# Deformable bodies in active nematics

**Thomas G. J. Chandler** & Saverio E. Spagnolie

Department of Mathematics, University of Wisconsin–Madison

**Read the papers & watch animations**

## Introduction

Fluid anisotropy can be observed in biofluids like mucus or, at a larger scale, self-aligning swarms of bacteria. A model fluid used to investigate such environments is a nematic liquid crystal (LC). Activity within these LCs generate large scale flows, while large deformable bodies tend to be stretched when immersed.

We use complex variables to analytically solve for the flow of an active LC and the deformation of its boundary. These solutions bring many novel insights into this complex problem.

## Active Nematic LCs

Active nematic LCs are described by a director field $\boldsymbol{n}(\boldsymbol{x},t)$ and a fluid velocity $\boldsymbol{u}(\boldsymbol{x},t)$. With weak activity and one elastic constant

$$\mathbf{P}(\boldsymbol{n})\cdot\nabla^2\boldsymbol{n}=\mathbf{0},\quad \boldsymbol{\nabla}\cdot(\boldsymbol{\sigma}_v+\boldsymbol{\sigma}_v+\boldsymbol{\sigma}_v-p\mathbf{I})=\mathbf{0},\quad \boldsymbol{\nabla}\cdot\boldsymbol{u}=0 \quad \text{in } D$$

$$\mathbf{P}(\boldsymbol{n})\cdot[K\hat{\boldsymbol{\nu}}\cdot\boldsymbol{\nabla}\boldsymbol{n}+W(\boldsymbol{n}\cdot\boldsymbol{n}_0)\boldsymbol{n}_0]=\mathbf{0} \quad\text{and}\quad \boldsymbol{u}=\mathbf{0} \quad\text{on } \partial D$$

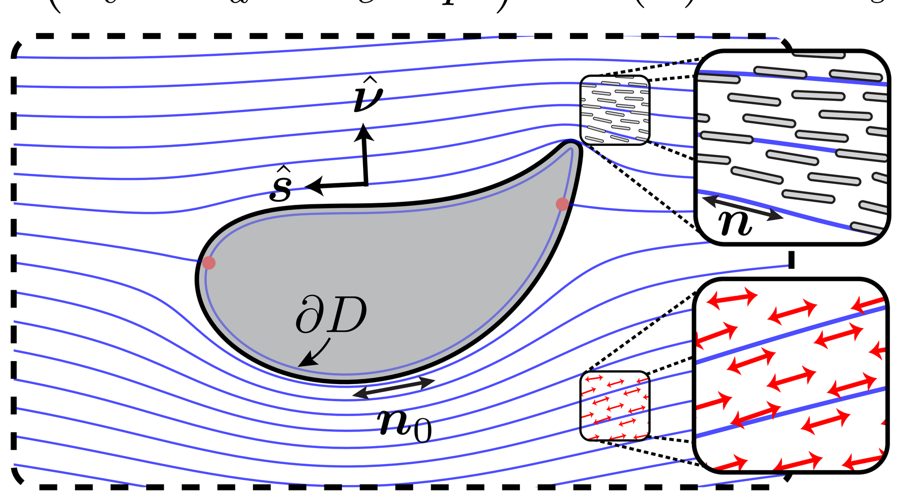
with $|\boldsymbol{n}|=1$ and $\mathbf{P}(\boldsymbol{n})=\mathbf{I}-\boldsymbol{n}\boldsymbol{n}$. For small anisotropic viscosities

| viscous stress | active stress | elastic stress |
|---|---|---|
| $\boldsymbol{\sigma}_v=\mu(\boldsymbol{\nabla}\boldsymbol{u}+\boldsymbol{\nabla}\boldsymbol{u}^T)$ | $\boldsymbol{\sigma}_a=2\alpha\boldsymbol{n}\boldsymbol{n}$ | $\boldsymbol{\sigma}_e=-K\boldsymbol{\nabla}\boldsymbol{n}\cdot\boldsymbol{\nabla}\boldsymbol{n}^T$ |

anchoring surface stress

The traction on $\partial D$ is $\boldsymbol{t}=\hat{\boldsymbol{\nu}}\cdot(\boldsymbol{\sigma}_v+\boldsymbol{\sigma}_a+\boldsymbol{\sigma}_e-p\mathbf{I})+\mathbf{P}(\hat{\boldsymbol{\nu}})\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}_s$

- $K$ - elastic constant
- $W$ - anchoring strength
- $\alpha$ - activity strength
- $\mu$ - solvent viscosity
- $\boldsymbol{n}_0$ - anchoring direction
- $L$ - length scale
- $K/\mu L$ - velocity scale




In 2D, we introduce a streamfunction $\psi(x,y)$ and director angle $\theta(x,y)$. These satisfy the dimensionless equations

$$\nabla^2\theta=0 \quad\text{and}\quad \nabla^4\psi=A\left[(\partial_{xx}-\partial_{yy})\sin 2\theta-2\partial_{xy}\cos 2\theta\right] \quad \text{in } D$$

$$\frac{\partial\theta}{\partial\nu}=\frac{w}{2}\sin\left[2(\theta-\theta_0)\right] \quad\text{and}\quad \psi=\frac{\partial\psi}{\partial\nu}=0 \quad\text{on } \partial D$$

for $\boldsymbol{u}=(\psi_y,-\psi_x)$, $\boldsymbol{n}=(\cos\theta,\sin\theta)$, $\boldsymbol{n}_0=(\cos\theta_0,\sin\theta_0)$, and

**activity strength** $A=\dfrac{\alpha L^2}{K}$ **anchoring strength** $w=\dfrac{WL}{K}$

Introducing a complex position $z=x+iy$ these can be solved exactly with locally-analytic functions $f(z)$, $g(z)$, and $h(z)$:

$$\theta=-\arg f''(z) \qquad \psi=\frac{A}{4}\operatorname{Im}\left[\bar{z}g(z)+h(z)+\overline{f(z)}/f''(z)\right]$$

provided the BCs are satisfied, the velocity is $u-iv=2i\psi'(z)$.

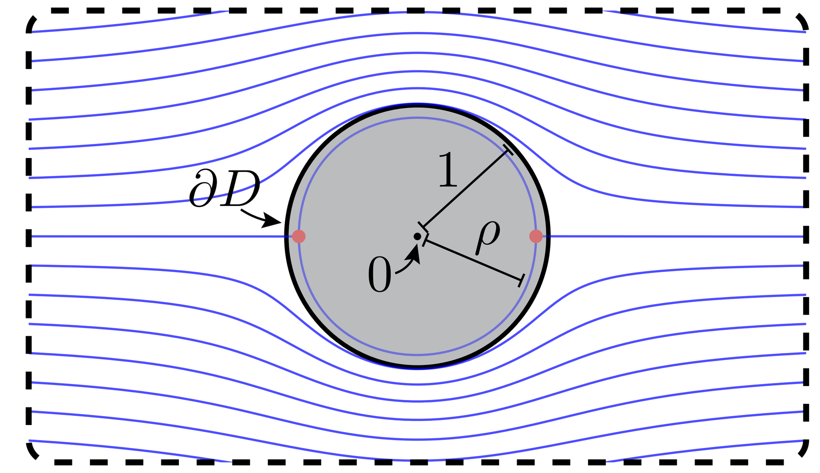
## Activity-induced flow

### An immersed cylinder with tangential anchoring

Here, the equilibrium director is

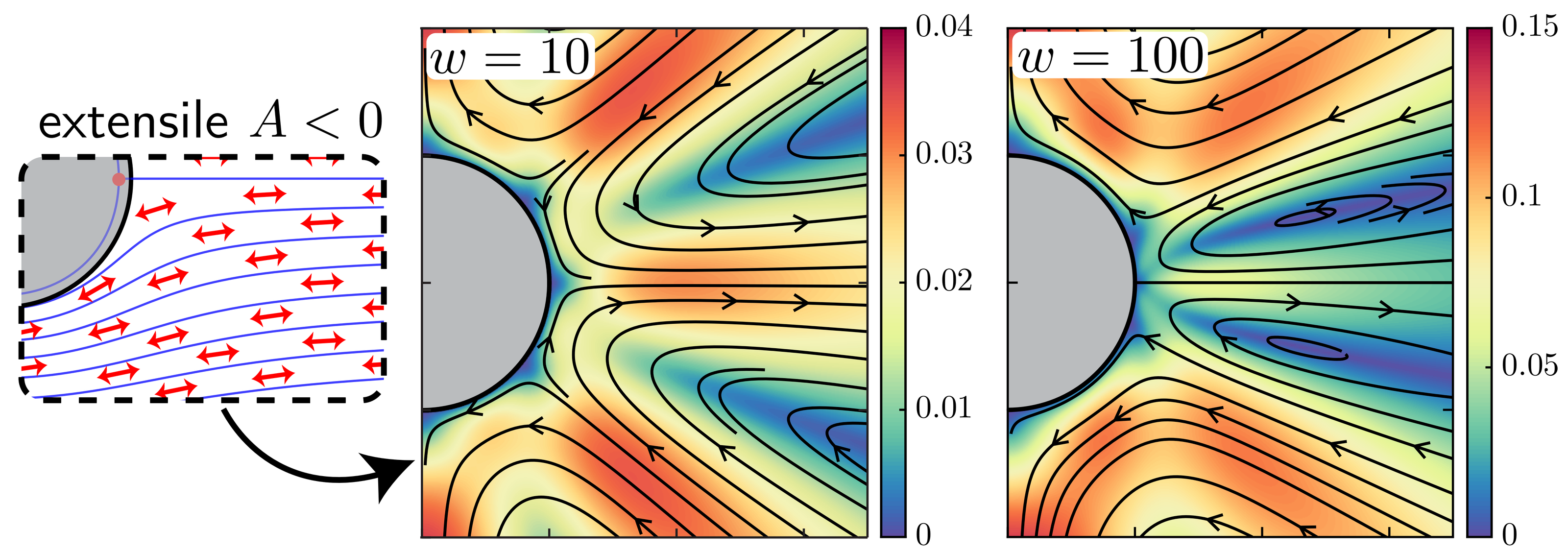
$$\theta=-\arg(1-\rho^2/z^2)$$

$$\rho=\left(\sqrt{1+4/w^2}-2/w\right)^{1/2}$$




which yields the streamfunction

$$\psi=\frac{A\rho^2}{8}\operatorname{Im}\left[\frac{(|z|^2-1)^2}{z^2(z^2-\rho^2)}+\frac{1-|z|^2+2\log|z|}{(z^2-\rho^2)/(2\rho^2)}\right]$$




The large scale flow is characterized by $w$, with plumes extending symmetrically to both the left and right of the cylinder.

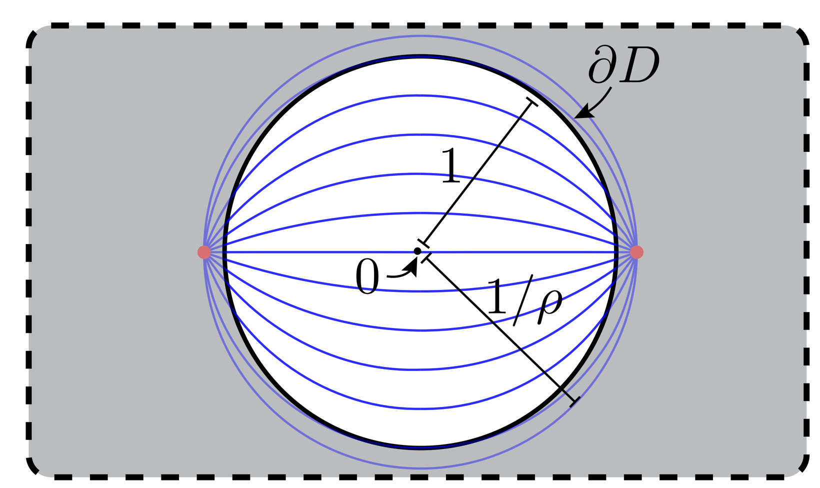
$$u+iv\sim(\rho^2\cos 2\alpha-\cos 4\alpha)\frac{A\rho^2e^{i\alpha}}{2|z|} \quad\text{as } |z|\to\infty \text{ for } \alpha=\arg z$$

### A bounding cylinder with tangential anchoring

Here, the equilibrium director is

$$\theta=\arg(1-\rho^2z^2)$$

$$\rho=\left(\sqrt{1+4/w^2}-2/w\right)^{1/2}$$

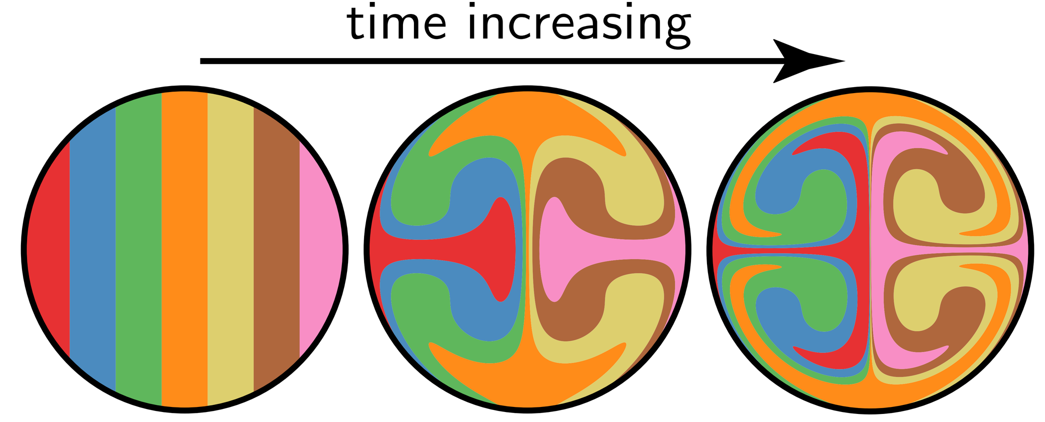



which yields the streamfunction

$$\psi=\frac{A}{8}(|z|^2-1)\operatorname{Im}\left[\frac{1-\rho z}{z^2}\log(1-\rho z)+\frac{1+\rho z}{z^2}\log(1+\rho z)\right]$$

time increasing

Confined LCs induce stirring flows with speed controlled by $A$ and $w$.



In both cases, the stress on the cylinder can be computed exactly. These deform soft boundaries, relaxing the LC. We assume the cylinder deforms according to plane strain in linear elasticity.

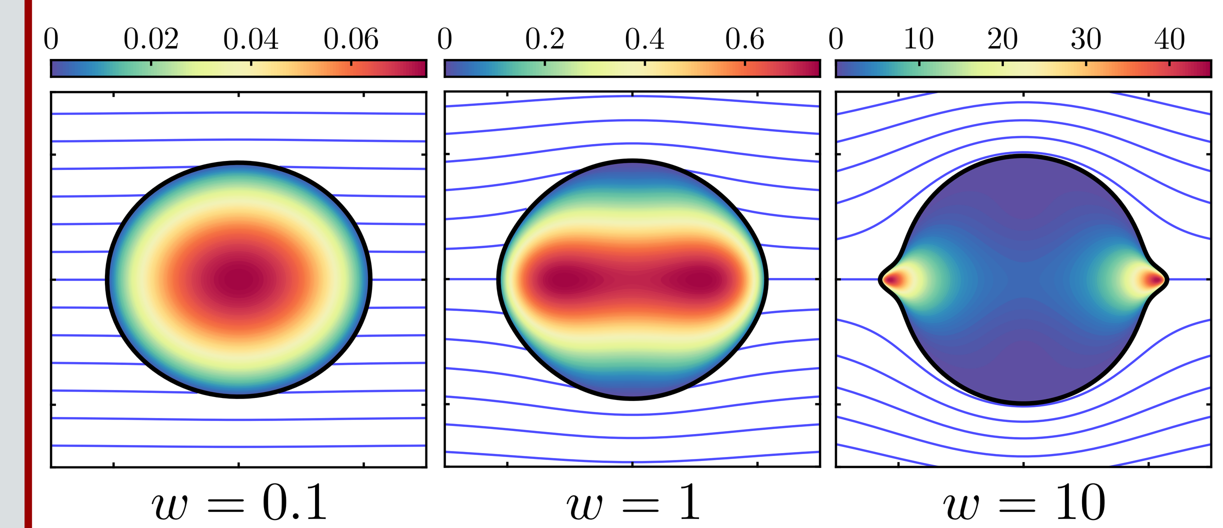
$\nu$ - Poisson's ratio $\qquad G$ - shear modulus

## Deformable boundaries

In linear elastic plane stress, the solid deformation and stress tensor can be written in terms of a biharmonic Airy stress function. Again, this can be found analytically using complex variables.

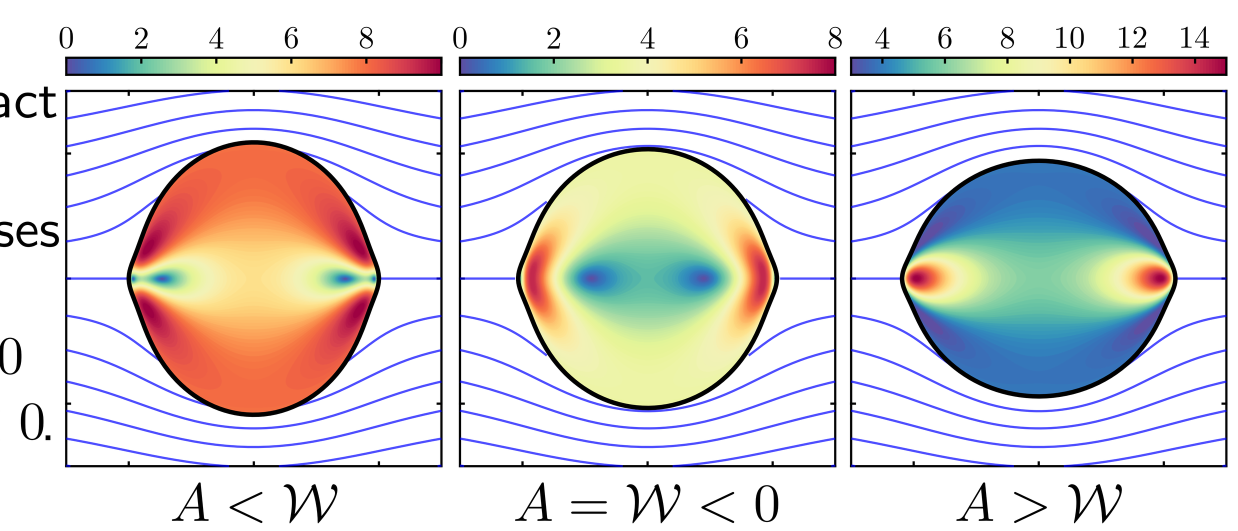
**elastic modulus** $M=\dfrac{GL^2}{K}$

### An immersed cylinder with tangential anchoring




Elastic stresses in passive LCs stretch the immersed body.

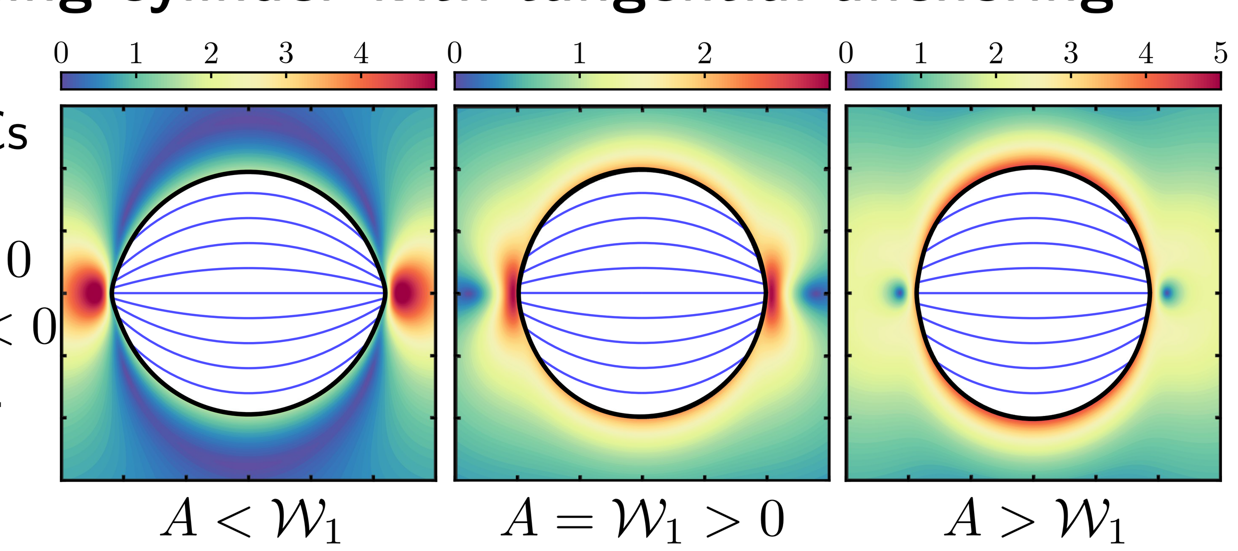
Active stresses act with or against the elastic stresses depending on if contractile $A>0$ or extensile $A<0$.




Aspect ratio is $\mathcal{A}_R=1+\dfrac{A-\mathcal{W}}{M}$ where $\mathcal{W}=-\dfrac{3w}{2(1+\nu)}+\ldots$

### A bounding cylinder with tangential anchoring

For confined LCs the response of contractile $A>0$ vs. extensile $A<0$ stress is flipped.




Aspect ratio is $\mathcal{A}_R=1+\dfrac{\mathcal{W}_1-A}{M/\mathcal{W}_2}$ where $\mathcal{W}_1=\dfrac{w}{2(1+\nu)}+\ldots$

## References

[1] **Chandler, T. G. J.** & Spagnolie, S. E. 2023. *J. Fluid Mech. 967*, A19. doi:10.1017/jfm.2023.488. arXiv:2301.10924.

[2] **Chandler, T. G. J.** & Spagnolie, S. E. 2024. *Phys. Rev. Fluids 9*, 110511. doi:10.1103/PhysRevFluids.9.110511.

[3] **Chandler, T. G. J.** & Spagnolie, S. E. 2024. *SIAM J. Appl. Math. (in press)* arXiv:2311.17708.

[4] **Chandler, T. G. J.** & Spagnolie, S. E. 2024. *(submitted)* arXiv:2409.15617.

people.math.wisc.edu/~tgchandler/ tgchandler@wisc.edu