Extremizers for L^p-improving convolution operators: existence and regularity

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Acknowledgements

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Preliminaries

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4. Normalized extremizing sequence: $f_n \in L^p$ such that $\|f_n\|_p = 1$ and $\lim_{n\to\infty} \|Tf_n\|_q = \|T\|_{p,q}$

Motivation

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2. Similar arguments can be used to prove the existence of minimal blowup solutions for mass and energy critical NLS (eg Killip-Visan ’13).
3. If we can explicitly identify extremizers (Flock ’16), then we can compute the operator norm (Drouot ’12).

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Theorem (Christ '11)

All extremizers are of the form $c(1 + |(x', x_d + \frac{1}{2}|x'|^2)|^2)^{-d/2}$ modulo some affine transformations.

Sphere

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Yes!

Main result

Let $\mathbb{X} \in \{\mathbb{R}^d, \mathbb{T}^d\}$.

Theorem (T. '23)

Let σ be a compactly-supported probability measure such that $|\widehat{\sigma}(\xi)| \lesssim \langle \xi \rangle^{-\alpha}$ for some $\alpha > 0$. If $1 < s < p < q < \infty$ are such that $\|T\|_{p,q} < \infty$ and $\|T\|_{s,q} < \infty$, then extremizing sequences for $T\colon L^p(\mathbb{X}) \to L^q(\mathbb{X})$ are precompact (modulo translation when $\mathbb{X} = \mathbb{R}^d$). In particular, extremizers exist.
Furthermore, for all extremizers f there exists a unimodular $\omega_0 \in \mathbb{C}$ such that

$$\inf_{x \in W} \omega_0 f(x) > 0$$

for all compact $W \subset \mathbb{X}$ and $f \in C^\infty_{loc}(\mathbb{X}) \cap L^\infty(\mathbb{X})$.

Concentration lemma

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Lemma

Let f_n be an extremizing sequence such that $\|f_n\|_p = 1$. Then there exists a sequence $x_n \in \mathbb{X}$ such that

$$\lim_{R\to\infty} \limsup_{n\to\infty} \|\mathbf{1}_{|f_n|>R} f_n\|_p + \|\mathbf{1}_{|\cdot|>R} f_n(\cdot - x_n)\|_p = 0.$$

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Lemma

If $Tf_n \to F$ and $f_n \rightharpoonup f$, then $F = Tf$ and $f_n \to f$.

Proof.

This follows by duality, the boundedness of T, and the convexity of L^p. □

This completes the existence part of the theorem.

Smoothness

Lemma (Euler-Lagrange equation)

All extremizers f with $\|f\|_p = 1$ satisfy

$$|f| = \|T\|_{p,q}^{-\frac{q}{q-p}} \left(T^*|Tf|^{q-1}\right)^{\frac{1}{p-1}} .$$

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Solution

Strong positivity and local estimates.

Positivity

Lemma

For all extremizers f there exists a unimodular $\omega_0 \in \mathbb{C}$ such that

$$\inf_{x \in W} \omega_0 f(x) > 0$$

for all compact $W \subset \mathbb{X}$.

Positivity

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Lemma

Let $a \neq 1$, $W \subset \mathbb{X}$ be compact, and f be a non-negative extremizer. Then there exists $\eta \in C^\infty$ such that $\partial^k \eta(0) = 0$ for all $k \geq 0$ and

$$f^a|_W \equiv \eta \circ f|_W.$$

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For all $\psi_1 \in C^\infty_{cpct}$*, there exists* $\psi_2 \in C^\infty_{cpct}$ *such that*

$$Tf|_{\operatorname{supp}\psi_1} \equiv T(\psi_2 f)|_{\operatorname{supp}\psi_1}.$$

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Lemma

There exists $\kappa > 0$ *such that for all* $s \geq 0$ *and* $\psi_1 \in C^\infty_{cpct}$,

$$\left\| \psi_1 \left(T^* |Tf|^{q-1} \right)^{\frac{1}{p-1}} \right\|_{W^{s+\kappa,p}} \lesssim \|\psi_2 f\|_{W^{s,p}}$$

for some $\psi_2 \in C^\infty_{cpct}$.

Thank you!