Extremizers for Convolution with Compact, Well-Curved Hypersurfaces

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Existence

Since the proof of existence is fairly standard in the world of sharp Fourier restriction, let's focus on the part specific to this problem. The compact support of σ means that the L^q norm approximately decouples.

Proposition

For all $\varepsilon > 0$ sufficiently small, there exists $0 < C < 1$ and $N > 0$ such that, for all f with $\|Tf\|_q \gtrsim A(p,q)\|f\|_p$ and $\|f\|_p = 1$, there exist non-negative functions $\{f_j\}_{j=1}^N$, unit cubes $\{Q_j\}_{j=1}^N$, and a function $r_N \in L^p$ with the following properties:

1. $\operatorname{supp} f_j \subset 3Q_j$;
2. $f_j = (f - \sum_{k=1}^{j-1} f_k)_{3Q_j}$;
3. $f = \sum_{j=1}^N f_j + r_N$; and
4. $\|Tr_N\|_q < \varepsilon$.

We use this as well as the other assumptions to localize and truncate any near extremizer, and then prove that extremizing sequences are precompact modulo translation.

References

[1] M. Christ and Q. Xue. "Smoothness of extremizers of a convolution inequality". In: *Journal de mathématiques pures et appliquées* 97.2 (2012), pp. 120–141.

[2] T Flock. "On extremizers for certain inequalities of the k-plane transform and related topics". PhD thesis. UC Berkeley, 2014.

[3] W. Littman. "Lp-Lq-estimates for singular integral operators arising from hyperbolic equations". In: *Proc. Sympos. Pure Math.* 23 (1973), pp. 479–481.

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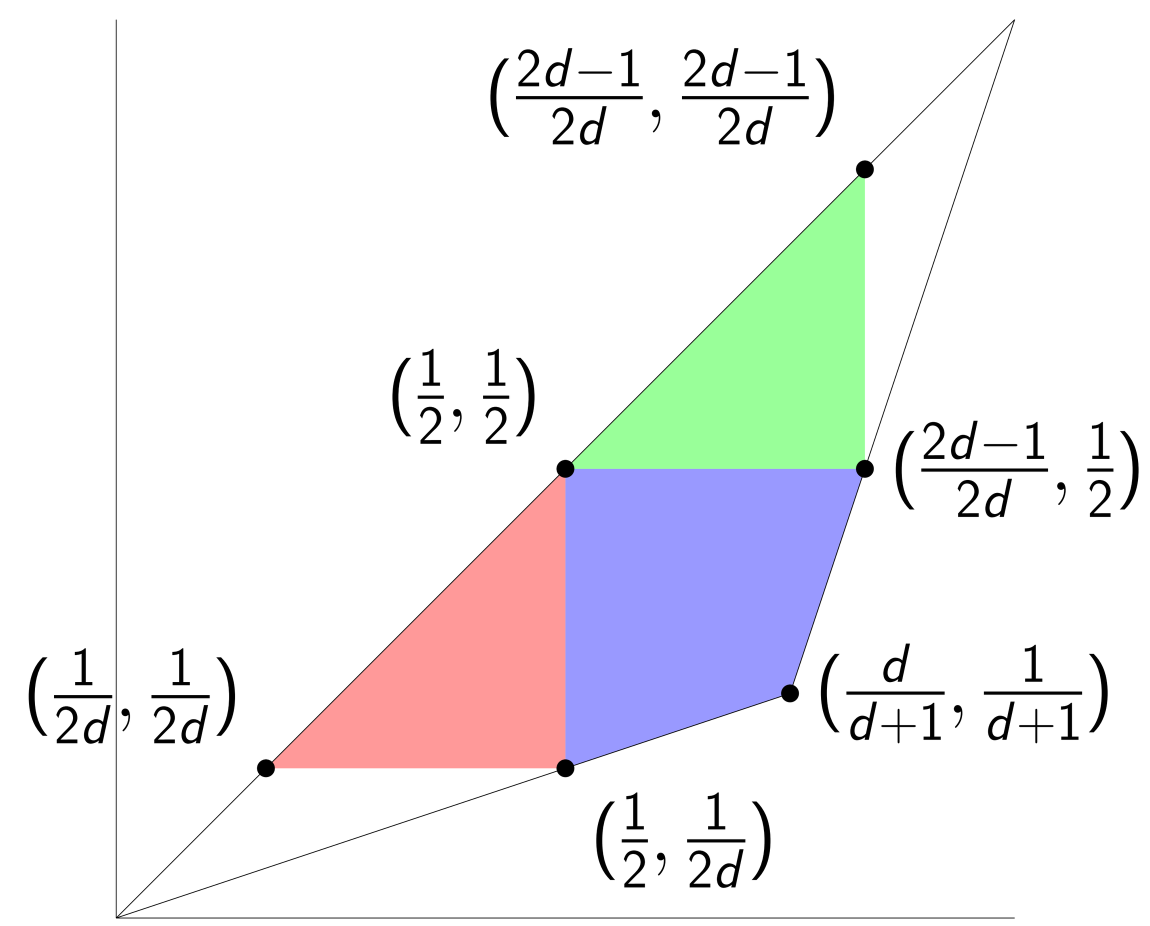
Let σ be a finite, compactly supported measure on $\mathbb{R}^d$ and define an operator on smooth functions by

$$Tf(x) = \int f(x-y)\,d\sigma(y).$$

When σ has Fourier decay (e.g. surface measure on a curved hypersurface), T is known to have $L^p \to L^q$ estimates for some $p < q$ ([3]). For truncated surface measures, we have bounds for exponents in the big triangle depicted in the figure below.



Theorem (Preparing)

1. Let σ be a finite measure with compact support and such that T is bounded on a neighborhood of $(\frac{1}{p}, \frac{1}{q})$ and $L^2 \to L^2_\gamma$ for some $\gamma > 0$. Then there exists a non-zero $f \in L^p$ such that
$$\|Tf\|_q = \|T\|_{p\to q}\|f\|_p.$$
2. If σ is surface measure on a compact subset of a hypersurface M such that M has $d-1$ principal curvatures bounded away from zero on $\operatorname{supp}\sigma$ and $(\frac{1}{p}, \frac{1}{q})$ lies in any of the colored regions, then $f \in C^\infty$.

Smoothness

Smoothness of extremizers has previously been proven for convolution with the paraboloid ([1]) as well as certain k-plane transforms ([2]). We follow in this vein, but analyzing the Euler-Lagrange equation. Via Hölder's inequality, it follows that all extremizers must satisfy

$$f = \lambda(T^*(Tf)^{q-1})^{\frac{1}{p-1}}.$$

The main difference with previous work is the consideration of $q-1, \frac{1}{p-1} \notin \mathbb{N}$. To overcome this obstacle, we use three main tools.

1. Every extremizer f is continuous, and uniformly bounded below on compact sets. This follows from the fact that $\sigma * \sigma$ is absolutely continuous with respect to Lebesgue measure. It is a key ingredient in the proof of the power rule estimates.
2. Although we can't take classical derivatives of the Euler-Lagrange equation, we can differentiate it weakly. We develop an algebraic notation to handle the long sums and products produced by Leibniz's rule.
3. Finally, we induct over $f \in C^\ell$. Derivative estimates take us from C^ℓ to $L^p_{\ell+1-\varepsilon}$. Applying the derivative estimates to the weak derivatives from the previous part proves that $f \in L^p_{\ell+2-\varepsilon}$. Finally, the specific region of boundedness of T and Sobolev embedding imply that $f \in C^{\ell+1}$.