End point estimates and Monge–Ampère equation with drifts

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0. About Richard Bellman and Madison, WI

Stanislaw Ulam writes:
"One day in my office in North Hall of the University of Wisconsin, a young and brilliant graduate student named **Richard Bellman** appeared and expressed a desire to work with me.... I remembered that in Princeton Lefschetz had some new scientifico-technological enterprise connected with the war efforts. I wrote to him about Bellman in a sort of Machiavellian way, saying that I had a very able student who was so good that he deserved considerable financial support, but I added that I doubt that Princeton could afford it. This immediately challenged Lefschetz, and he offered Bellman a position.... Two years later, **Dick Bellman** appeared in Los Alamos in uniform as a member of special engineering detachment...."

1. Main theorem

Theorem (Nazarov–Reznikov–Vasyunin–Volberg)

There exists an A_1 weight w such that
$\|H : L^1(w) \to L^{1,\infty}(w)\| \geq c[w]_{A_1} \log^{1/4}(1 + [w]_{A_1})$

Let us fix the notation: $Q := [w]_{A_1} := \sup_x \frac{Mw(x)}{w(x)}$. Notice that $Q < \infty$ iff for every interval (cube) I, one has

$$\langle w \rangle_I \leq C \inf_{x \in I} w(x) \,.$$

The smallest C is $[w]_{A_1} =: Q \geq 1$.
In other words, for any sufficiently large Q one can find a weight w, a function f, and a number $\lambda > 0$ such that

$$w\{x : Hf(x) > \lambda\} \geq c\lambda^{-1} Q \log^{1/4} Q \int |f(x)| w(x) dx \,. \qquad (1)$$

2. A brief history

Muckenhoupt 40 years ago posed two problems:
1) prove (or disprove) that

$$w\{x : Hf(x) > \lambda\} \le c\lambda^{-1} \int |f(x)| Mw(x) dx\,. \qquad (2)$$

2) If this inequality is correct, then for any $w \in A_1$, with $Q = [w]_{A_1}$ one will have automatically

$$w\{x : Hf(x) > \lambda\} \le c\lambda^{-1} Q \int |f(x)| w(x) dx\,. \qquad (3)$$

Suppose inequality (2) is **incorrect**, then prove (or disprove) (3). There can be 3 possible answers: a) (2) is correct, b) (2) fails, but (3) holds (in other words, there is no counterexample for "smooth" weights), c) (3) fails. Obvious: if (3) fails then (2) fails. But there is no other obvious claim.

3. A brief history

Maria Reguera and Christoph Thiele disproved (2) in 2009. That was a sophisticated counterexample, but the weight w was very much irregular, and very far from being from A_1. So the so-called "weak Muckenhoupt conjecture" or A_1-conjecture was still open:

$$w \in A_1 \Rightarrow \; w\{x : Hf(x) > \lambda\} \le c\lambda^{-1} Q \int |f(x)| w(x) dx \text{ ???} \quad (4)$$

As a Theorem on slide 1 or (1) shows, weak Muckenhoupt conjecture gets also disproved: the claim above is false, and one can detect a logarithmic blow-up–see $\log^{1/4} Q$ in (1) on slide 1.
What is known for the estimate from above for $\|H : L^1(w) \to L^{1,\infty}(w)\|$ for $[w]_{A_1} = Q < \infty, Q >> 1$?

Theorem (Lerner–Ombrosi–Pérez)

$$w\{x : Hf(x) > \lambda\} \le c\lambda^{-1} Q \log Q \int |f(x)| w(x) dx \,. \qquad (5)$$

4. Dyadic singular operators first

Our measure space throughout this article will be $(X, \mathfrak{A}, dx)$, where σ-algebra $\mathfrak{A}$ is generated by a standard dyadic filtration $\mathcal{D} = \cup_k \mathcal{D}_k$ on $\mathbb{R}$. We consider the martingale transform (and the square function transform) related to this homogeneous dyadic filtration. For our case of dyadic lattice on the line we have that $|\Delta_J f|$ is constant on J, and

$$\Delta_J f = \frac{1}{2}[(\langle f\rangle_{J_+} - \langle f\rangle_{J_-})\mathbf{1}_{J_+} + (\langle f\rangle_{J_-} - \langle f\rangle_{J_+})\mathbf{1}_{J_-}]\,.$$

The square function transform: $(S\varphi)^2(x) = \sum_{J\in\mathcal{D}} |\Delta_J\varphi|^2 \mathbf{1}_J(x)\,.$
Recall that the martingale transform is the operator given by $(|\varepsilon_J| \le 1)$:

$$T\varphi = \sum_{J\in cD} \varepsilon_J \Delta_J \varphi\,.$$

$$\frac{1}{|I|} w\{x\in I : \sum_{J\in\mathcal{D}(I)} \varepsilon_J(\varphi, h_J)h_J(x) > \lambda\} \le C_{[w]_{A_1}} \frac{\langle|\varphi|w\rangle_I}{\lambda}\,. \qquad (6)$$

5. Results for Martingale Transform

Theorem (NRVV)

There is a positive absolute constant c and a weight $w \in A_1$ such that constant $C_{[w]_{A_1}}$ from (6) satisfies

$$C_{[w]_{A_1}} \geq c[w]_{A_1}(\log[w]_{A_1})^{1/4}\,.$$

Theorem (LOP)

For any weight $w \in A_1$ constant $C_{[w]_{A_1}}$ from (6) satisfies $C_{[w]_{A_1}} \leq c[w]_{A_1}\log[w]_{A_1}\,.$

6. Bellman function of a problem

To find the "some estimates on" $C_{[w]_{A_1}}$ we use again the Bellman function technique. The idea is to reformulate the infinitely dimensional problem of optimization of $C_{[w]_{A_1}}$, that is finding of the "smallest" $C_{[w]_{A_1}}$ that works for all inequalities (6), in terms of the growth estimate on a certain function of only finite number of variables (5 in this case).
Here it is. It will depend on number $Q \geq 1$.

$$\mathbf{B}(F, w, m, f, \lambda) := \mathbf{B}_Q(F, w, m, f, \lambda) :=$$
$$\sup \frac{1}{|I|}\omega\{x \in I : \sum_{J \subseteq I, J \in D} \varepsilon_J(\varphi, h_J)h_J(x) > \lambda\}, \tag{7}$$

where the sup is taken over all $\varepsilon_J, |\varepsilon_J| \leq 1, J \in D(I)$, and over all $\varphi \in L^1(I, \omega\, dx)$ such that $F := \langle|\varphi|\,\omega\rangle_I$, $f := \langle\varphi\rangle_I$, $w = \langle\omega\rangle_I, m \leq \inf_I \omega$, and ω are all dyadic A_1 weights, such that $[w]_{A_1} \leq Q$.

This function is obviously defined in the convex subdomain of $\mathbb{R}^5$:

$$\Omega := \{(F, w, m, f, \lambda) \in \mathbb{R}^5 : F \geq |f|\, m, \;\; m \leq w \leq Q\, m\}\,. \tag{8}$$

$$s\mathbf{B}(\frac{F}{s}, \frac{w}{s}, \frac{m}{s}, f, \lambda) = \mathbf{B}(F, w, m, f, \lambda)\,,$$

$$\mathbf{B}(tF, w, m, tf, t\lambda) = \mathbf{B}(F, w, m, f, \lambda)\,.$$

Introducing new variables $\alpha = \frac{F}{m\lambda}, \beta = \frac{w}{m}, \gamma = \frac{f}{\lambda}$ we can see that

$$\frac{1}{m}\mathbf{B}(F, w, m, f, \lambda) = B(\frac{F}{m\lambda}, \frac{w}{m}, \frac{f}{\lambda}) =: B(\alpha, \beta, \gamma)\,, \tag{9}$$

where function $B(\alpha, \beta, \gamma) = \mathbf{B}(\alpha, \beta, 1, \gamma, 1)$. B is defined in the domain

$$G := \{(\alpha, \beta, \gamma) : |\gamma| \leq \alpha, 1 \leq \beta \leq Q\}\,. \tag{10}$$

8. Properties of $\mathbf{B}_Q$: a special form of concavity

Theorem

Let $P, P_+, P_- \in \Omega$, $P = (F, w, \min(m_+, m_-), f, \lambda)$,
$P_+ = (F + A, w + u, m_+, f + a, \lambda + ta)$,
$P_- = (F - A, w - u, m_-, f - a, \lambda - ta)$, $0 \le t \le 1$. *Then*

$$\mathbf{B}(P) - \frac{1}{2}(\mathbf{B}(P_+) + \mathbf{B}(P_-)) \ge 0\,. \tag{11}$$

At the same time, if
$P, P_+, P_- \in \Omega$, $P = (F, w, \min(m_+, m_-), f, \lambda)$,
$P_+ = (F + A, w + u, m_+, f + a, \lambda - ta)$,
$P_- = (F - A, w - u, m_-, f - a, \lambda + ta)$, $0 \le t \le 1$. *Then*

$$\mathbf{B}(P) - \frac{1}{2}(\mathbf{B}(P_+) + \mathbf{B}(P_-)) \ge 0\,. \tag{12}$$

In particular $B(\alpha, \beta, \gamma)$ of slide 7 is concave: just put $t = 0$ here.

9. Properties of $\mathbf{B}_Q$: a special form of concavity

In particular, with fixed m, and with all points being inside Ω we get for all $t \in [0,1]$

$$\begin{aligned}\mathbf{B}(F,w,m,f,\lambda) \geq \frac{1}{4}\big(&\mathbf{B}(F-dF,w-dw,m,f-d\lambda,\lambda-td\lambda)+\\ &\mathbf{B}(F-dF,w-dw,m,f+d\lambda,\lambda-td\lambda)+\\ &\mathbf{B}(F+dF,w+dw,m,f-d\lambda,\lambda+td\lambda)+\\ &\mathbf{B}(F+dF,w+dw,m,f+d\lambda,\lambda+td\lambda)\big)\,.\end{aligned} \tag{13}$$

In fact, only $t=0$ and $t=1$ should be looked upon. Let us look at $t=1$ case. In lines one and four $f_+ - f_- = \lambda_+ - \lambda_-$. In lines two and three $f_+ - f_- = -(\lambda_+ - \lambda_-)$. In both case $|f_+ - f_-| = |\lambda_+ - \lambda_-|$.

Remark

1) Differential notation $dF, dw, d\lambda$ just mean small numbers, 2) in (13) we loose a bit of information (in comparison with (11),(12)), but this is exactly (13) that we are going to use in the future.

Fix $P, P_+, P_- \in \Omega$. Let φ_+, φ_-, ω_+, ω_- be functions and weights giving the supremum in $\mathbf{B}(P_+), \mathbf{B}(P_-)$ respectively up to a small number $\eta > 0$. Using the fact that $\mathbf{B}$ does not depend on I, we think that φ_+, ω_+ is on I_+ and φ_-, ω_- is on I_-. Consider

$$\varphi(x) := \begin{cases} \varphi_+(x)\,,\ x \in I_+ \\ \varphi_-(x)\,,\ x \in I_- \end{cases} ;\ \omega(x) := \begin{cases} \omega_+(x)\,,\ x \in I_+ \\ \omega_-(x)\,,\ x \in I_- \end{cases}$$

Put $a := \Delta_I \varphi = \frac{1}{2}(P_{+,4} - P_{-,4})$. Notice that for $x \in I_+, \varepsilon_I = -t$,

$$\frac{1}{|I|}\omega_+\{x \in I_+ : \sum_{J \subseteq I_+, J \in D} \varepsilon_J(\varphi, h_J)h_J(x) > \lambda\} =$$

$$\frac{1}{|I|}\omega_+\{x \in I_+ : \sum_{J \subseteq I_+, J \in D} \varepsilon_J(\varphi, h_J)h_J(x) > \lambda + ta\}$$

$$= \frac{1}{2|I_+|}\omega_+\{x \in I_+ : \sum_{J \subseteq I_+, J \in D} \varepsilon_J(\varphi_+, h_J)h_J(x) > P_{+,5}\} \geq \frac{1}{2}B(P_+) - \eta\,.$$

11. Sketch of the proof

Similarly, for $x \in I_-$ we get if $\varepsilon_I = -t$, $0 \le t \le 1$,

$$\frac{1}{|I|}\omega_-\{x \in I_- : \sum_{J \subseteq I, J \in D} \varepsilon_J(\varphi, h_J)h_J(x) > \lambda\} =$$

$$\frac{1}{|I|}\omega_-\{x \in I_- : \sum_{J \subseteq I_-, J \in D} \varepsilon_J(\varphi, h_J)h_J(x) > \lambda - ta\}$$

$$= \frac{1}{2|I_-|}\omega_-\{x \in I_- : \sum_{J \subseteq I_-, J \in D} \varepsilon_J(\varphi_-, h_J)h_J(x) > P_{-,5}\} \ge \frac{1}{2}B(P_-) - \eta\,.$$

Combining the two left hand sides we obtain for $\varepsilon_I = -1$

$$\frac{1}{|I|}\omega\{x \in I_+ : \sum_{J \subseteq I, J \in D} \varepsilon_J(\varphi, h_J)h_J(x) > \lambda\} \ge \frac{1}{2}(B(P_+) + B(P_-)) - 2\eta\,.$$

Obviously $P_3 = \min(P_{3,-}, P_{3,+}) = \min(\min_{I_-} \omega_-, \min_{I_+} \omega_+)$,
$P_5 = \lambda$,

$$\langle|\varphi|\omega\rangle_I = F = P_1, \ \langle\omega\rangle_I = w = P_2, \ \langle\varphi\rangle_I = f = P_4 \,. \qquad (14)$$

Let us use now the simple information (14): if we take the supremum in the left hand side over all functions φ, such that $\langle|\varphi|\,\omega\rangle_I = F, \langle\varphi\rangle_I = f, \langle\omega\rangle_I = w$, and weights ω: $\langle\omega\rangle_I = w$, in dyadic A_1 with A_1-norm at most Q, and supremum over all $\varepsilon_J = \pm s$, $s \in [0,1]$, (only $\varepsilon_I = -1$ stays fixed), we get a quantity smaller or equal than the one, where we have the supremum over all functions φ, such that $\langle|\varphi|\,\omega\rangle = F, \langle\varphi\rangle_I = f, \langle\omega\rangle = w$, and weights ω: $\langle\omega\rangle = w$, in dyadic A_1 with A_1-norm at most Q, and an unrestricted supremum over all $\varepsilon_J = \pm s$, $s \in [0,1]$, $\varepsilon_I = -t$, $0 \le t \le 1$. The latter quantity is of course $\mathbf{B}(F, w, m, f, \lambda)$. So we proved (11).
To prove (12) we repeat verbatim the same reasoning, only keeping now $\varepsilon_I = t$, $0 \le t \le 1$. We are done with "fancy concavity" proof.

13. Property in m: function $t \to \frac{1}{t}B(t\alpha, t\beta, \gamma)$ is increasing

Function **B** is obviously decreasing in m. In fact, if m decreases (all other coordinates vein fixed) then the collection of weights increases, and the supremum increases. It is not difficult to see that **B** is also continuous.

$$\frac{1}{m}\mathbf{B}(F, w, m, f, \lambda) = B(\frac{F}{m\lambda}, \frac{w}{m}, \frac{f}{\lambda}) =: B(\alpha/m, \beta/m, \gamma)\,, \quad (15)$$

So $t \to \frac{1}{t}B(t\alpha, t\beta, \gamma)$ is increasing.

14. Two more properties, domain and symmetry

It is easy to see from the definition of **B** that it is even in its variable f. Therefore,

$$B(\alpha, \beta, \gamma) = B(\alpha, \beta, -\gamma)\,.$$

Notice that the concavity of B (in γ) and this symmetry together imply that $\gamma \to B(\cdot, \cdot, \gamma)$ is decreasing on $\gamma \in [0, \alpha]$.
The domain of definition of B is

$$G_Q := \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 : 1 \leq \beta \leq Q,\ |\gamma| \leq \alpha\}\,.$$

For function with all these properties the following holds.

Theorem

There are absolute positive constant c such that for some point $(\alpha, \beta, \gamma) \in G$

$$B(\alpha, \beta, \gamma) \geq cQ(\log Q)^{1/4}\alpha\,. \tag{16}$$

15. Idea of the proof

Now a couple of words about the idea of the proof of Theorem of slide 14. Ideally we would like to find the formula for B (and therefore for $\mathbf{B}$ because of (15)). To proceed we rewrite the second property of $\mathbf{B}$ as a PDE on B. Then we try to find the boundary conditions on B on ∂G, and then we may hope to solve this PDE. Unfortunately there are many roadblocks on this path, starting with the fact that the second property of $\mathbf{B}$ is not a PDE, it is rather a partial differential inequality in discrete form. We will write it down as a pointwise partial differential inequality, but for that we will need a subtle result of Aleksandrov. We also can find boundary values of B, see some of them in next slides. However, the main difficulty is that our partial differential expression is in $3D$.

16. Unweighted case

We first consider the simplest case of $m = \omega = 1$ identically. The we are left with function $\mathcal{B}el(F, f, \lambda) = \mathbf{B}(F, 1, 1, f, \lambda)$, which is defined in a convex domain $\Omega_0 \subset \mathbb{R}^3$:
$\Omega_0 := \{(F, f, \lambda) \in \mathbb{R}^3 : |f| \leq F\}$, and whose concavity properties are described in

Theorem

Let $P, P_+, P_- \in \Omega_0, P = (F, f, \lambda)$, $P_+ = (F + A, f + a, \lambda + ta)$, $P_- = (F - A, f - a, \lambda - ta), t \in [0, 1]$. Then

$$\mathcal{B}el(P) - \frac{1}{2}(\mathcal{B}el(P_+) + \mathcal{B}el(P_-)) \geq 0\,. \tag{17}$$

At the same time, if $P, P_+, P_- \in \Omega_0, P = (F, f, \lambda)$, $P_+ = (F + A, f + a, \lambda - ta)$, $P_- = (F - A, f - a, \lambda + ta, t \in [0, 1]$. Then

$$\mathcal{B}el(P) - \frac{1}{2}(\mathcal{B}el(P_+) + \mathcal{B}el(P_-)) \geq 0\,. \tag{18}$$

17. Unweighted case

Let us make the change of variables, $(F, f, \lambda) \to (F, y_1, y_2)$:

$$y_1 := \frac{1}{2}(\lambda + f), \; y_2 := \frac{1}{2}(\lambda - f).$$

Denote

$$M(F, y_1, y_2) := B(F, y_1 - y_2, y_1 + y_2) = \mathcal{B}el(F, f, \lambda).$$

In terms of function M:

Theorem

The function M is defined in the domain $G := \{(F, y_1, y_2) : |y_1 - y_2| \le F\}$, and for each fixed y_2, $M(F, y_1, y_2)$ is concave in (F, y_1) and for each fixed y_1, $M(F, y_1, y_2)$ is concave in (F, y_2).

The properties of M remind strongly the properties of Burkholder function.

18. Unweighted case

In the unweighted situation we can find **B** (or M) precisely.

Theorem

$$\mathcal{B}el(F,f,\lambda) = \begin{cases} 1, & if\ \lambda \le F\,, \\ 1 - \frac{(\lambda-F)^2}{\lambda^2-f^2} & if\ \lambda > F\,. \end{cases} \tag{19}$$

This result means that we found a boundary value of the Bellman function $\mathbf{B}(F,w,m,f,\lambda)$ of the weighted problem on the part of its boundary, namely we found this function of 5 variables on $\{P \in \partial\Omega : w = P_2 = P_3 = m\}$.

$$\mathbf{B}(F,m,m,f,\lambda) = m \begin{cases} 1, & \text{if } \lambda \le F\,, \\ 1 - \frac{(\lambda-F)^2}{\lambda^2-f^2} & \text{if } \lambda > F\,. \end{cases} \tag{20}$$

Thus, the boundary values of B:

$$B(\alpha,1,\gamma) = \begin{cases} 1, & \text{if } \alpha \ge 1\,, \\ 1 - \frac{(1-\alpha)^2}{1-\gamma^2} & \text{if } 0 \le |\gamma| \le \alpha < 1\,. \end{cases} \tag{21}$$

18a. Unweighted case: a small miracle

Let $\mathcal{B}el_0(F, f, \lambda) =$ the same function as on slide 11 but ε_I are allowed to be only ± 1.

Theorem

$$\mathcal{B}el_0(F, f, \lambda) = \begin{cases} 1, \ \text{if } \lambda \le F\,, \\ 1 - \frac{(\lambda - F)^2}{\lambda^2 - f^2} \ \text{if } \lambda > F \end{cases} = \mathcal{B}el(F, f, \lambda) \qquad (22)$$

By definition $\mathcal{B}el_0 \le \mathcal{B}el$: $\varepsilon_I = \pm 1$ versus $\varepsilon_I \in [-1, 1]$. In Banach space norm such martingale transforms obviously have the same norm. But we work now with $L^{1,\infty}$. By Sten–Weiss lemma $\|MT_{[-1,1]}\|_{L^{1,\infty}} \le 2(2 + \log 2 \sum k 2^{-k}) \|MT_{\pm 1}\|_{L^{1,\infty}}$. But we got from the Theorem above that the norms are equal:

$$\|MT_{[-1,1]}\|_{L^{1,\infty}} = \|MT_{\pm 1}\|_{L^{1,\infty}}\,.$$

How to get this equality without the use of Bellman functions?

19. Why Aleksandrov's theorem is necessary below

We can mollify $\mathbf{B}$ to make it smooth and still to have its "fancy concavity properties". But then we loose homogeneity, and cannot reduce $\mathbf{B}$ to B. We can mollify $\mathbf{B}$ to keep its homogeneity–just choose the mollifier depending on the point–but then we loose its "fancy concavity property". In short, we have a problem with the mollification. This is why Aleksandrov's theorem is very useful now.

20. From discrete inequality to differential inequality via Aleksandrov's theorem

We saw on slide 8 that b is concave. By the result of Aleksandrov, B has all second derivatives almost everywhere, this means that for a. e. $x \in G^\circ$ and all small vectors $h \in \mathbb{R}^3$,

$$B(x+h) = B(x) + \nabla B(x) \cdot h + \langle H_B(x) \cdot h, h \rangle + o(|h|^2)\,, \quad (23)$$

where H_B is the Hessian matrix of B. On the other hand, the "fancy concavity property" of slide 9 can be rewritten in terms of B as follows: $B(\frac{F}{\lambda}, \beta, \frac{f}{\lambda}) -$

$$\frac{1}{4}\Big[B(\frac{F-dF}{\lambda-d\lambda}, \beta-d\beta, \frac{f-d\lambda}{\lambda-d\lambda}) + B(\frac{F-dF}{\lambda-d\lambda}, \beta-d\beta, \frac{f+d\lambda}{\lambda-d\lambda}) +$$
$$B(\frac{F+dF}{\lambda+d\lambda}, \beta+d\beta, \frac{f-d\lambda}{\lambda+d\lambda}) + B(\frac{F+dF}{\lambda+d\lambda}, \beta+d\beta, \frac{f+d\lambda}{\lambda+d\lambda})\Big] \geq 0\,. \quad (24)$$

21. From discrete inequality to differential inequality via Aleksandrov's theorem

Theorem

For almost every point $P = (\alpha, \beta, \gamma) =: (\frac{F}{\lambda}, \beta, \frac{f}{\lambda}) \in G^\circ$ *and every vector* $(dF, d\beta, d\lambda) \in \mathbb{R}^3$ *we have*

$$
\begin{aligned}
&-\alpha^2 B_{\alpha\alpha}(P)\left(\frac{dF}{F} - \frac{d\lambda}{\lambda}\right)^2 - \beta^2 B_{\beta\beta}(P)\left(\frac{d\beta}{\beta}\right)^2 - \\
&(1+\gamma^2)B_{\gamma\gamma}(P)\left(\frac{d\lambda}{\lambda}\right)^2 - 2\alpha\beta B_{\alpha\beta}(P)\left(\frac{dF}{F} - \frac{d\lambda}{\lambda}\right)\frac{d\beta}{\beta} + \\
&2\beta\gamma B_{\beta\gamma}(P)\frac{d\beta}{\beta}\frac{d\lambda}{\lambda} + 2\alpha\gamma B_{\alpha\gamma}(P)\left(\frac{dF}{F} - \frac{d\lambda}{\lambda}\right)\frac{d\lambda}{\lambda} + \\
&2\alpha B_\alpha(P)\left(\frac{dF}{F} - \frac{d\lambda}{\lambda}\right)\frac{d\lambda}{\lambda} - 2\gamma B_\gamma(P)\left(\frac{d\lambda}{\lambda}\right)^2 \geq 0\,.
\end{aligned} \tag{25}
$$

22. From discrete inequality to differential inequality via Aleksandrov's theorem

Let us call by $\mathcal{N}$ the matrix of the quadratic form in (25). After a rather straightforward operation $\mathcal{N} \to \mathcal{M} := A^* \mathcal{N} A$ with an invertible matrix A we can write down the non-negativity of the differential form in (25) as the a.e. in G° non-negativity of the following matrix

$$\mathcal{M}_1 := \begin{bmatrix} -\alpha^2 B_{\alpha\alpha}, & -\alpha\beta B_{\alpha\beta}, & \alpha\gamma B_{\alpha\gamma} + \alpha B_\alpha \\ -\alpha\beta B_{\alpha\beta}, & -\beta^2 B_{\beta\beta}, & \beta\gamma B_{\beta\gamma} \\ \alpha\gamma B_{\alpha\gamma} + \alpha B_\alpha, & \beta\gamma B_{\beta\gamma}, & -(1+\gamma^2) B_{\gamma\gamma} - 2\gamma B_\gamma \end{bmatrix} \geq 0\,. \tag{26}$$

$$\mathcal{M}_2 := \begin{bmatrix} -\alpha^2 B_{\alpha\alpha}, & -\alpha\beta B_{\alpha\beta}, & -\alpha\gamma B_{\alpha\gamma} \\ -\alpha\beta B_{\alpha\beta}, & -\beta^2 B_{\beta\beta}, & -\beta\gamma B_{\beta\gamma} \\ -\alpha\gamma B_{\alpha\gamma}, & -\beta\gamma B_{\beta\gamma}, & -\gamma^2 B_{\gamma\gamma} \end{bmatrix} \geq 0\,. \tag{27}$$

24. From discrete inequality to differential inequality via Aleksandrov's theorem

Taking half-sum of (26) and (27), we obtain the following non-negativity:

$$\mathcal{M} := \begin{bmatrix} -\alpha^2 B_{\alpha\alpha}, & -\alpha\beta B_{\alpha\beta}, & \frac{1}{2}\alpha B_\alpha \\ -\alpha\beta B_{\alpha\beta}, & -\beta^2 B_{\beta\beta}, & 0 \\ \frac{1}{2}\alpha B_\alpha, & 0, & -(\frac{1}{2}+\gamma^2)B_{\gamma\gamma} - \gamma B_\gamma \end{bmatrix} \geq 0\,. \tag{28}$$

It is now natural to restrict the quadratic form of this matrix on certain $2D$ hyperplanes in the $3D$ tangent space Tan_p of the graph $\Gamma := \{p := (P, B(P)), P \in G^\circ\}$ at a given point p. Namely, let us consider the quadratic form of matrix $\mathcal{M}$ in (26) on vectors of the form

$$(\xi, \xi, \eta)\,. \tag{29}$$

Then, using the notation

$$\psi(\alpha,\beta,\gamma) := \psi_B(\alpha,\beta,\gamma) := -\alpha^2 B_{\alpha\alpha} - 2\alpha\beta B_{\alpha\beta} - \beta^2 B_{\beta\beta}\,, \quad (30)$$

we get the a.e. in G° non-negativity of the following matrix

$$\begin{bmatrix} \psi(\alpha,\beta,\gamma), & \frac12 \alpha B_\alpha \\ \frac12 \alpha B_\alpha, & -(\frac12+\gamma^2)B_{\gamma\gamma} - \gamma B_\gamma \end{bmatrix} \ge 0\,. \quad (31)$$

Or,

$$\begin{bmatrix} \psi(\alpha,\beta,\gamma), & \frac12 \alpha B_\alpha \\ \frac12 \alpha B_\alpha, & -(\frac12+\gamma^2)^{1/2}[(\frac12+\gamma^2)^{1/2} B_\gamma]_\gamma \end{bmatrix} \ge 0\,. \quad (32)$$

Or, as $\gamma << 1$

$$\begin{bmatrix} \psi(\alpha,\beta,\gamma), & \frac12 \alpha B_\alpha \\ \frac12 \alpha B_\alpha, & -[(\frac12+\gamma^2)^{1/2} B_\gamma]_\gamma \end{bmatrix} \ge 0\,. \quad (33)$$

26. Mollification of B

Definition

Consider a subdomain of G,

$$G_1 := \{(\alpha, \beta, \gamma) \in G : |\gamma| < \frac{1}{2}\alpha, 2 < \beta < Q\}\,.$$

Denote temporarily $P_t := (t\alpha, t\beta, \gamma)$, $(\alpha, \beta, \gamma) \in G_1$, $1/2 \le t \le 1$. Then we get for every such t and every point P_t the following inequality for all $(\xi, \eta) \in \mathbb{R}^2$:

$$\xi^2[\psi(P_t)] + \xi\eta(\alpha t B_\alpha(P_t)) + \eta^2(-[(\frac{1}{2} + \gamma^2)^{1/2} B_\gamma]_\gamma(P_t)) \ge 0\,. \tag{34}$$

27. Mollified B is H

Denote $H(P) = 2\int_{1/2}^{1} B(P_t)dt$. Notice several simple facts. First of all

$$\alpha H_\alpha = 2\int_{1/2}^{1} \alpha t B(t\alpha, t\beta, \gamma)dt,\ \ \alpha^2 H_{\alpha\alpha} = 2\int_{1/2}^{1} (\alpha t)^2 B_{\alpha\alpha}(t\alpha, t\beta, \gamma)dt\,.$$

$$\psi_H = -\alpha^2 H_{\alpha\alpha} - 2\alpha\beta H_{\alpha\beta} - \beta^2 H_{\beta\beta} = 2\int_{1/2}^{1} \psi_B(t\alpha, t\beta, \gamma)dt\,.$$

Now integrate (34) on the interval $t \in [1/2, 1]$. The previous simple observations allow us now to rewrite this as a pointwise inequality for function H on domain G_1 introduced in Definition on slide 26:

$$\xi^2[\psi_H(P)] + \xi\eta(\alpha H_\alpha(P)) + \eta^2(-[(\frac{1}{2} + \gamma^2)^{1/2} B_\gamma]_\gamma(P)) \geq 0\,. \tag{35}$$

28. Why H and not B?

The reader wonders why we are so keen to replace (34) by a virtually the same (35)? The answer is because we can give a very good pointwise estimate on $\psi_H(P), P \in G_1$. Unfortunately we cannot give any pointwise estimate on $\psi(P), P \in G$.

$$R := \sup \frac{B(P)}{\alpha} \quad P = (\alpha, \beta, \gamma) \in G\,. \tag{36}$$

Our goal formulated in (16) is to prove $R \geq cQ(\log Q)^{\varepsilon}$. We are still not too close, but notice that automatically $B(P) \leq R\alpha,\ P = (\alpha, \beta, \gamma) \in G,.$

Lemma (Main)

If $P = (\alpha, \beta, \gamma)$ is such that $|\gamma| \leq \frac{1}{8}\alpha$ and $\beta > 100$ then

$$\psi_H(P) = 2\int_{1/2}^{1} \psi(t\alpha, t\beta, \gamma)dt \leq CR(|\gamma| + \frac{\alpha}{\beta})\,,$$

where C is an absolute constant.

29. The proof of the Main Lemma

Consider function

$$\varphi(t) := B(t\alpha, t\beta, \gamma) \tag{37}$$

for a. e. $(\alpha, \beta, \gamma) \in G_1$. It is concave.
Let us first prove that

$$\int_{1/2}^{1} -\varphi''(t)dt \leq CR(|\gamma| + \frac{\alpha}{\beta}). \tag{38}$$

This would imply

$$\int_{1/2}^{1} \psi(t\alpha, t\beta, \gamma)dt \leq CR(|\gamma| + \frac{\alpha}{\beta}),$$

because we have

$$\psi(t\alpha, t\beta, \gamma) = -t^2\varphi''(t).$$

To prove (38) let us consider an auxiliary function $r(t) := \varphi(1)t - \varphi(t)$. It is defined for $t \in [\max(\frac{|\gamma|}{\alpha}, \frac{1}{\beta}), 1]$. At 1 it vanishes, it is convex, and it attains its maximum on its left end-point $t_0 = \max(\frac{|\gamma|}{\alpha}, \frac{1}{\beta})$. The last statement follows from the fact that $\varphi(t)/t$ is increasing: property of B from slide 13.
So on $[t_0, 1]$

$$r(t) \le r(t_0) \le \varphi(1)t_0 \le R\alpha t_0 \le R\alpha(\frac{|\gamma|}{\alpha} + \frac{1}{\beta})\,. \qquad (39)$$

As $\varphi(t)/t$ is increasing, we have $t\varphi'(t) - \varphi(t) \ge 0$, and thus $r'(1) \le 0$. Let us write down the Taylor formula for convex function $r(t)$ in the integral form, keeping in mind that $r(1) = 0$, $r'(1) \le 0$: $r(t_0) = (t_0 - 1)r'(1) + \int_{t_0}^1 dt \int_t^1 r''(s)ds$. Fubini's theorem, (39), and $r'(1) \le 0$ imply

$$\int_{t_0}^1 (s - t_0)r''(s)ds \le R\alpha(\frac{|\gamma|}{\alpha} + \frac{1}{\beta})\,.$$

But $t_0 \le \frac{1}{8}$ by the assumptions of the lemma. So $\int_{1/2}^1 r''(s)ds \le \frac{8}{3}R\alpha(\frac{|\gamma|}{\alpha} + \frac{1}{\beta})$. Hence, as $r'' = -\varphi''$, we get proof.

31. The obstacle condition

Let us temporarily take for granted the following inequality, where c_1, c_2 are absolute positive constants:

$$\alpha \le c_2 \frac{\beta}{R} \Rightarrow H_\alpha(\alpha, \beta, \gamma) \ge c_1 \beta\,,\ \ \beta \in (1, Q/2]\,. \tag{40}$$

32. Ending the proof

Put

$$G_3 = \{P \in G : |\gamma| \le \frac{1}{1000}\alpha, \beta > 100\} .$$

By positivity of quadratic form on slide 27, we conclude that for any $P = (\alpha, \beta, \gamma) \in G_3$

$$[\psi_H] \cdot [-(\frac{1}{2} + \gamma^2)^{1/2} B_\gamma]_\gamma \ge \frac{1}{4}\alpha^2 H^2_\alpha . \qquad (41)$$

Using the Main Lemma we obtain

$$\psi_H \le CR(\gamma + \frac{\alpha}{\beta}) .$$

Now we combine this inequality with the ones on slides 39 and 27 obtain

$$-[(\frac{1}{2} + \gamma^2)^{1/2} B_\gamma]_\gamma \ge c_3 \frac{\alpha^2\beta^2}{R(\frac{\alpha}{\beta} + \gamma)} . \qquad (42)$$

Integrate (and use $\gamma << 1$)

$$-H_\gamma \ge c_6 \frac{\alpha^2\beta^2}{R} \log\left(1 + \frac{\beta}{\alpha}\gamma\right) .$$

33. Ending the proof

Integrate again:

$$H(\alpha,\beta,0)-H(\alpha,\beta,\gamma)\geq c_6\frac{\alpha^3\beta}{R}\left[\left(1+\frac{\beta}{\alpha}\gamma\right)\log\left(1+\frac{\beta}{\alpha}\gamma\right)-\frac{\beta}{\alpha}\gamma\right]$$
$$\geq c_7\frac{\alpha^2\beta^2}{R}\gamma\log\left(\frac{\beta}{\alpha}\gamma\right)\,, \tag{43}$$

the last inequality holds true because $\frac{\beta}{\alpha}=cR$, and because from now on we will fix α, γ and β:

$$\alpha=c_0\frac{\beta}{R},\ \beta=\frac{Q}{4},\ \gamma=c_1\frac{\beta}{R}\,, c_1<<c_0. \tag{44}$$

We just obtained the following inequality

$$\frac{\alpha^2\beta^2}{R}\gamma\log\left(\frac{\beta}{\alpha}\gamma\right)\leq C(H(\alpha,\beta,0)-H(\alpha,\beta,\gamma))\,. \tag{45}$$

34. Ending the proof

Being even in γ on $\gamma \in [-\alpha, \alpha]$ and concave, H automatically decreases for $\gamma \in [0, \alpha]$, concavity and non-negativity of H give $H(\alpha, \beta, \gamma) \geq (1 - \frac{\gamma}{\alpha})H(\alpha, \beta, 0)$. This allows us to estimate the right hand side of (45), and we have

$$\frac{\alpha^2\beta^2}{R}\gamma \log\left(\frac{\beta}{\alpha}\gamma\right) \leq C(H(\alpha, \beta, 0) - H(\alpha, \beta, \gamma)) \leq C\frac{\gamma}{\alpha}H(\alpha, \beta, 0)\,.$$

Taking into consideration one more time that $H(\alpha, \beta, \gamma) \leq R\alpha$ by the definition of R in (36) and by the construction of H, we get

$$\frac{\alpha^2\beta^2}{R}\gamma \log\left(\frac{\beta}{\alpha}\gamma\right) \leq C(H(\alpha, \beta, 0) - H(\alpha, \beta, \gamma)) \leq CR\gamma\,.$$

Or, as by our choice of α, β, γ, $\frac{\beta}{\alpha}\gamma \asymp cQ$, we get

$$\frac{Q^4}{R^4} \log\left(\frac{\beta}{\alpha}\gamma\right) \leq C \Rightarrow R \geq cQ(\log Q)^{\frac{1}{4}} \qquad (46)$$

Let us consider the largest $\widetilde{\alpha} \in [\alpha, 1]$, where $\alpha = \frac{Q}{24R}$ such that the following holds

$$H(\widetilde{\alpha}, \frac{Q}{4}, 0) = \frac{Q}{24}, \text{ then } H(\widetilde{\alpha}, \frac{Q}{4}, \gamma) \leq \frac{Q}{24}, \gamma \in [0, \widetilde{\alpha}] . \tag{47}$$

Two cases may occur.
Case 1: $\widetilde{\alpha} \geq \frac{Q^{1/2}}{24R^{1/2}}$. Then with these new data, but without any other changes,

$$c\frac{Q^3}{R^3} \log\left(\frac{cQ}{\widetilde{\alpha}}\gamma\right) = c\frac{Q^3}{R^3} \log\left(\frac{cQR^{1/2}}{Q^{1/2}} \cdot \frac{cQ^{1/2}}{R^{1/2}}\right) \leq C . \tag{48}$$

This implies

$$R \geq cQ \log^{1/3} Q . \tag{49}$$

Case 2: $\widetilde{\alpha} \le \frac{Q^{1/2}}{24R^{1/2}}$. At $\alpha_1 := \min(\frac{Q}{48R}, \frac{2}{3}\widetilde{\alpha})$ we have

$$H(\alpha_1, \frac{Q}{4}, \gamma) \le \frac{Q}{48}\,.$$

But we saw that $\widetilde{\alpha} \ge \frac{Q}{24R}$ by its definition. Hence, $\alpha_1 = \frac{Q}{48\alpha}$.
Comparing with (47) we conclude that

$$\widetilde{\alpha} H_\alpha(\alpha_1, \frac{Q}{4}, \gamma) \ge (\widetilde{\alpha} - \alpha_1) H_\alpha(\alpha_1, \frac{Q}{4}, \gamma) \ge$$

$$H(\widetilde{\alpha}, \frac{Q}{4}, \gamma) - H(\alpha_1, \frac{Q}{4}, \gamma) \ge (1 - \frac{\gamma}{\widetilde{\alpha}}) H(\widetilde{\alpha}, \frac{Q}{4}, 0) - \frac{Q}{48} \ge$$

$$(1 - \frac{\gamma}{\widetilde{\alpha}}) H(\widetilde{\alpha}, \frac{Q}{4}, 0) - \frac{Q}{48} \ge (1 - \frac{\gamma}{\widetilde{\alpha}}) \frac{Q}{24} - \frac{Q}{48} = \frac{Q}{144}\,,$$

if $\gamma \in [0, \frac{2}{3}\alpha_1]$.
Using $\widetilde{\alpha} \le \frac{Q^{1/2}}{24R^{1/2}}$, we get the improved estimate on the derivative:

$$\forall \gamma \in [0, \frac{2}{3}\alpha_1] \;\; H_\alpha(\alpha_1, \frac{Q}{4}, \gamma) \ge c Q^{1/2} R^{1/2} \qquad (50)$$

$$\Rightarrow \; c\frac{Q^2}{R^2}\frac{QR}{R} \log\left(\frac{cQ}{\alpha_1}\gamma\right) \le CR, \Rightarrow R \ge cQ \log^{1/3} Q\,.$$

36. Isoperimetric inequalities and Monge–Ampère with drift

What follows is a joint work with Paata Ivanisvili.

Theorem

If a real valued function $M(x,y)$ is such that $M(x,\sqrt{y}) \in C^2(\Omega \times \mathbb{R}_+)$ and it satisfies the differential inequalities

$$\begin{bmatrix} M_{xx} + \frac{M_y}{y} & M_{xy} \\ M_{xy} & M_{yy} \end{bmatrix} \le 0 \quad \text{and} \quad M_y \le 0, \tag{51}$$

then for any $f \in C_0^\infty(\mathbb{R}^n;\ \Omega)$ we have

$$\int_{\mathbb{R}^n} M(f, \|\nabla f\|) d\gamma \le M\left(\int_{\mathbb{R}^n} f d\gamma, 0\right). \tag{52}$$

$$M(x,y) = x\ln x - \frac{y^2}{2x}, \quad x>0 \quad \text{and} \quad y \ge 0. \tag{53}$$

Notice that $M(x,y)$ satisfies (51). Indeed, $M_y = -\frac{y}{x} \le 0$ and

$$\begin{bmatrix} M_{xx} + \frac{M_y}{y} & M_{xy} \\ M_{xy} & M_{yy} \end{bmatrix} = \begin{bmatrix} -\frac{y^2}{x^3} & \frac{y}{x^2} \\ \frac{y}{x^2} & -\frac{1}{x} \end{bmatrix} \le 0. \tag{54}$$

Log-Sobolev inequality of Gross states that

$$\int_{\mathbb{R}^n} |f|^2 \ln |f|^2 d\gamma - \left(\int_{\mathbb{R}^n} |f|^2 d\gamma\right) \ln \left(\int_{\mathbb{R}^n} |f|^2 d\gamma\right) \le 2\int_{\mathbb{R}^n} \|\nabla f\|^2 d\gamma \tag{55}$$

whenever the right hand side of (55) is well-defined and finite for complex-valued f.

Beckner:
For $f \in L^2(d\gamma)$ and $1 \le p \le 2$ we have

$$\int |f|^2 d\gamma - \left(\int |f|^p d\gamma\right)^{2/p} \le (2-p)\int_{\mathbb{R}^n} \|\nabla f\|^2 d\gamma \qquad (56)$$

For $p = 1$ this is $\int |f|^2 d\gamma - \left(\int |f| d\gamma\right)^2 \le \int_{\mathbb{R}^n} \|\nabla f\|^2 d\gamma$. This shows that the spectral gap i.e. the first nontrivial eigenvalue of the self-adjoint positive operator $L = -\Delta + x \cdot \nabla$ in $L^2(\mathbb{R}^n, d\gamma)$ is bounded from below by 1.
$M(x,y) = x^{\frac{2}{p}} - \frac{2-p}{p^2} x^{\frac{2}{p}-2} y^2$ where $x, y \ge 0$ $1 \le p \le 2$. Notice that $M_y = -\frac{2(2-p)}{p^2} x^{\frac{2}{p}-2} y \le 0$ and

$$\begin{bmatrix} M_{xx} + \frac{M_y}{y} & M_{xy} \\ M_{xy} & M_{yy} \end{bmatrix} = \begin{bmatrix} -\frac{2(2-p)(1-p)(2-3p)x^{\frac{2}{p}-4} y^2}{p^4} & -\frac{4(2-p)(1-p)x^{\frac{2}{p}-3} y}{p^3} \\ -\frac{4(2-p)(1-p)x^{\frac{2}{p}-3} y}{p^3} & -\frac{4(2-p)x^{\frac{2}{p}-2}}{p^2} \end{bmatrix} \le 0 \qquad (57)$$

Bobkov:
For a Lipschitz function $f : \mathbb{R}^n \to [0,1]$, we have

$$I\left(\int_{\mathbb{R}^n} f d\gamma\right) \le \int_{\mathbb{R}^n} \sqrt{I^2(f) + \|\nabla f\|^2} d\gamma\,, \tag{58}$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-x^2/2} dx$, and $I(x) := \Phi'(\Phi^{-1}(x))$.
Testing (58) for $f(x) = 1_A$ where A is a Borel subset of $\mathbb{R}^n$ one obtains Gaussian isoperimetry: for any Borel measurable set $A \subset \mathbb{R}^n$

$$\gamma^+(A) \ge \Phi'(\Phi^{-1}(\gamma(A)))\,, \tag{59}$$

where $\gamma^+(A) := \liminf_{\varepsilon\to 0} \frac{\gamma(A_\varepsilon)-\gamma(A)}{\varepsilon}$ denotes Gaussian perimeter of A, here $A_\varepsilon = \{x \in \mathbb{R}^n : \mathrm{dist}_{\mathbb{R}^n}(A,x) < \varepsilon\}$.

$$M(x,y) = -\sqrt{I^2(x)+y^2} \quad \text{where} \quad x \in [0,1], \quad y \geq 0. \qquad (60)$$

Then $M_y = \frac{-y}{\sqrt{I^2(x)+y^2}} \leq 0$ and

$$\begin{bmatrix} M_{xx} + \frac{M_y}{y} & M_{xy} \\ M_{xy} & M_{yy} \end{bmatrix} = \begin{bmatrix} -\frac{(I'(x))^2 y^2}{(I^2(x)+y^2)^{3/2}} + \frac{I(x)I''(x)+1}{\sqrt{I^2(x)+y^2}} & y\frac{I(x)I'(x)}{(I^2(x)+y^2)^{3/2}} \\ y\frac{I(x)I'(x)}{(I^2(x)+y^2)^{3/2}} & -\frac{I^2(x)}{(I^2(x)+y^2)^{3/2}}. \end{bmatrix} \qquad (61)$$

Notice that $I''(x)I(x) = -1$, therefore (61) is negative semidefinite.

In general finding $M(x,y)$ will be based purely on solving PDEs. First notice that in log-Sobolev (55) and in Bobkov's inequality (58) determinant of the matrices (54) and (61) are zero. In Beckner–Sobolev inequality (56) determinant of (57) is zero if and only if $p=1,2$. We will seek $M(x,y)$ among those functions which in addition with (51) also satisfy *Monge–Ampére equation with a drift*:

$$\det \begin{bmatrix} M_{xx}+\frac{M_y}{y} & M_{xy} \\ M_{xy} & M_{yy} \end{bmatrix} = M_{xx}M_{yy}-M_{xy}^2+\frac{M_y M_{yy}}{y}=0 \qquad (62)$$

for $(x,y)\in\Omega\times\mathbb{R}_+$.

42. Reduction to the exterior differential systems and backwards heat equation

Let us make the following observation: consider

$$(x, y, p, q) = (x, y, M_x(x, y), M_y(x, y))$$

in *xypq*-space. This is a surface Σ in 4-space on which $\Upsilon = \mathrm{d}x \wedge \mathrm{d}y$ is nonvanishing but to which the two 2-forms

$$\Upsilon_1 = \mathrm{d}p \wedge \mathrm{d}x + \mathrm{d}q \wedge \mathrm{d}y \quad \text{and} \quad \Upsilon_2 = (y\mathrm{d}p + q\mathrm{d}x) \wedge \mathrm{d}q$$

pull back to be zero. Consider a simply connected surface Σ in *xypq*-space (with $y > 0$) on which Υ is nonvanishing but to which Υ_1 and Υ_2 pullback to be zero. The 1-form $p\mathrm{d}x + q\mathrm{d}y$ pull back to Σ to be closed (since Υ_1 vanishes on Σ) and hence exact, and therefore there exists a function $m : \Sigma \to \mathbb{R}$ such that $\mathrm{d}m = p\mathrm{d}x + q\mathrm{d}y$ on Σ. We then have (at least locally), $m = M(x, y)$ on Σ and, by its definition, we have $p = M_x(x, y)$ and $q = M_y(x, y)$ on the surface. Υ_2 vanishes when pulled back to Σ implies that $M(x, y)$ satisfies the desired equation (62).

Thus, we have encoded the given PDE as an exterior differential system on $\mathbb{R}^4$. Note, that we can make a change of variables on the open set where $q < 0$: Set $y = qr$ and let $t = \frac{1}{2}q^2$. then, using these new coordinates on this domain, we have

$$\Upsilon_1 = \mathrm{d}p \wedge \mathrm{d}x + \mathrm{d}t \wedge \mathrm{d}r \quad \text{and} \quad \Upsilon_2 = (r\mathrm{d}p + \mathrm{d}x) \wedge \mathrm{d}t.$$

Now, when we take an integral surface Σ on these 2-forms on which $\mathrm{d}p \wedge \mathrm{d}t$ is not vanishing, it can be written locally as a graph of the form

$$(p, t, x, r) = (p, t, u_p(p, t), u_t(p, t))$$

(since Σ is an integral of Υ_1), where $u(p, t)$ satisfies $u_t + u_{pp} = 0$ (since on Σ $0 = \Upsilon_2 = u_t \mathrm{d}p \wedge \mathrm{d}t + \mathrm{d}u_p \wedge \mathrm{d}t = (u_t + u_{pp})\mathrm{d}p \wedge \mathrm{d}t$). Thus, "generically" our PDE is equivalent to the backwards heat equation, up to a change of variables.

44. Parametrization of Bellman function M

Thus the function $M(x, y)$ can be parametrized as follows:

$$x = u_p\left(p, \frac{1}{2}q^2\right); \quad y = qu_t\left(p, \frac{1}{2}q^2\right);$$

$$M(x, y) = pu_p\left(p, \frac{1}{2}q^2\right) + q^2 u_t\left(p, \frac{1}{2}q^2\right) - u\left(p, \frac{1}{2}q^2\right), \quad (63)$$

where

$$u_t + u_{pp} = 0\,.$$

$M(x, \sqrt{y}) \in C^2(\Omega \times \mathbb{R}_+)$ therefore $M_y(x, 0) = 0$. By choosing $y = 0$ in (63), we have $q = 0$, and we obtain the boundary condition:

$$M(x, 0) = M_x(x, 0) \cdot x - M_y(x, 0) \cdot y|_{=0} - u(M_x(x, 0), 0)\,.$$

Or, if to denote boundary function $M(x, 0)$ by $f(x)$, then u has initial conditions ($t = 0$, that is $q^2 = (M_y(x, 0))^2 = 0$):

$$u(f'(x), 0) = xf'(x) - f(x)\,,\ f(x) = M(x, 0)\,.$$

Non-negativity of matrix also implies one more condition

In this case inequality (55) shows us sharp lower bounds of the expression $\left(\int g d\gamma\right) \ln \left(\int g d\gamma\right)$. Therefore, we should take $M(x,0) = x \ln x$. Boundary condition then can be rewritten as $u(\ln x + 1, 0) = x$ or $u(p,0) = e^{p-1}$ for all $p \in \mathbb{R}$. If we set $D = \frac{\partial^2}{\partial p^2}$ then

$$u(p,t) = e^{-tD} e^{p-1} = \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} e^{p-1} = e^{p-t-1} \quad \text{for all} \quad t \geq 0.$$

Clearly $u(p,t)$ satisfies (64) because $\det(\mathrm{Hess}\, u) = 0$. Notice that we have $u_t < 0$,

$$\begin{cases} x = e^{p-\frac{q^2}{2}-1}; \\ y = -q e^{p-\frac{q^2}{2}-1}; \end{cases} \qquad \text{then} \qquad \begin{cases} q = -\frac{y}{x}; \\ p = \ln x + \frac{y^2}{2x^2} + 1. \end{cases}$$

Therefore we obtain

$$M(x,y) = xp + qy - u\left(p, \frac{1}{2}q^2\right) = x \ln x + \frac{y^2}{2x} + x - \frac{y^2}{x} - x = x \ln x - \frac{y^2}{2x}.$$

In this case we are interested for the sharp lower bounds of the expression $-I(\int f d\gamma)$ in terms of $\int M(f, \|\nabla f\|)d\gamma$. We have $M(x,0) = -I(x)$. Boundary condition takes the form

$$u(p,0) = p\Phi(p) + \Phi'(p) \quad \text{for all} \quad p \in \mathbb{R}. \tag{65}$$

In fact, $M_x(x,0) = -I'(x)$ and $-I'(x) = \Phi^{-1}(x)$: $I'(x) = \left[e^{-\frac{[\Phi^{-1}]^2}{2}}\right]'$ and $(\Phi^{-1})' = e^{\frac{[\Phi^{-1}]^2}{2}}$. Now we will try to find the usual heat extension of $u(p,0)$ (call it $\tilde{u}(p,t)$) which satisfies $\tilde{u}_{pp} = \tilde{u}_t$, and then we try to consider the formal candidate $u(p,t) := \tilde{u}(p,-t)$. The heat extension of $\Phi'(p) = \frac{1}{\sqrt{2\pi}}e^{-p^2/2}$ is $\frac{1}{\sqrt{2\pi}\sqrt{1+2t}}e^{-\frac{p^2}{2(1+2t)}}$. Heat extension of $\Phi(p)$ is $\Phi\left(\frac{p}{\sqrt{1+2t}}\right)$. Indeed, the heat extension of the function $1_{(-\infty,0]}(p)$ at time $t = 1/2$ is $\Phi(p)$. By the semigroup property the heat extension of $\Phi(p)$ at time t will be the heat extension of $1_{(-\infty,0]}(p)$ at time $1/2 + t$ which equals to $\Phi\left(\frac{p}{\sqrt{1+2t}}\right)$.

Therefore, the heat extension of $p\Phi(p)$ can be found as follows:

$$\frac{2t}{\sqrt{2\pi}\sqrt{1+2t}}e^{-\frac{p^2}{2(1+2t)}} + p\Phi\left(\frac{p}{\sqrt{1+2t}}\right).$$

Thus we obtain that

$$\tilde{u}(p,t) = \sqrt{1+2t}\,\Phi'\left(\frac{p}{\sqrt{1+2t}}\right) + p\Phi\left(\frac{p}{\sqrt{1+2t}}\right).$$

This expression is well defined even for $t \in (0, -1/2)$. Therefore if we set

$$u(p,t) = \tilde{u}(p,-t) = \sqrt{1-2t}\,\Phi'\left(\frac{p}{\sqrt{1-2t}}\right) + p\Phi\left(\frac{p}{\sqrt{1-2t}}\right) \quad \text{for} \quad p$$

Direct computations show that $u(p,t)$ satisfies $u_t + u_{pp} = 0$, the boundary condition (65) and (64) because

$\det(\mathrm{Hess}\, u) = -\left(\frac{\Phi'(\frac{p}{\sqrt{1-2t}})}{1-2t}\right)^2 < 0$. We have $u_t = -\frac{\Phi'(\frac{p}{\sqrt{1-2t}})}{\sqrt{1-2t}} < 0$ and $u_p = \Phi\left(\frac{p}{\sqrt{1-2t}}\right)$. Therefore,

$$\begin{cases} x = \Phi\left(\frac{p}{\sqrt{1-q^2}}\right); \\ y = qr = qu_t = \frac{-q}{\sqrt{1-q^2}}\Phi'(\frac{p}{\sqrt{1-q^2}}); \end{cases} \quad \text{then} \quad \begin{cases} \Phi^{-1}(x) = \frac{p}{\sqrt{1-q^2}}; \\ y = \frac{-q}{\sqrt{1-q^2}}\Phi'(\Phi^{-1}(x)) \end{cases}$$

From the last equalities we obtain $M_y = q = -\frac{y}{\sqrt{I^2(x)+y^2}}$ and $M_x = p = \frac{I(x)\Phi^{-1}(x)}{\sqrt{I^2(x)+y^2}}$ where we remind that $I(x) = \Phi'(\Phi^{-1}(x))$. Then it is clear that

$$M(x,y) = -\sqrt{I^2(x) + y^2}.$$