Fluid–body interactions in elastic anisotropic fluids

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Introduction

Fluid anisotropy is found all across biology, from biofluids like mucus to swarms of active bacteria. A model fluid used to investigate such enviroments is a nematic liquid crystal (LC). Large colloids undergo shape-dependent interactions and body deformations when immersed in such environments.

We use complex variables to analytically solve for interactions in two-dimensional LCs. Allowing the study of body shape/orientation, body deformations, interaction dynamics, and much more.

Mathematical formulation

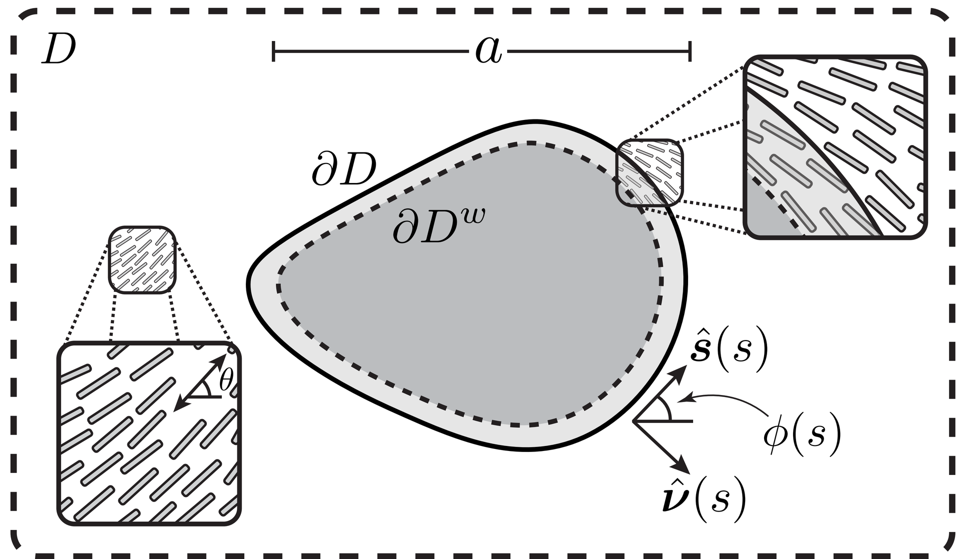
Active nematic LCs are described by a director field $\boldsymbol{n}(\boldsymbol{x},t)$ and fluid velocity $\boldsymbol{u}(\boldsymbol{x},t)$, which satisfy the Ericksen–Leslie equations:

$$\frac{D\boldsymbol{n}}{Dt} = (\boldsymbol{I}-\boldsymbol{n}\boldsymbol{n})\cdot\left(\boldsymbol{n}\cdot\nabla\boldsymbol{u} + \frac{K}{\gamma}\nabla^2\boldsymbol{n}\right) \qquad |\boldsymbol{n}| = 1$$

$$\nabla\cdot[\boldsymbol{\sigma}^e(\boldsymbol{n}) + 2\alpha\boldsymbol{n}\boldsymbol{n} + \boldsymbol{\sigma}^v(\boldsymbol{n},\boldsymbol{u}) - p\boldsymbol{I}] = \boldsymbol{0} \qquad \nabla\cdot\boldsymbol{u} = 0$$

elastic stress — active stress — (anistropic) viscous stress

- K - Frank elastic constant
- W - anchoring strength
- α - activity strength
- a - length scale
- μ - solvent viscosity
- μ_i - anisotropic viscosities
- γ - rotational viscosity



In 2D, with $\gamma, \mu_i \ll \mu$ and $a^2\gamma\alpha \ll \mu K$, the problem reduces to

$$\nabla^2\theta = 0 \text{ and } \mu\nabla^4\psi = \alpha\left[(\partial_{xx}-\partial_{yy})\sin(2\theta) - 2\partial_{xy}\cos(2\theta)\right] \text{ in } D$$

$$K\frac{\partial\theta}{\partial\nu} = \frac{W}{2}\sin\left[2(\theta-\phi)\right] \text{ and } \nabla\psi = 0 \text{ on } \partial D$$

for $\boldsymbol{u} = (\psi_y, -\psi_x)$, $\boldsymbol{n} = (\cos\theta, \sin\theta)$, and tangent angle ϕ.

Effective boundary (eff. bdy.) technique

Finding $\theta(x,y)$ is equivalent to finding a locally analytic $f(z)$ with $\theta(x,y) = -\arg f'(z)$, $z = x+iy$, and

$$\left(|f_s|^2\right)_s + w\,\mathrm{Im}\left[(f_s)^2\right] = 0 \quad \text{on } \partial D$$

For large anchoring strengths ($w := aW/K \gg 1$) one can instead consider an effective domain D^w subject to

$$\mathrm{Im}\, f = \text{const.} + \mathcal{O}(1/w^3) \quad \text{on } \partial D^w$$

where ∂D^w is ∂D displaced by $-\hat{\boldsymbol{\nu}}/w - \hat{\boldsymbol{t}}_s/w^2$.

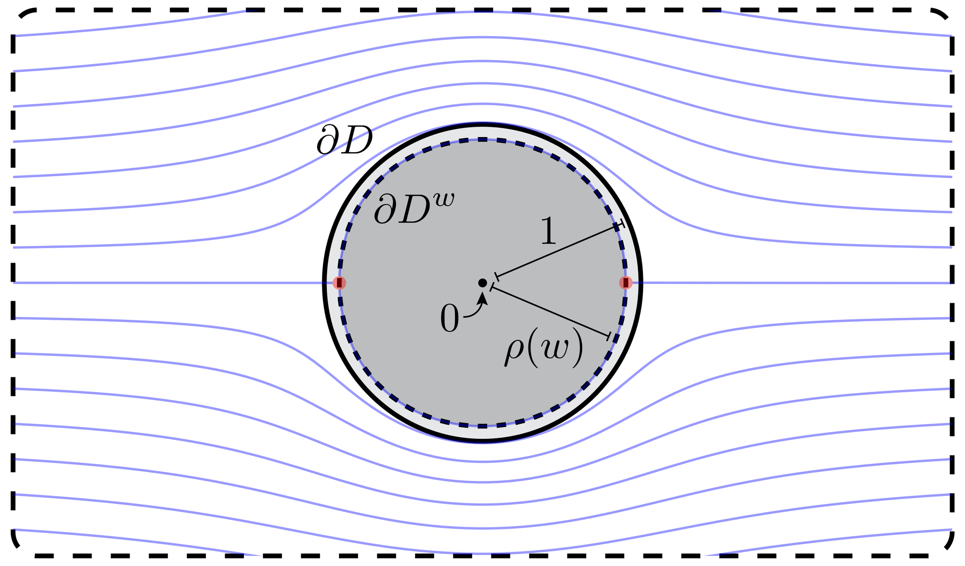
Elastic interactions

A cylinder in a static liquid crystal

For $w \gg 1$, the eff. bdy. is a cylinder of radius

$$\rho(w) \sim 1 - 1/w + 1/w^2$$

The equations yield the director field up to an angle γ that sets the positions of surface defects:



$$\theta(z) = 2\arg z - \arg\left[z - \rho(w)e^{i\gamma}\right] - \arg\left[z + \rho(w)e^{-i\gamma}\right]$$

The free energy can be computed using Cauchy's residue theorem:

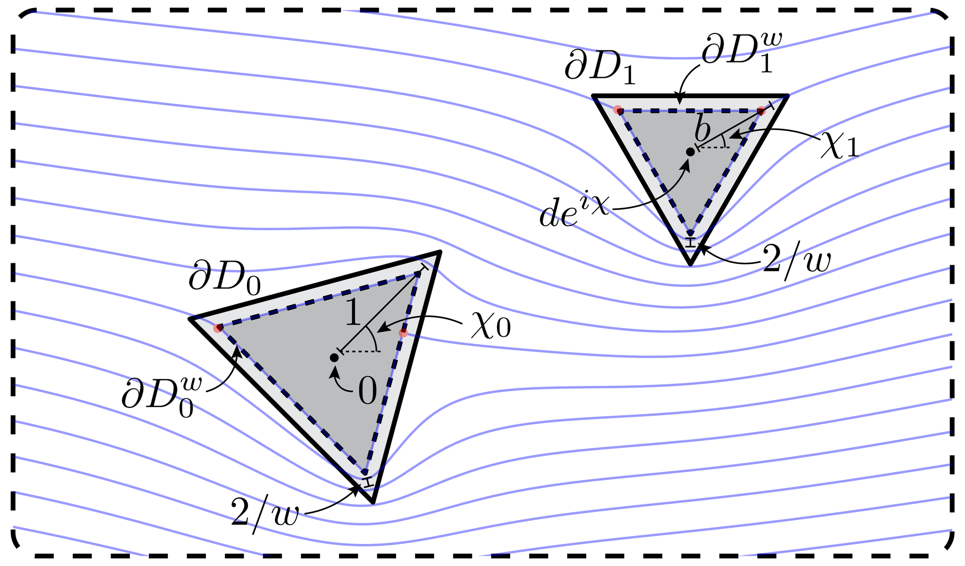
$$\frac{\mathcal{E}}{K} = \frac{\pi}{2}w\left(1-\rho^2\right) - \pi\log\left|\left(1-\rho^2\right)\left(1+\rho^2 e^{i\gamma}\right)\right|$$

This is minimized when $\gamma = 0$ (the field is up–down symmetric).

Two triangles in a static liquid crystal

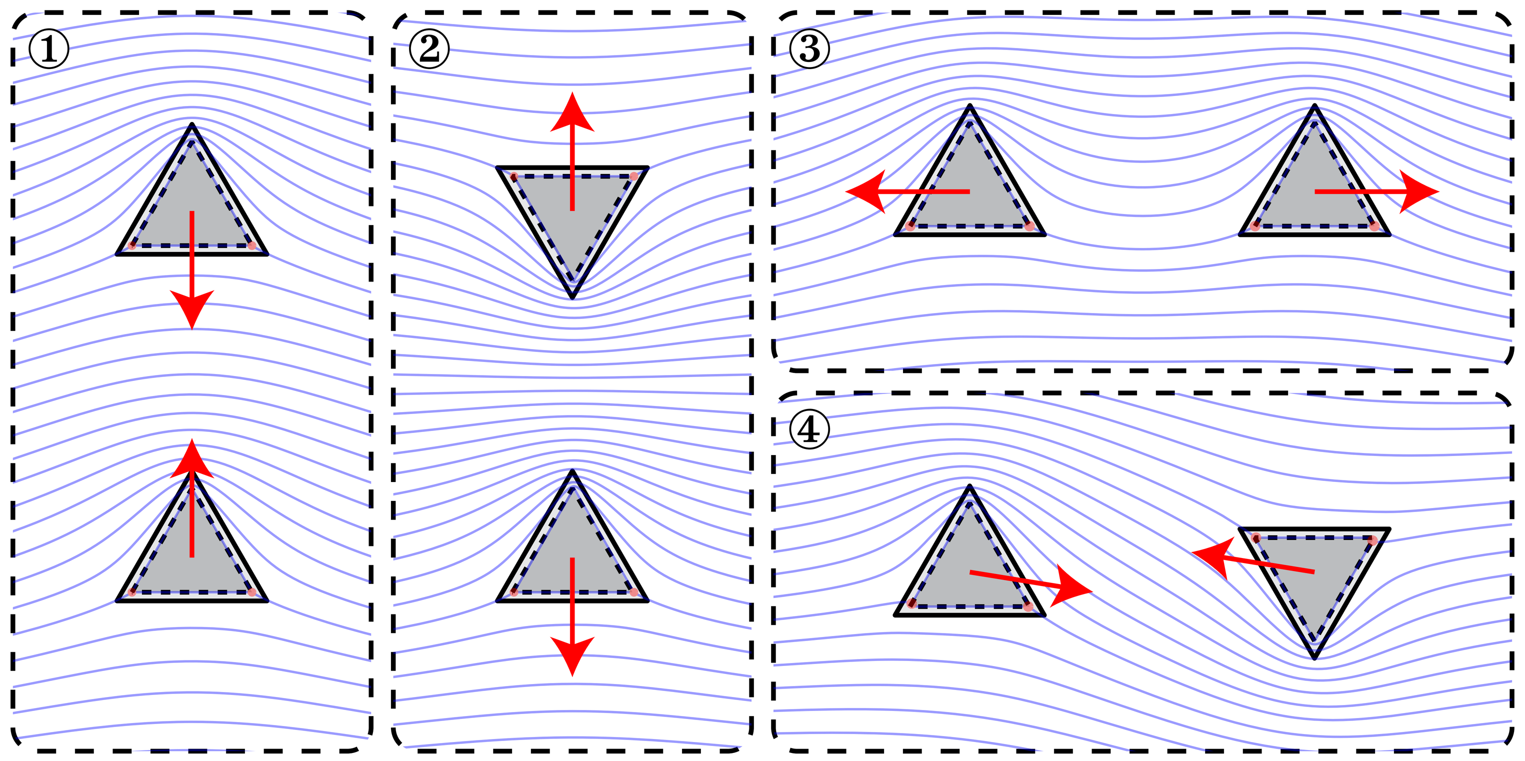
For $w \gg 1$, the eff. bdys. are smaller triangles.

The equations can be solved by conformally mapping onto an annulus and applying the method of images.



The net force acting on each body is given by the elastic traction:

$$F_x - iF_y = \oint_{\partial D_0} \hat{\boldsymbol{\nu}}\cdot\boldsymbol{\sigma}^e(\theta)\cdot(\boldsymbol{x}-i\boldsymbol{y})\,\mathrm{d}s = \frac{K}{2}\oint_{\partial D_0}\left(\frac{f''(z)}{f'(z)}\right)^2\mathrm{d}z$$



As $d\to\infty$, Cauchy's residue theorem yields $C = 17.202\ldots$

① $F_x - iF_y \sim -iKC^2/(\pi d^3)$

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④ $F_x - iF_y \sim KC^2/(\pi d^3)$

Elastic vs. active tractions

Activity in the LC induces a flow. For small activity strengths, the static LC solution yields an ODE for the streamfunction:

$$\frac{\partial^4\psi}{\partial z^2\partial\bar{z}^2} = \frac{\alpha}{4\mu}\mathrm{Im}\left[\frac{\partial^2}{\partial z^2}\left(\frac{\bar{f}'(\bar{z})}{f'(z)}\right)\right] \quad \text{for} \quad \theta(z) = -\arg f'(z)$$

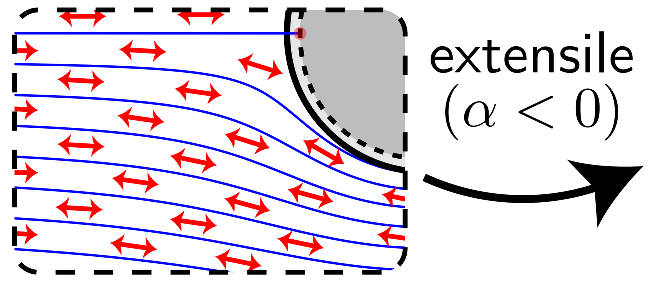
A cylinder in an active liquid crystal

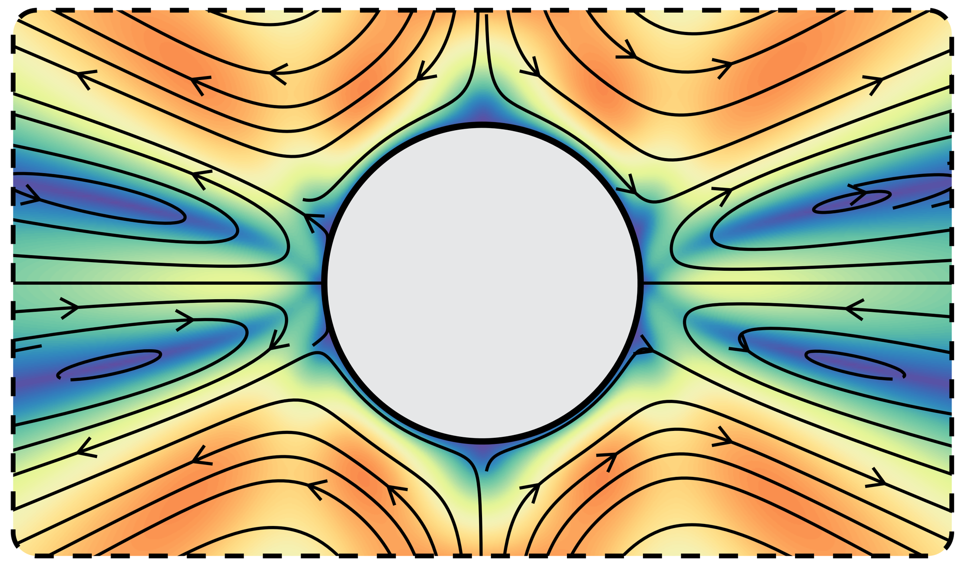
Potential solution for the unit cylinder yields the streamfunction

$$\psi(z) = \frac{\alpha\rho^2}{2\mu}\mathrm{Im}\left[\frac{(|z|^2-1)^2 + 2\rho^2 z^2(1-|z|^2+2\log|z|)}{z^2(z^2-\rho^2)}\right]$$

The fluid velocity is

$$u - iv = 2i\psi'(z)$$

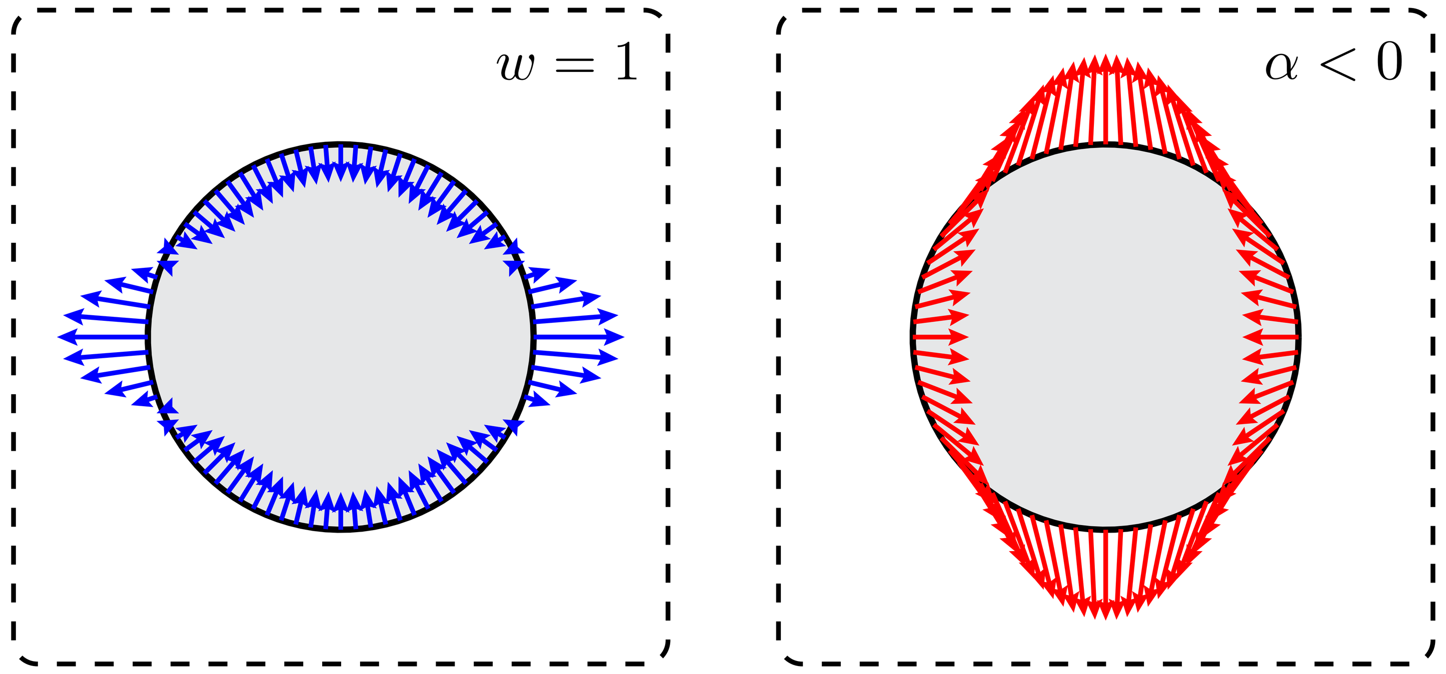




The elastic LC and fluid flow each yield a surface traction. If extensile ($\alpha < 0$), these tractions oppose each other:

Elastic traction $\hat{\boldsymbol{\nu}}\cdot\boldsymbol{\sigma}^e(\theta)$

Active traction $\hat{\boldsymbol{\nu}}\cdot\left[\mu(\nabla\boldsymbol{u}+\nabla\boldsymbol{u}^T) + 2\alpha\boldsymbol{n}\boldsymbol{n} - p\boldsymbol{I}\right]$



Shapes of deformable bodies lie in the balance

References

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[2] **Chandler, T.G.J.** & Spagnolie, S.E. 2023. Exact and approximate solutions for elastic interactions in a nematic liquid crystal. *Submitted.* arXiv:2311.17708.

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