Extremizers for Adjoint Restriction to a Pair of Reflected Paraboloids

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Symmetries

The operators $\mathcal{E}$ and $\mathcal{E}_-$ posses Lie groups of symmetries that must be accounted for if we wish to prove convergence. The set of symmetries corresponding to scaling, frequency translation (spacetime modulation), and spacetime translation spans all elements that generate non-compact orbits in the Lie group, so for compactness purposes, we only need to account for these symmetries.

Definition

Let $\mathbf{S}_+ \subset \mathrm{Isom}(L^p(\mathbb{R}^d))$ and $\mathbf{T}_+ \subset \mathrm{Isom}(L^q(\mathbb{R}^{d+1}))$ be the subgroups generated by scaling, frequency translation, and spacetime translation. They are distinguished by the fact that $\mathcal{E} \circ \mathbf{S}_+ = \mathbf{T}_+ \circ \mathcal{E}$ and each element generates non-compact orbits in L^p or L^q.
We can define the groups $\mathbf{S}_-$ and $\mathbf{T}_-$ similarly for the operator $\mathcal{E}_-$.

Profile Decomposition

We use the technique of profile decomposition as in [3] or [2] to analyze each paraboloid independently. This method relies on a refined version of the restriction estimate to show that any near-extremizer is composed of several concentrated pieces supported on some collection of dyadic cubes, and a remainder whose extension is small in L^q. The technique also tells us that one of these bubbles is $\gtrsim 1$ in L^p.
Using tools such as the fact that $q > p$ and Tao's bilinear restriction estimate, this result can be strengthened to address an extremizing sequence.

Proposition

Let $\{f_n\} \subset L^2(\mathbb{R}^d)$ be bounded. Then after passing to a subsequence, there exists $J^* \in \mathbb{N} \cup \{\infty\}$; functions $\phi^j \in L^2$ for all $j < J^*$; remainders $w_n^J \in L^2$ for all $J < J^*$; and symmetries $S_n^j \subset \mathbf{S}_+$ for all $j < J^*$ such that for all $J < J^*$,

$$f_n = \sum_{j=1}^{J} S_n^j \phi^j(\xi) + w_n^J,$$

1. $\lim_{J \to J^*} \limsup_{n \to \infty} \|\mathcal{E} w_n^J\|_q = 0,$
2. $\sup_J \lim_{n \to \infty} \left[\|f_n\|_2^2 - \sum_{j=1}^J \|\phi^j\|_2^2 - \|w_n^J\|_2^2 \right] = 0,$
3. $\sup_J \lim_{n \to \infty} \left[\|\mathcal{E} f_n\|_q^q - \sum_{j=1}^J \|\mathcal{E}\phi^j\|_q^q - \|\mathcal{E} w_n^J\|_q^q \right] = 0,$

and for all $j \neq k$, $(S_n^j)(S_n^k)^{-1} \rightharpoonup 0$ in the weak operator topology.

Results

We will consider the restriction estimate associated with affine surface measure on the union of the standard paraboloid in $\mathbb{R}^{d+1}$ and its reflection across the plane $\{(\tau, \xi) \in \mathbb{R}^{d+1} : \tau = 0\}$ given by

$$\begin{aligned} \mathcal{E}_\pm(f,g)(t,x) &:= \mathcal{E}f + \mathcal{E}_- g \\ &:= \int_{\mathbb{R}^d} e^{i(t,x)\cdot(|\xi|^2,\xi)} f(\xi) + e^{i(t,x)\cdot(-|\xi|^2,\xi)} g(\xi)\, d\xi. \end{aligned}$$

For exponents (p,q) such that the original extension operator for the paraboloid is bounded, $\mathcal{E}_\pm : \ell^p(L^p) \to L^q$ is bounded by Minkowski's inequality. Stovall ([3]) proved that extremizers exist for adjoint restriction for the single standard paraboloid, provided that the exponent pair is "non-endpoint." In particular, this includes the Stein-Tomas exponent. Let $p = 2$, $q = \frac{2(d+2)}{d}$, and

$$A_p^\pm := \sup_{f,g \in L^p} \frac{\|\mathcal{E}f + \mathcal{E}_- g\|_q}{(\|f\|_p^p + \|g\|_p^p)^{1/p}}.$$

Theorem

1. Let $\{(f_n, g_n)\} \subset L^2 \times L^2$ be such that $\|f_n\|_2^2 + \|g_n\|_2^2 = 1$ for all n and
$$\lim_{n\to\infty} \|\mathcal{E}f_n + \mathcal{E}g_n\|_q = A_2^\pm.$$
If
$$\left(\frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma(\frac{q+1}{2})}{\Gamma(\frac{q+2}{2})} \right)^{1/q} 2^{1/p'} A_2 < A_2^\pm, \qquad (1)$$
then there exist $\{Q_n\} \subset \mathbf{S}_+ \cap \mathbf{S}_-$ and $f, g \in L^2$ such that along a subsequence,
    1. $\|f - Q_n f_n\|_2 \to 0$;
    2. $\|g - Q_n g_n\|_2 \to 0$; and therefore
    3. $\|\mathcal{E}f + \mathcal{E}_- g\|_q = A_2^\pm$.
2. For all $d \in \mathbb{N}$, there exist $f, g \in L^2$ normalized so that $\|f\|_2^2 + \|g\|_2^2 = 1$ such that
$$\|\mathcal{E}f + \mathcal{E}_- g\|_q = A_2^\pm.$$

References

[1] D. Foschi. "Maximizers for the Strichartz inequality". In: *J. Eur. Math. Soc.* 9.4 (2007), pp. 739–774.

[2] M. Visan R. Killip. *Evolution Equations*. Vol. 17. Providence, RI: Clay Math. Proc., 2013, pp. 325–437.

[3] B. Stovall. "Extremizability of Fourier restriction to the paraboloid". In: *Adv. Math.* 360 (2020).

Bubble Pairing

When we consider the double paraboloid and an extremizing sequence (f_n, g_n), we find that any frequency translation applied to f_n must be the negative of that applied to g_n to guarantee that $\|\mathcal{E}(Sf, Rg)\|_q$ remains constant. By employing the bilinear estimate, we prove that only bubbles with a "partner" bubble on the other paraboloid (i.e. the symmetries are not asymptotically orthogonal) can contribute sizably to $\|\mathcal{E}_\pm(f_n, g_n)\|_q$. In addition, using the fact that $\frac{2(d+2)}{d} > 2$, we prove that as $n \to \infty$, one of the bubbles accounts for the full L^2 norm for both f_n and g_n. This means that there exists a sequence $\eta_n \in \mathbb{R}^d$ such that f_n is, morally, supported near η_n and g_n near $-\eta_n$.
Assume (1) holds. If $\eta_n \to \infty$, we can show that the limit of $\|\mathcal{E}_\pm(f_n, g_n)\|_q$ is less than or equal to the lefthand side of the inequality in the Theorem. Since this contradicts the assumption that (f_n, g_n) is an extremizing sequence, then there exists an η_0 such that $\eta_n \to \eta_0$ along a subsequence. By a standard argument using the smoothing properties of $\mathcal{E}_\pm$ and the strict convexity of L^2, we prove convergence to an extremizer.
If we assume instead that (1) is replaced with equality, let f be an extremizer for $\mathcal{E}$ ([3]) and $g_\theta(\xi) = e^{i\theta}\overline{f}(-\xi)$. Then we have

$$\frac{1}{2\pi} \int_0^{2\pi} \int |\mathcal{E}f + \mathcal{E}_- g_\theta|^q = 2^{q/2} (A_2^\pm)^q.$$

Since the inner integral is continuous as a function of θ, there exists a θ_0 such that

$$\|\mathcal{E}_\pm(f, g_{\theta_0})\|_q = A_2^\pm,$$

which completes the Theorem.

Numerics

Interestingly, we are able to verify the strict inequality for $d \in \{1, 2\}$. For these dimensions, it is known that radial Gaussians are extremizers for $\mathcal{E}$ at the Stein-Tomas exponent ([1]). Using the identity $\mathcal{E}_- f = \overline{\mathcal{E}\overline{f}}$, we are able to calculate that

$$\frac{1}{2\pi} \int_0^{2\pi} \|\mathcal{E}_\pm(e^{-|\xi|^2 - i\theta/2}, e^{-|\xi|^2 + i\theta/2})\|_q^q = 2^{q/2} (A_2^\pm)^q.$$

The integrand is continuous in θ by the dominated convergence theorem, so it suffices to show that it is not constant to get a θ_0 that provides an example proving (1). We prove that the values for $\theta = 0$ and $\theta = \frac{\pi}{2}$ are different using Sage.