Extremizers for Adjoint Restriction to a Pair of Reflected Paraboloids

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Symmetries

The operators $\mathcal{E}$ and $\mathcal{E}'$ possess Lie groups of symmetries that must be accounted for if we wish to prove convergence. The set of symmetries corresponding to scaling, frequency translation (spacetime modulation), and spacetime translation spans all elements that generate non-compact orbits in the Lie group, so for compactness purposes, we only need to account for these symmetries.

Definition

Let $\mathcal{S}_s \subset \text{Isom}(L^q(\mathbb{R}^d))$ and $\mathcal{T}_s \subset \text{Isom}(L^q(\mathbb{R}^{d+1}))$ be the subgroups generated by scaling, frequency translation, and spacetime translation. They are distinguished by the fact that $\mathcal{E} \circ \mathcal{S}_s = \mathcal{T}_s \circ \mathcal{E}$ and each element generates non-compact orbits in $L^q$ or $L^q$.

We can define the groups $\mathcal{S}_s$ and $\mathcal{T}_s$ similarly for the operator $\mathcal{E}'$.

Profile Decomposition

We use the technique of profile decomposition as in [3] or [2] to analyze each paraboloid independently. This method relies on a refined version of the restriction estimate to show that any near-extremizer is composed of several concentrated pieces supported on some collection of dyadic cubes, and a remainder whose extension is small in $L^q$. The technique also tells us that one of these bubbles is $\geq 1$ in $L^q$. Using tools such as the fact that $\mathcal{E}$ is supported on some collection of dyadic cubes, and a remainder whose extension is small in $L^q$. Assume (1) holds. If $\eta_0 \to \infty$, we can show that the limit of $\|\mathcal{E} \mathcal{E}_n(f, g_n)\|_q$ is less than or equal to the left-hand side of the inequality in the Theorem. Since this contradicts the assumption that $(f_n, g_n)$ is an extremizing sequence, then there exists an $\eta_0$ such that $\eta_0 \to \eta_0$ along a subsequence. By a standard argument using the smoothing properties of $\mathcal{E}$, and the strict convexity of $L^q$, we prove convergence to an extremizer.

Numerics

Interestingly, we are able to verify the strict inequality for $d \in \{1,2\}$. For these dimensions, it is known that radial Gaussians are extremizers for $\mathcal{E}$ at the Stein-Tomas exponent (1). Using the identity $\mathcal{E} = \mathcal{E}'$, we are able to calculate that

$$\frac{1}{2\pi} \int_0^{2\pi} \|\mathcal{E} \mathcal{E}_n(f, g_n)\|_q = 2^{3/2}(A_n^e)^d.$$ 

The integrand is continuous in $\theta$ by the dominated convergence theorem, so it suffices to show that it is not constant to get a $\theta_0$ that provides an example proving (1). We prove that the values for $\theta = 0$ and $\theta = \frac{\pi}{2}$ are different using Sage.

References


Results

We will consider the restriction estimate associated with affine surface measure on the union of the standard paraboloid in $\mathbb{R}^{d+1}$ and its reflection across the plane $\{\tau, \xi \in \mathbb{R}^{d+1} : \tau = 0\}$ given by

$$\mathcal{E}_n(f, g)(t, x) = \mathcal{E} f + \mathcal{E} g,$$

$$= \int_{\mathbb{R}^d} e^{i(t\xi + t^2/2)} f(\xi) + e^{i(t\xi - t^2/2)} g(\xi) \, d\xi.$$

For exponents $(p, q)$ such that the original extension operator for the paraboloid is bounded, $\mathcal{E}_n : L^p(\mathbb{R}^d) \to L^q$ is bounded by Minkowski’s inequality. Stovall (3) proved that extremizers exist for adjoint restriction for the single standard paraboloid, provided that the exponent pair is “non-endpoint.” In particular, this includes the Stein–Tomas exponent. Let $p = 2, q = \frac{2(d+2)}{d+4}$, and

$$A_0^e := \sup_{\mathcal{E} \mathcal{E}_n(f, g) \in \ell^q} \|\mathcal{E} f + \mathcal{E} g\|_q.$$

Theorem

Let $\{f_n, g_n\} \subset L^2 \times L^2$ be such that $\|f_n\|_2^2 + \|g_n\|_2^2 = 1$ for all $n$ and

$$\lim_{n \to \infty} \|\mathcal{E} f_n + \mathcal{E} g_n\|_q = A_0^e.$$

If

$$\left(1 + \frac{1}{\sqrt{p}} \right)^{1/q} 2^{1/p} A_0^e < A_0^e,$$

then there exist $\{Q_n\} \subset \mathcal{S}_s \cap \mathcal{T}_s$ and $f, g \in L^2$ such that along a subsequence,

1. $|f - Q_n f_n| \to 0$;
2. $|g - Q_n g_n| \to 0$; and therefore
3. $\|\mathcal{E} f + \mathcal{E} g\|_q \to A_0^e$.

For all $d \in \mathbb{N}$, there exist $f, g \in L^2$ normalized so that $\|f\|_2^2 + \|g\|_2^2 = 1$ such that

$$\|\mathcal{E} f + \mathcal{E} g\|_q = A_0^e.$$