Nonlinear Fourier Series via Blaschke products

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(joint with Raphy Coifman)

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Fourier series. Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic and

$$f(z) = a_0 + a_1 z + a_2 z^2 + \dots$$

On the boundary of the unit disk: Fourier Series in L^2_+

$$f(e^{it}) = a_0 + a_1 e^{it} + a_2 e^{2it} + \dots$$

Recursively.

$$f(z) = f(0) + (f(z) - f(0))$$

We can factor and get $f(z) - f(0) = z \cdot g(z)$.

$$\begin{aligned} f(z) &= f(0) + zg(z) \\ &= f(0) + z(g(0) + g(z) - g(0)) \\ &= f(0) + zg(0) + z^2 h(z) \\ &= f(0) + zg(0) + z^2 h(0) + \dots \end{aligned}$$

Recursively.

$$f(z) = f(0) + (f(z) - f(0))$$

We can factor out and get $f(z) - f(0) = z \cdot g(z)$.

Idea (Coifman, mid 90s). Factor *all* the roots inside the unit disk. There is a canonical way of doing this.

Blaschke products

$$z^m \prod_{i=1}^{k} \frac{z - a_i}{1 - \overline{a_i} z}$$

where $a_i \in \mathbb{D}$

Theorem

Any holomorphic $F : \mathbb{C} \to \mathbb{C}$ can be written as

$$F = B \cdot G,$$

where B is a Blaschke product and G has no roots inside $\mathbb{D}$.

Blaschke products – moving roots around

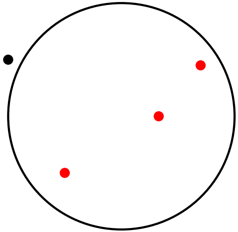


Figure: Roots of $F = BG$

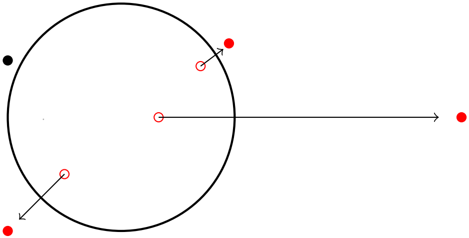


Figure: Roots of G

$$\left(z - \frac{1}{3}\right)(z-3) = \frac{z - \frac{1}{3}}{1 - \frac{z}{3}}\left(1 - \frac{z}{3}\right)(z-3)$$

If the roots are all close to the boundary, not much is happening.

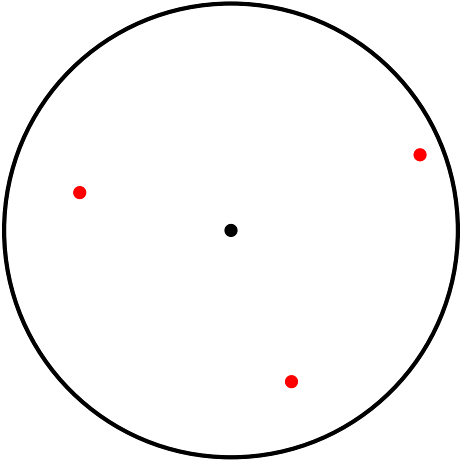
Unwinding series (Coifman, mid 1990s)

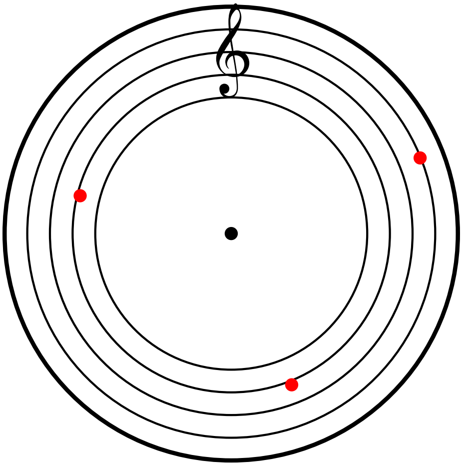
On the boundary of the unit disk $|B| = 1$.

$$F = \underbrace{B}_{\sim\text{phase}} \quad \underbrace{G}_{\sim\text{amplitude}}$$

$$\begin{aligned} F &= BG \\ &= B(G(0) + (G(z) - G(0))) \\ &= G(0)B + B(G(z) - G(0))) \\ &= G(0)B + B(B_1 G_1) \\ &= G(0)B + G_1(0)BB_1 + G_2(0)BB_1B_2 + G_3(0)BB_1B_2B_3 + \dots \end{aligned}$$

- Some sort of nonlinear Fourier series? Fun math popping up?
- Finding roots is difficult... $\implies$ (G & M Weiss algorithm)
- Convergence?





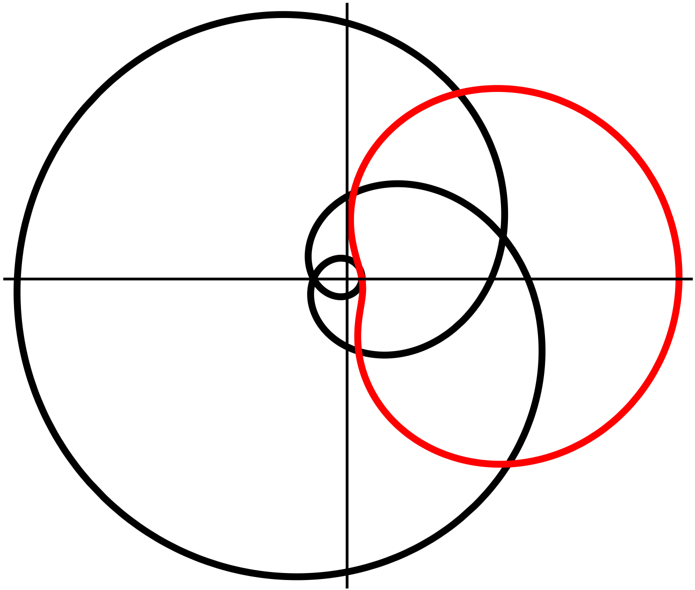


Figure: $F(\partial\mathbb{D})$ and $G(\partial\mathbb{D})$

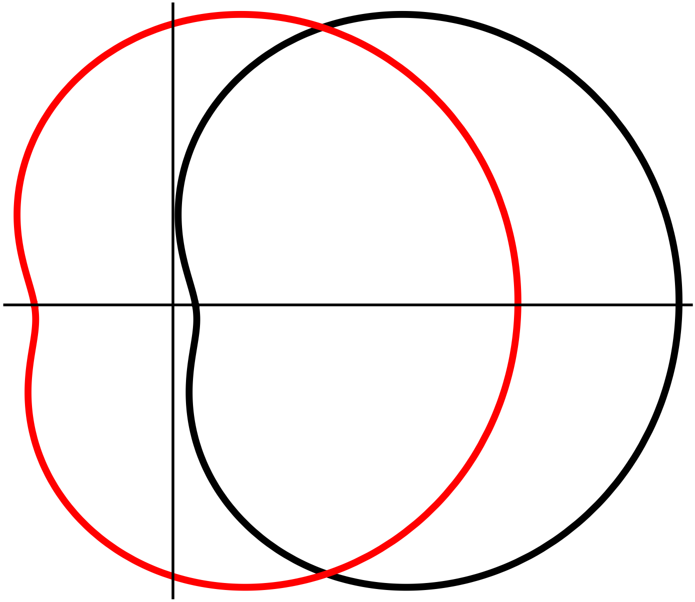


Figure: $G(\partial\mathbb{D})$ and $(G - G(0))(\partial\mathbb{D})$

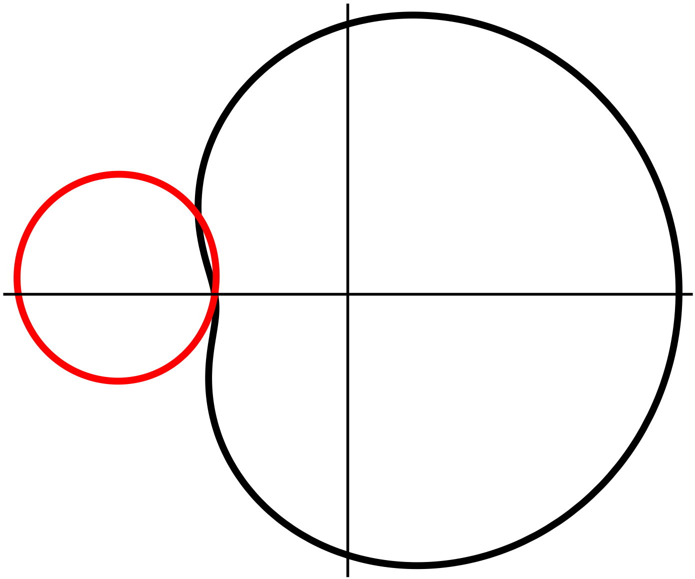


Figure: $(G - G(0))(\partial\mathbb{D})$ and its outer function

Existing work: Michel Nahon, PhD Yale 2000

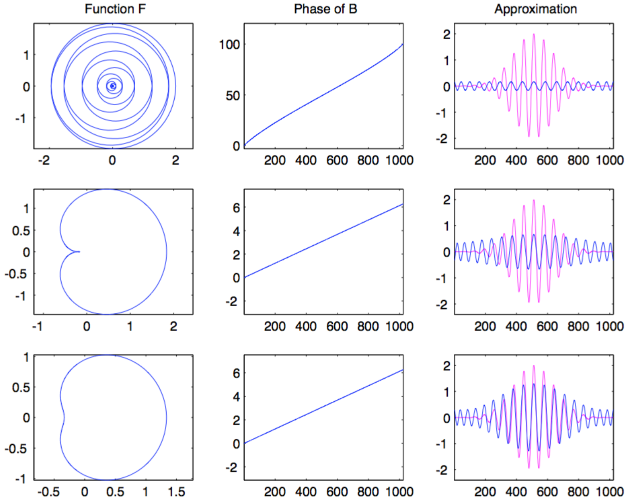
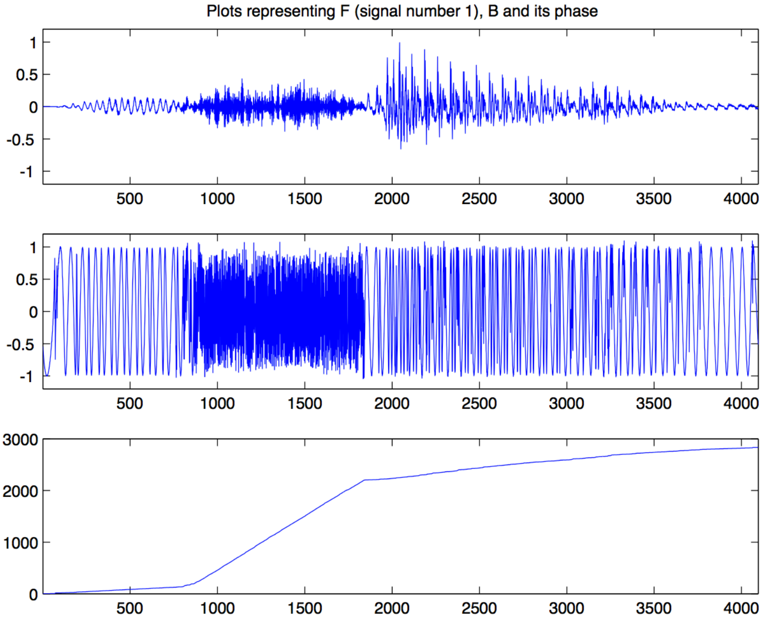
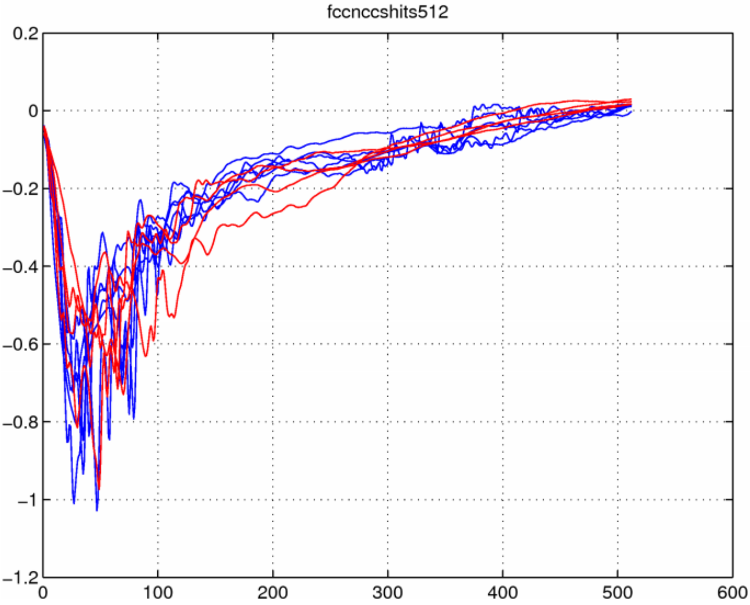


Figure 1.20: *Orthogonal decomposition of the modulated Gaussian signal* $F : \theta \mapsto e^{-(\theta-\theta_0)^2} \cdot e^{in\theta}$ *in three steps. The right column shows the evolution of the approximation.*

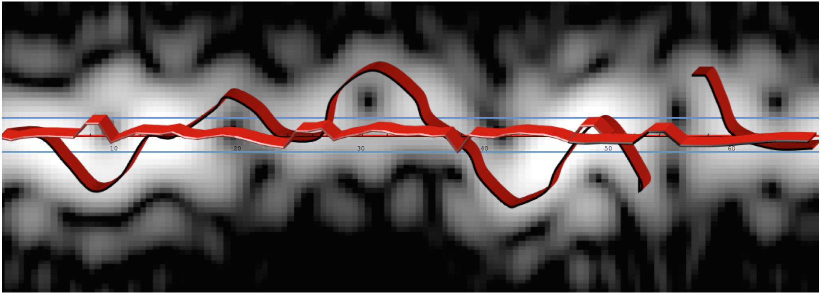
"Michel" Nahon (PhD Thesis 2000)



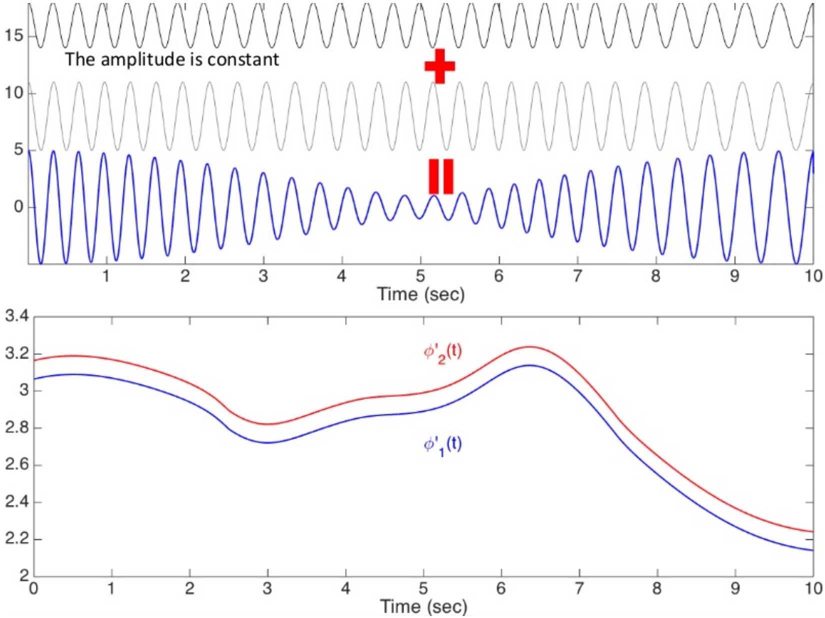
Acoustic Underwater Scattering (Letelier & Saito, 2009)



Doppler Effect (Healy, 2009)



Ongoing work (joint with Hau-tieng Wu, Toronto)



Existing results

(Guido & Mary Weiss, 1963)

One can find $F = B \cdot G$ without computing the roots of F .

Theorem (Tao Qian, 2012)

The sequence converges in $\mathcal{H}^2$ for initial data in $\mathcal{H}^2$. The convergence is at least as fast as that of Fourier series.

Proof.

Write $F(z) - F(0) = z \cdot B \cdot G$.

$$F = a_0 + a_1 z B_1 + a_2 z^2 B_1 B_2 + \cdots + a_n z^n B_1 \cdot \ldots B_n G.$$

Observation: All these terms are mutually orthogonal in $L^2(\partial\mathbb{D})$. The last term is additionally orthogonal to

$$1, z, z^2, \ldots, z^{n-1}.$$

□

Main result

Let $0 = \gamma_0 \leq \gamma_1 \leq \dots$ be an arbitrary monotonically increasing sequence of real numbers and let X be the subspace of $L^2(\mathbb{T})$ for which

$$\left\| \sum_{n \geq 0} a_n z^n \right\|_X^2 := \sum_{n \geq 0} \gamma_n |a_n|^2 < \infty.$$

We define a norm Y

$$\left\| \sum_{n \geq 0} a_n z^n \right\|_Y^2 := \sum_{n \geq 0} (\gamma_{n+1} - \gamma_n) |a_n|^2.$$

Examples.

$$\gamma_n = n^{2s} \implies X = H^s, Y = H^{s-\frac{1}{2}}$$

$$\gamma_n = n \implies X = \mathcal{D}irichlet\ space, Y = L^2$$

$$\gamma_n = \chi_{n \geq A} \implies X = P_{\geq A}, Y = P_{=A}$$

(Coifman and S., 2015)

If holomorphic F has a Blaschke factorization $F = B \cdot G$, then

$$\|G(e^{i\cdot})\|_X \leq \|F(e^{i\cdot})\|_X.$$

Moreover, if $F(\alpha) = 0$ for some $\alpha \in \mathbb{D}$, we even have

$$\|G(e^{i\cdot})\|_X^2 \leq \|F(e^{i\cdot})\|_X^2 - (1 - |\alpha|^2) \left\| \frac{G(e^{i\cdot})}{1 - \overline{\alpha} z} \right\|_Y^2.$$

Corollary. Initial data in $X \implies$ convergence in Y

(Coifman and S., 2015)

$$\|G(e^{i\cdot})\|_X^2 \leq \|F(e^{i\cdot})\|_X^2 - (1-|\alpha|^2)\left\|\frac{G(e^{i\cdot})}{1-\overline{\alpha}z}\right\|_Y^2.$$

Iinteresting special case: $X = \mathcal{D}$ and $Y = L^2$. We recover

(Carleson's formula for Blaschke product, 1960)

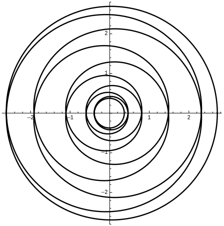
Assume F is holomorphic with roots $\{\alpha_i : i \in I\}$ in $\mathbb{D}$ and $F = B \cdot G$, then

$$\int_{\mathbb{D}} |F'(z)|^2 dz = \int_{\mathbb{D}} |G'(z)|^2 dz + \frac{1}{2}\int_{\partial\mathbb{D}} |G|^2 \sum_{i\in I} \frac{1-|a_i|^2}{|z-\alpha_i|^2}.$$

The big open question

In reality, convergence seems to happen much, much faster.

$$\int_{\mathbb{D}} |F'(z)|^2 dz = \int_{\mathbb{D}} |G'(z)|^2 dz + \frac{1}{2} \int_{\partial\mathbb{D}} |G|^2 \sum_{i \in I} \frac{1 - |a_i|^2}{|z - \alpha_i|^2}$$

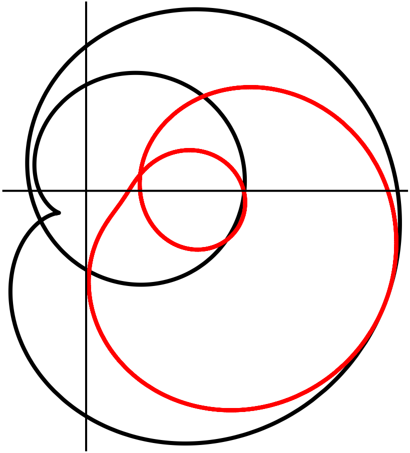


If the roots are nicely spread

$$\sum_{i \in I} \frac{1 - |a_i|^2}{|z - \alpha_i|^2} \sim \text{winding number.}$$

$$(\text{winding number}) \int_{\partial\mathbb{D}} |G|^2 \sim \int_{\mathbb{D}} |F'(z)|^2 dz$$

$$\implies \text{exponential convergence}$$



Thank you!