for each $m = 1, 2, \ldots$, $\{\xi_1^n, \ldots, \xi_m^n\} \Rightarrow \{\xi_1, \ldots, \xi_m\}$ in S^m.

Then $\{\xi_i\}$ is exchangeable and setting $\xi_i^n = s_0 \in S$ for $i > N_n$, $\{\Xi^n, \xi_1^n, \xi_2^n \ldots\} \Rightarrow \{\Xi, \xi_1, \xi_2, \ldots\}$ in $\mathcal{P}(S) \times S^\infty$, where Ξ is the deFinetti measure for $\{\xi_i\}$.

If for each m, $\{\xi_1^n, \ldots, \xi_m^n\} \to \{\xi_1, \ldots, \xi_m\}$ in probability in S^m, then $\Xi^n \to \Xi$ in probability in $\mathcal{P}(S)$.

Remarks

a) The set $D_\Xi = \{t : E[\Xi\{x : x(t) \neq x(t-)\}] > 0\}$ is at most countable.

b) If for $i \neq j$, with probability one, X_i and X_j have no simultaneous discontinuities, then $D_\Xi = \emptyset$ and convergence of X^n to X in $D_E[0,\infty)^\infty$ implies convergence in $D_{E^\infty}[0,\infty)$.